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A

COURSE

OF

MATHEMATICS;

FOR THE

USE OF ACADEMIES

AS WELL AS

PRIVATE TUITION.

IN TWO VOLUMES.

BY

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THE FIFTH AMERICAN, FROM THE NINTH LONDON EDITION,

WITH MANY CORRECTIONS AND IMPROVEMENTS.

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CONTENTS

OF

VOL. II.

A

COURSE

OF

MATHEMATICS, &c.

PLANE TRIGONOMETRY CONSIDERED ANALYTICALLY.

ART. 1. THERE are two methods which are adopted by mathematicians in investigating the theory of Trigonometry: the one *Geometrical*, the other *Algebraical*. In the former, the *various relations of the sines, cosines, tangents*, &c. of single or multiple *arcs or angles, and those of the sides and* angles of triangles, are deduced immediately from the figures to which the several inquiries are referred; each individual case requiring its own particular method, and resting on evidence peculiar to itself. In the latter, the nature and properties of the linear-angular quantities (sines, tangents, &c.) being first defined, some general relation of these quantities, or of them in connexion with a triangle, is expressed by one or more algebraical equations; and then every other theorem or precept, of use in this branch of science, is developed by the simple reduction and transformation of the primitive equation. Thus, the rules for the three fundamental cases in Plane Trigonometry, which are deduced by three independent geometrical investigations, in the first volume of this Course of Mathematics, are obtained algebraically, by forming, between the three data and the three unknown quantities, three equations, and obtaining, in expressions of known terms, the value of each of the unknown quantities, the others being exterminated by the usual processes. Each of these general methods has its peculiar advantages. The geometrical method carries conviction at every step; and by

keeping the objects of inquiry constantly before the eye of the student, serves admirably to guard him against the admission of error : the algebraical method, on the contrary, requiring little aid from first principles, but merely at the commencement of its career, is more properly mechanical than mental, and requires frequent checks to prevent any deviation from truth. The geometrical method is direct, and rapid, in producing the requisite conclusions at the outset of trigonometrical science ; but slow and circuitous in arriving at those results which the modern state of the science requires : while the algebraical method, though sometimes circuitous in the developement of the mere elementary theorems, is very rapid and fertile in producing those curious and interesting formulæ, which are wanted in the higher branches of pure analysis, and in mixed mathematics, especially in Physical Astronomy. This mode of developing the theory of Trigonometry is, consequently, well suited for the use of the more advanced student : and is therefore introduced here with as much brevity as is consistent with its nature and utility.

2. To save the trouble of turning very frequently to the 1st volume, a few of the principal definitious, there given, are here repeated, as follows :

The SINE of an arc, is the perpendicular let fall from one of its extremities upon the diameter of the circle which passes through the other extremity.

The COSINE of an arc, is the sine of the complement of that arc, and is equal to the part of the radius comprised between the centre of the circle and the foot of the sine.

The TANGENT of an arc, is a line which touches the circle in one extremity of that arc, and is continued from thence till it meets a line drawn from or through the centre and through the other extremity of the arc.

The SECANT of an arc, is the radius drawn through one of the extremities of that arc, and prolonged till it meets the tangent drawn from the other extremity.

The VERSED SINE of an arc, is that part of the diameter of the circle which lies between the beginning of the arc and the foot of the sine.

The COTANGENT, COSECANT, and COVERSED SINE of an arc, are the tangent, secant, and versed sine, of the complement of such arc.

3. Since arcs are proper and adequate measures of plane angles, (the ratio of any two plane angles being constantly equal to the ratio of the two arcs of any circle whose centre is the angular point, and which are intercepted by the lines

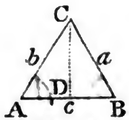
whose inclinations form the angle), it is usual, and it is perfectly safe, to apply the above names without circumlocution as though they referred to the angles themselves; thus, when we speak of the sine, tangent, or secant, of an angle, we mean the sine, tangent, or secant, of the arc which measures that angle; the radius of the circle employed being known.

4. It has been shown in the 1st vol. (pa. 382), that the tangent is a fourth proportional to the cosine, sine, and radius; the secant, a third proportional to the cosine and radius; the cotangent, a fourth proportional to the sine, cosine, and radius; and the cosecant a third proportional to the sine and radius. Hence, making use of the obvious abbreviations, and converting the analogies into equations, we have

$\tan = \frac{\text{rad.} \times \text{sine}}{\text{cos.}}$, $\text{cot.} = \frac{\text{rad.} \times \text{cos.}}{\text{sine}}$, $\text{sec.} = \frac{\text{rad}^2.}{\text{cos.}}$, $\text{cosec.} = \frac{\text{rad}^2.}{\text{sine}}$. Or, assuming unity for the rad. of the circle, these will become $\tan. = \frac{\sin.}{\cos.}$ ... $\cot. = \frac{\cos.}{\sin.} = \frac{1}{\tan.}$ ... $\sec. = \frac{1}{\cos.}$ ... $\text{cosec.} = \frac{1}{\sin.}$.

These preliminaries being borne in mind, the student may pursue his investigations.

5. Let ABC be any plane triangle, of which the side BC opposite the angle A is denoted by the small letter a, the side AC opposite the angle B by the small letter b, and the side AB opposite the angle C by the small letter c, and CD perpendicular to AB: then is $c = a \,.\, \cos.\, B + b \,.\, \cos.\, A$.



For, since $AC = b$, AD is the cosine of A to that radius; consequently, supposing radius to be unity, we have $AD = b \,.\, \cos.\, A$. In like manner it is $BD = a \,.\, \cos.\, B$. Therefore, $AD + BD = AB = c = a \,.\, \cos.\, B + b \,.\, \cos.\, A$. By pursuing similar reasoning with respect to the other two sides of the triangle, exactly analogous results will be obtained. Placed together, they will be as below:

$$\left.\begin{aligned} a &= b \,.\, \cos.\, C + c \,.\, \cos.\, B \\ b &= a \,.\, \cos.\, C + c \,.\, \cos.\, A \\ c &= a \,.\, \cos.\, B + b \,.\, \cos.\, A \end{aligned}\right\} \quad (\text{I.})$$

6. Now, if from these equations it were required to find expressions for the angles of a plane triangle, when the sides are given; we have only to multiply the first of these equations by a, the second by b, the third by c, and to subtract

each of the equations thus obtained from the sum of the other two. For thus we shall have

$$\left.\begin{array}{ll} b^2+c^2-a^2=2bc\,.\cos.\,\textsc{a}, \text{ whence } & \cos.\,\textsc{a}=\dfrac{b^2+c^2-a^2}{2bc} \\ a^2+c^2-b^2=2ac\,.\cos.\,\textsc{b}, \ldots & \cos.\,\textsc{b}=\dfrac{a^2+c^2-b^2}{2ac} \\ a^2+b^2-c^2=2ab\,.\cos.\,\textsc{c}, \ldots & \cos.\,\textsc{c}=\dfrac{a^2+b^2-c^2}{2ab} \end{array}\right\}\text{(II.)}$$

7. More convenient expressions than these will be deduced hereafter: but even these will often be found very convenient, when the sides of triangles are expressed in integers, and tables of sines and tangents, as well as a table of squares, (like that in our first vol.) are at hand.

Suppose, for example, the sides of the triangle are $a=320$, $b=562$, $c=800$, being the numbers given in prop. 4, pa. 161, of the Introduction to the Mathematical Tables: then we have

$b^2+c^2-a^2=853444$	log. =	5·9311751
$2bc$ . . $=899200$	log. =	5·9538080
The remainder being log. cos. A, or of 18°20′ =		9·9773671
Again, $a^2+c^2-b^2=426556$ . . .	log. =	5·6299760
$2ac$ . . . $=512000$ . . .	log. =	5·7092700
The remainder being log cos. B, or of 33°35′ =		9·9207060

Then $180°-(18°20'+33°35')=128°5'=$ C; where all the three angles are determined in 7 lines.

8. If it were wished to get expressions for the sines, instead of the cosines, of the angles; it would merely be necessary to introduce into the preceding equations (marked II.), instead of cos. A, cos. B, &c. their equivalents cos. A $=\sqrt{(1-\sin^2.\,\textsc{a})}$, cos. B $=\sqrt{(1-\sin^2.\,\textsc{b})}$, &c. For then, after a little reduction, there would result,

$$\left.\begin{array}{l} \sin\,\textsc{a}=\frac{1}{2bc}\sqrt{2a^2b^2+2a^2c^2+2b^2c^2-(a^4+b^4+c^4)} \\ \sin.\,\textsc{b}=\frac{1}{2ac}\sqrt{2a^2b^2+2a^2c^2+2b^2c^2-(a^4+b^4+c^4)} \\ \sin.\,\textsc{c}=\frac{1}{2ab}\sqrt{2a^2b^2+2a^2c^2+2b^2c^2-(a^4+b^4+c^4)} \end{array}\right\}$$

Or, resolving the expression under the radical into its four constituent factors, substituting s for $a+b+c$, and reducing, the equations will become

$$\left.\begin{array}{l} \sin.\,\textsc{a}=\frac{2}{bc}\sqrt{\tfrac{1}{2}s(\tfrac{1}{2}s-a)(\tfrac{1}{2}s-b)(\tfrac{1}{2}s-c)} \\ \sin.\,\textsc{b}=\frac{2}{ac}\sqrt{\tfrac{1}{2}s(\tfrac{1}{2}s-a)(\tfrac{1}{2}s-b)(\tfrac{1}{2}s-c)} \\ \sin.\,\textsc{c}=\frac{2}{ab}\sqrt{\tfrac{1}{2}s(\tfrac{1}{2}s-a)(\tfrac{1}{2}s-b)(\tfrac{1}{2}s-c)} \end{array}\right\}\text{(III.)}$$

$\frac{a+b-c+c-c}{2}=\frac{a+b+c-2c}{2}=\frac{s-2c}{2}=(\tfrac{1}{2}s-$

$\frac{4}{4}\times\frac{1}{2bc}\sqrt{2a^2b^2+2a^2c^2+2b^2c^2-(a^4+b^4+c^4)}=\frac{1}{4}\times\frac{4}{2bc}\sqrt{2a^2b^2+2a^2c^2+2b^2c^2-(a^4+b^4}$

$\sqrt{\frac{2a^2b^2+2a^2c^2+2b^2c^2-(a^4+b^4+c^4)}{16}}=\frac{2}{a}\sqrt{\frac{(a+b+c)}{2}\frac{(a+b-c)}{2}\frac{(a-b+c)}{2}\frac{(-a+b+c)}{2}}=$

$\frac{1}{a}\sqrt{s(\tfrac{1}{2}s-a)(\tfrac{1}{2}s-b)(\tfrac{1}{2}s-c)}$

These equations are moderately well suited for computation in their latter form; they are also perfectly symmetrical: and as indeed the quantities under the radical are identical, and are constituted of known terms, they may be represented by the same character; suppose K: then shall we have

$$\sin. A = \frac{2K}{bc} \ldots \sin. B = \frac{2K}{ac} \ldots \sin. C = \frac{2K}{ab} \ldots (iii.)$$

Hence we may immediately deduce a very important theorem: for, the first of these equations, divided by the second, gives $\frac{\sin. A}{\sin. B} = \frac{a}{b}$, and the first divided by the third gives $\frac{\sin. A}{\sin. C} = \frac{a}{c}$; whence since two equal fractions denote an equation, we have

$$\sin. A : \sin. B : \sin. C \propto a : b : c \ldots (IV.)$$

Or, in words, *the sides of plane triangles are proportional to the sines of their opposite angles.* (See th. 1 Trig. vol. i).

9. Before the remainder of the theorems, necessary in the solution of plane triangles, are investigated, the fundamental proposition in the theory of sines, &c. must be deduced, and the method explained by which Tables of these quantities, confined within the limits of the quadrant, are made to extend to the whole circle, or to any number of quadrants whatever. *In order to this, expressions must* be first obtained for the sines, cosines, &c. of the sums and differences of any two arcs or angles. Now, it has been found (I.) that $a = b \cdot \cos. C + c \cdot \cos. B$. And the equations (IV.) give $b = a \cdot \frac{\sin. B}{\sin. A} \ldots c = a \cdot \frac{\sin. C}{\sin. A}$. Substituting these values of b and c for them in the preceding equation, and multiplying the whole by $\frac{\sin. A}{a}$, it will become

$$\sin. A = \sin. B \cdot \cos. C + \sin. C \cdot \cos. B.$$

But, in every plane triangle, the sum of the three angles is equal to two right angles; therefore, B and C are equal to the supplement of A: and, consequently, since an angle and its supplement have the same sine (cor. 1, p. 379, vol. i), we have $\sin. (B + C) = \sin. B \cdot \cos. C + \sin. C \cdot \cos. B$.

10. If, in the last equation, C become subtractive, then would sin. C manifestly become subtractive also, while the cosine of C would not change its sign, since it would still continue to be estimated on the same radius in the same direction. Hence the preceding equation would become

$$\sin. (B - C) = \sin. B \cdot \cos. C - \sin. C \cdot \cos. B.$$

11. Let c′ be the complement of c, and $\frac{1}{4}\bigcirc$ be the quarter of the circumference: then will $c' = \frac{1}{4}\bigcirc - c$, sin. c′ = cos. c, and cos. c′ = sin. c. But (art. 10), sin. (B — c′) = sin. B. cos. c′ — sin. c′ cos. B. Therefore, substituting for sin. c′, cos. c′, their values, there will result sin. (B — c′) = sin. B. sin. c — cos. B . cos c. But because $c' = \frac{1}{4}\bigcirc - c$, we have sin. (B—c′) = sin. (B + c — $\frac{1}{4}\bigcirc$) = sin. [(B + c)— $\frac{1}{4}\bigcirc$] = — sin. [$\frac{1}{4}\bigcirc$ — (B + c)] = — cos. (B + c). Substituting this value of sin. (B — c′) in the equation above, it becomes cos. (B + c) = cos. B . cos. c — sin. B . sin. c.

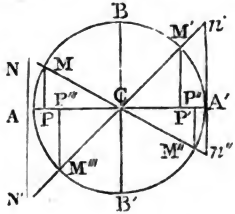
12. In this latter equation, if c be made subtractive, sin. c will become — sin. c, while cos. c will not change: consequently the equation will be transformed to the following, viz. cos. (B — c) = cos. B . cos c + sin. B . sin. c.

If, instead of the angles B and c, the angles had been A and B; or, if A and B represented the *arcs* which measure those angles, the results would evidently be similar: they may therefore be expressed generally by the two following equations, for the sines and cosines of the sums or differences of any two arcs or angles.

$$\left.\begin{aligned}\sin.(A \pm B) &= \sin. A . \cos. B \pm \sin. B . \cos. A.\\ \cos.(A \pm B) &= \cos. A . \cos. B \mp \sin. A . \sin. B^{*}.\end{aligned}\right\} \quad (V.)$$

13. We are now in a state to trace completely the mutations of the sines, cosines, &c. as they relate to arcs in the various parts of a circle; and thence to perceive that tables which apparently are included within a quadrant, are, in fact, applicable to the whole circle.

Imagine that the radius MC of the circle, in the marginal figure, coinciding at first with AC, turns about the point C (in the same manner as a rod would turn on a pivot) and thus forming successively with AC all possible angles: the point M at its extremity passing over all the points of the circumference ABA′B′A, or describing the whole circle. Tracing this motion attentively, it will appear, that at the point A, where the arc is nothing, the sine is nothing also, while the cosine does not differ from the radius. As the radius MC recedes from AC, the sine PM keeps increasing, and the cosine CP decreasing, till the



* See, for a different mode of investigating these and other useful formulæ, vol. i. pp. 393—396.

describing point M has passed over a quadrant, and arrived at B: in that case, PM becomes equal to CB the radius, and the cosine CP vanishes. The point M continuing its motion beyond B, the sine P'M' will diminish, while the cosine C'P', which now falls on the *contrary* side of the centre C will increase. In the figure, P'M' and CP' are respectively the sine and cosine of the arc A'M', or the sine and cosine of ABM', which is the supplement of A'M' to $\frac{1}{2}\bigcirc$, half the circumference: whence it follows that an obtuse angle (measured by an arc greater than a quadrant) has *the same sine and cosine as its supplement;* the cosine, however, being reckoned subtractive or negative, because it is situated contrariwise with regard to the centre C.

When the describing point M has passed over $\frac{1}{2}\bigcirc$, or half the circumference, and has arrived at A', the sine P'M' vanishes, or becomes nothing, as at the point A, and the cosine is again equal to the radius of the circle. Here the angle ACM has attained its maximum limit; but the radius CM may still be supposed to continue its motion, and pass *below* the diameter AA'. The sine, which will then be P''M'', will consequently fall below the diameter, and will augment as M moves along the third quadrant, while on the contrary CP'', the cosine, will diminish. In this quadrant too, both sine and cosine must be considered as negative; the former being on a contrary side of the diameter, the latter a contrary side of the centre, *to what each was respectively in the first* quadrant. At the point B', where the arc is three-fourths of the circumference, $\frac{3}{4}\bigcirc$, the sine P''M'' becomes equal to the radius CB, and the cosine CP'' vanishes. Finally, in the fourth quadrant, from B' to A, the sine P'''M''', always *below* AA', diminishes in its progress, while the cosine CP''', which is then found on the same side of the centre as it was in the first quadrant, augments till it becomes equal to the radius CA. Hence, the sine in this quadrant is to be considered as negative or subtractive, the cosine as positive. If the motion of M were continued through the circumference again, the circumstances would be exactly the same in the fifth quadrant as in the first, in the sixth as in the second, in the seventh as in the third, in the eighth as in the fourth: and the like would be the case in any subsequent revolutions.

14 If the mutations of the *tangent* be traced in like manner, it will be seen that its magnitude passes from nothing to infinity in the first quadrant; becomes negative, and decreases from infinity to nothing in the second; becomes positive again, and increases from nothing to infinity in the third quadrant; and lastly, becomes negative again, and decreases from infinity to nothing, in the fourth quadrant.

15. These conclusions admit of a ready confirmation, and others may be deduced, by means of the analytical expressions in arts. 4 and 12. Thus, if A be supposed equal to $\frac{1}{4}\bigcirc$, in equa. V, it will become

$$\cos.\ (\tfrac{1}{4}\bigcirc \pm \text{B}) = \cos.\ \tfrac{1}{4}\bigcirc\ .\ \cos.\ \text{B} \mp \sin.\ \tfrac{1}{4}\bigcirc\ .\ \sin.\ \text{B},$$
$$\sin.\ (\tfrac{1}{4}\bigcirc \pm \text{B}) = \sin.\ \tfrac{1}{4}\bigcirc\ .\ \cos.\ \text{B} \pm \sin.\ \text{B}\ .\ \cos.\ \tfrac{1}{4}\bigcirc.$$

But $\sin.\ \frac{1}{4}\bigcirc = \text{rad.} = 1$; and $\cos.\ \frac{1}{4}\bigcirc = 0$:

so that the above equations will become

$$\cos.\ (\tfrac{1}{4}\bigcirc \pm \text{B}) = \mp \sin.\ \text{B}.$$
$$\sin.\ (\tfrac{1}{4}\bigcirc \pm \text{B}) = \cos.\ \text{B}.$$

From which it is obvious, that if the sine and cosine of an arc, less than a quadrant, be regarded as positive, the cosine of an arc greater than $\frac{1}{4}\bigcirc$ and less than $\frac{1}{2}\bigcirc$ will be negative, but its sine positive. If B also be made $= \frac{1}{4}\bigcirc$; then shall we have $\cos.\ \frac{1}{2}\bigcirc = -1$; $\sin.\ \frac{1}{2}\bigcirc = 0$.

Suppose next, that in the equa. V, $\text{A} = \frac{1}{2}\bigcirc$; then shall we obtain

$$\cos.\ (\tfrac{1}{2}\bigcirc \pm \text{B}) = -\cos.\ \text{B}\ .$$
$$\sin.\ (\tfrac{1}{2}\bigcirc \pm \text{B}) = \mp \sin.\ \text{B};$$

which indicates, that every arc comprised between $\frac{1}{2}\bigcirc$ and $\frac{3}{4}\bigcirc$, or that terminates in the third quadrant, will have its sine and its cosine both negative. In this case too, when $\text{B} = \frac{1}{4}\bigcirc$, or the arc terminates at the *end* of the third quadrant, we shall have $\cos.\ \frac{3}{4}\bigcirc = 0$, $\sin.\ \frac{3}{4}\bigcirc = -1$.

Lastly, the case remains to be considered in which $\text{A} = \frac{3}{4}\bigcirc$, or in which the arc terminates in the fourth quadrant. Here the primitive equations (V.) give

$$\cos.\ (\tfrac{3}{4}\bigcirc \pm \text{B}) = \pm \sin.\ \text{B}\ .$$
$$\sin.\ (\tfrac{3}{4}\bigcirc \pm \text{B}) = -\cos.\ \text{B};$$

so that in all arcs between $\frac{3}{4}\bigcirc$ and $\bigcirc$, the cosines are positive and the sines negative.

16. The changes of the tangents, with regard to positive and negative, may be traced by the application of the preceding results to the algebraic expression for the tangent: viz. $\tan. = \frac{\sin.}{\cos.}$. For it is hence manifest, that when the sine and cosine are either both positive or both negative, the tangent will be positive; which will be the case in the first and third quadrants. But when the sine and cosine have different signs, the tangents will be negative, as in the second and fourth quadrants. The algebraic expression for the cotangent, viz. $\cot. = \frac{\cos.}{\sin.}$, will produce exactly the same results.

The expressions for the secants and cosecants, viz. sec. =

$\frac{1}{\text{cos.}}$, cosec. $= \frac{1}{\text{sin.}}$ show, that the signs of the secants are the same as those of the cosines; and those of the cosecants the same as those of the sines.

The *magnitude* of the tangent at the end of the first and third quadrants will be infinite; because in those places the sign is equal to radius, the cosine equal to zero, and therefore $\frac{\text{sin.}}{\text{cos.}} = \infty$ (infinity). Of these, however, the former will be reckoned positive, the latter negative.

17. The magnitudes of the cotangents, secants, and cosecants, may be traced in like manner; and the results of the 13th, 14th, and 15th articles, recapitulated and tabulated as below.

	0°	90°	180°	270°	360°	
Sin.	0	R	0	− R	0	
Tan.	0	∞	0	− ∞	0	
Sec.	R	∞	− R	− ∞	R	(VI.)
Cos.	R	0	− R	0	R	
Cot.	∞	0	− ∞	0	∞	
Cosec.	∞	R	− ∞	− R	∞	

The changes of signs are these:

quadrants				sin.	cos.	tan.	cot.	sec.	cosec.	
1st.	5th.	9th.	13th.	+	+	+	+	+	+	
2d.	6th.	10th.	14th.	+	−	−	−	−	+	(VII.)
3d.	7th.	11th.	15th.	−	−	+	+	−	−	
4th.	8th.	12th.	16th.	−	+	−	−	+	−	

We have been thus particular in tracing the mutations, both with regard to value and algebraic signs, of the principal trigonometrical quantities, because a knowledge of them is absolutely necessary in the application of trigonometry to the solution of equations, and to various astronomical and physical problems.

18. We may now proceed to the investigation of other expressions relating to the sums, differences, multiples, &c. of arcs; and in order that these expressions may have the more generality, give to the radius any value R, instead of confining it to unity. This indeed may always be done in an expression, however complex, by merely rendering all the terms homogeneous; that is, *by multiplying each term by such a power of* R *as shall make it of the same dimension, as the term in the equation which has the highest dimension.* Thus, the expression for a triple arc

$\sin. 3A = 3\sin. A - 4\sin^3. A$ (radius $= 1$)
becomes when radius is assumed $= R$,
$R^2 \sin. 3A = R^2 \, 3\sin. A - 4\sin^3. A$

$$\text{or } \sin. 3A = \frac{3R^2\sin. A - 4\sin^3. A}{R^2}.$$

Hence then, if consistently with this precept, R be placed for a denominator of the second member of each equation v (art. 12), and if A be supposed equal to B, we shall have

$$\sin. (A + A) = \frac{\sin. A \cdot \cos. A + \sin. A \cdot \cos. A}{R}$$

$$\text{That is, } \sin. 2A = \frac{2\sin. A \cdot \cos. A}{R}.$$

And, in like manner, by supposing B to become successively equal to 2A, 3A, 4A, &c. there will arise

$$\left.\begin{aligned} \sin. 3A &= \frac{\sin. A \cdot \cos. 2A + \cos. A \cdot \sin. 2A}{R} \\ \sin. 4A &= \frac{\sin. A \cdot \cos. 3A + \cos. A \cdot \sin. 3A}{R} \\ \sin. 5A &= \frac{\sin. A \cdot \cos. 4A + \cos. A \cdot \sin. 4A}{R} \end{aligned}\right\} \quad \text{(VIII.)}$$

And, by similar processes, the second of the equations just referred to, namely, that for cos. $(A + B)$, will give successively,

$$\left.\begin{aligned} \cos. 2A &= \frac{\cos^2. A - \sin^2. A}{R} \\ \cos. 3A &= \frac{\cos. A \cdot \cos. 2A - \sin. A \cdot \sin. 2A}{R} \\ \cos. 4A &= \frac{\cos. A \cdot \cos. 3A - \sin. A \cdot \sin. 3A}{R} \\ \cos. 5A &= \frac{\cos. A \cdot \cos. 4A - \sin. A \cdot \sin. 4A}{R} \end{aligned}\right\} \quad \text{(IX.)}$$

19. If, in the expressions for the successive multiples of the sines, the values of the several cosines in terms of the sines were substituted for them; and a like process were adopted with regard to the multiples of the cosines, other expressions would be obtained, in which the multiple sines would be expressed in terms of the radius and sine, and the multiple cosines in terms of the radius and cosine.

$$\left.\begin{aligned} \text{As } \sin. A &= s \\ \sin. 2A &= 2s \sqrt{(R^2 - s^2)} \\ \sin. 3A &= 3s - 4s^3 \\ \sin. 4A &= (4s - 8s^3) \sqrt{(R^2 - s^2)} \\ \sin. 5A &= 5s - 20s^3 + 16s^5 \\ \sin. 6A &= (6s - 32s^3 + 32s^5) \sqrt{(R^2 - s^2)} \\ &\&c. \ \&c. \end{aligned}\right\} \quad \text{(X.)}$$

$$\left.\begin{array}{l} \text{Cos. A} = c \\ \text{cos. 2A} = 2c^2 - 1 \\ \text{cos. 3A} = 4c^3 - 3c \\ \text{cos. 4A} = 8c^4 - 8c^2 + 1 \\ \text{cos. 5A} = 16c^5 - 20c^3 + 5c \\ \text{cos. 6A} = 32c^6 - 48c^4 + 18c^2 - 1 \\ \&c.\ \&c.^* \end{array}\right\} \quad \text{(XI.)}$$

Other very convenient expressions for multiple arcs may be obtained thus:

Add together the expanded expressions for sin. (B + A), sin. (B − A), that is,

add - sin. (B + A) = sin. B . cos. A + cos. B . sin. A,
to - - sin. (B − A) = sin. B . cos. A − cos. B . sin. A;
there results sin. (B + A) + sin. (B − A) = 2 cos. A . sin. B:
whence, - sin. (B + A) = 2 cos. A . sin. B − sin. (B − A).

Thus again, by adding together the expressions for cos. (B + A) and cos. (B − A), we have

cos. (B + A) + cos. (B − A) = 2 cos. A . cos. B;
whence, cos. (B + A) = 2 cos. A . cos. B − cos. (B − A).

Substituting in these expressions for the sine and cosine of B + A, the successive values A, 2A, 3A, &c, instead of B; the following series will be produced.

$$\left.\begin{array}{l} \text{sin. 2A} = 2 \text{ cos. A . sin. A.} \\ \text{sin. 3A} = 2 \text{ cos. A . sin. 2A} - \text{sin. A.} \\ \text{sin. 4A} = 2 \text{ cos. A . sin. 3A} - \text{sin. 2A.} \\ \text{sin. } n\text{A} = 2 \text{ cos. A . sin. } (n-1)\text{ A} - \text{sin. } (n-2)\text{ A.} \end{array}\right\} \quad (x.)$$

$$\left.\begin{array}{l} \text{cos. 2A} = 2 \text{ cos. A . cos. A} - \text{cos. 0 } (=1). \\ \text{cos. 3A} = 2 \text{ cos. A . cos. 2A} - \text{cos. A.} \\ \text{cos. 4A} = 2 \text{ cos. A . cos. 3A} - \text{cos. 2A.} \\ \text{cos. } n\text{A} = 2 \text{ cos. A . cos. } (n-1)\text{ A} - \text{cos. } (n-2)\text{ A.} \end{array}\right\} \quad (xi.)$$

Several other expressions for the sines and cosines of multiple arcs, might readily be found; but the above are the most useful and commodious.

20. From the equation sin. 2A $= \frac{2 \text{ sin. A . cos. A}}{\text{R}}$, it will be easy, when the sine of an arc is known, to find that of its half. For, substituting for cos. A its value $\sqrt{(\text{R}^2 - \text{sin}^2.\ \text{A})}$, there will arise sin. 2A $= \frac{2 \text{ sin. A} \sqrt{(\text{R}^2 - \text{sin}^2.\ \text{A})}}{\text{R}}$. This squared gives

$$\text{R}^2 \text{ sin.}^2\ 2\text{A} = 4\text{R}^2 \text{ sin.}^2\ \text{A} - 4 \text{ sin.}^4\ \text{A}.$$

* Here we have omitted the powers of R that were necessary to render all the terms homologous, merely that the expression might be brought in upon the page; but they may easily be supplied, when needed, by the rule in art. 18.

Here taking sin. A for the unknown quantity, we have a quadratic equation, which solved after the usual manner, gives

$$\sin. A = \pm\sqrt{\tfrac{1}{2}R^2 \pm \tfrac{1}{2}R\sqrt{R^2 - \sin^2. 2A}}.$$

If we make $2A = A'$, then will $A = \frac{1}{2}A'$; and consequently the last equation becomes

$$\left.\begin{aligned} \sin. \tfrac{1}{2}A' &= \pm\sqrt{\tfrac{1}{2}R^2 \pm \tfrac{1}{2}R\sqrt{R^2 - \sin^2. 2A'}} \\ \text{or } \sin. \tfrac{1}{2}A' &= \pm\tfrac{1}{2}\sqrt{2R^2 \pm 2R\cos. A'}: \end{aligned}\right\} \text{(XII.)}$$

by putting cos. A' for its value $\sqrt{R^2 - \sin.^2 A'}$, multiplying the quantities under the radical by 4, and dividing the whole second number by 2. Both these expressions for the sine of half an arc or angle will be of use to us as we proceed.

21. If the values of sin. $(A + B)$ and sin. $(A - B)$, given by equa. V, be added together, there will result

$$\sin(A + B) + \sin(A - B) = \frac{2\sin A \cdot \cos B}{R}; \text{ whence}$$

$$\sin A \cdot \cos B = \tfrac{1}{2}R \cdot \sin(A+B) + \tfrac{1}{2}R\sin(A-B) \ldots \text{(XIII.)}$$

Also, taking sin $(A - B)$ from sin $(A + B)$, gives

$$\sin(A + B) - \sin(A - B) = \frac{2\sin B \cdot \cos A}{R}; \text{ whence,}$$

$$\sin B \cdot \cos A = \tfrac{1}{2}R \cdot \sin(A+B) - \tfrac{1}{2}R \cdot \sin(A-B) \ldots \text{(XIV.)}$$

When $A = B$, both equa. XIII and XIV, become

$$\cos A \cdot \sin A = \tfrac{1}{2}R\sin 2A \ldots \text{(XV.)}$$

22. In like manner, by adding together the primitive expressions for cos $(A + B)$, cos $(A - B)$, there will arise

$$\cos(A + B) + \cos(A - B) = \frac{2\cos A \cdot \cos B}{R}; \text{ whence,}$$

$$\cos A \cdot \cos B = \tfrac{1}{2}R \cdot \cos(A+B) + \tfrac{1}{2}R \cdot \cos(A-B) \text{ (XVI.)}$$

And here, when $A = B$, recollecting that when the arc is nothing the cosine is equal to radius, we shall have

$$\cos^2 A = \tfrac{1}{2}R \cdot \cos. 2A + \tfrac{1}{2}R^2 \ldots \text{(XVII.)}$$

23. Deducting cos $(A + B)$ from cos $(A - B)$, there will remain

$$\cos(A - B) - \cos(A + B) = \frac{2\sin A \cdot \sin B}{R}; \text{ whence,}$$

$$\sin A \cdot \sin B = \tfrac{1}{2}R \cdot \cos(A-B) - \tfrac{1}{2}R \cdot \cos(A+B) \text{ (XVIII.)}$$

When $A = B$, this formula becomes

$$\sin^2 A = \tfrac{1}{2}R^2 - \tfrac{1}{2}R \cdot \cos 2A \ldots \text{(XIX.)}$$

24. Multiplying together the expressions for sin $(A + B)$ and sin $(A - B)$, equation V, and reducing, there results

$$\sin(A + B) \cdot \sin(A - B) = \sin^2 A - \sin^2 B.$$

And, in like manner, multiplying together the values of cos $(A + B)$ and cos $(A - B)$, there is produced

$$\cos(A + B) \cdot \cos(A - B) = \cos^2 A - \cos^2 B.$$

Here, since $\sin^2 A - \sin^2 B$, is equal to $(\sin A + \sin B) \times$

$(\sin A - \sin B)$, that is, to the rectangle of the sum and difference of the sines; it follows, that the first of these equations converted into an analogy, becomes

$\sin(A-B) : \sin A - \sin B :: \sin A + \sin B : \sin(A+B)$ (XX.)

That is to say, *the sine of the difference of any two arcs or angles, is to the difference of their sines, as the sum of those sines is to the sine of their sum.*

If A and B be to each other as $n+1$ to n, then the preceding proportion will be converted into $\sin A : \sin(n+1)A - \sin nA :: \sin(n+1)A + \sin nA : \sin(2n+1)A$. . . . (XXI.)

These two proportions are highly useful in computing a table of sines; as will be shown in the practical examples at the end of this chapter.

25. Let us suppose $A + B = A'$, and $A - B = B'$; then the half sum and the half difference of these equations will give respectively $A = \frac{1}{2}(A'+B')$, and $B = \frac{1}{2}(A'-B')$. Putting these values of A and B, in the expressions of $\sin A \cdot \cos B$, $\sin B \cdot \cos A$, $\cos A \cdot \cos B$, $\sin A \cdot \sin B$, obtained in arts. 21, 22, 23, there would arise the following formulæ:

$$\sin\tfrac{1}{2}(A'+B') \cdot \cos\tfrac{1}{2}(A'-B') = \tfrac{1}{2}R(\sin A' + \sin B'),$$
$$\sin\tfrac{1}{2}(A'-B') \cdot \cos\tfrac{1}{2}(A'+B') = \tfrac{1}{2}R(\sin A' - \sin B'),$$
$$\cos\tfrac{1}{2}(A'+B') \cdot \cos\tfrac{1}{2}(A'-B') = \tfrac{1}{2}R(\cos A' + \cos B'),$$
$$\sin\tfrac{1}{2}(A'+B') \cdot \sin\tfrac{1}{2}(A'-B') = \tfrac{1}{2}R(\cos B' - \cos A').$$

Dividing the second of these formulæ by the first, there will be had

$$\frac{\sin\frac{1}{2}(A'-B') \cdot \cos\frac{1}{2}(A'+B')}{\sin\frac{1}{2}(A'+B') \cdot \cos\frac{1}{2}(A'-B')} = \frac{\sin\frac{1}{2}(A'-B')}{\cos\frac{1}{2}(A'-B')} \cdot \frac{\cos\frac{1}{2}(A'+B')}{\sin\frac{1}{2}(A'+B')} = \frac{\sin A' - \sin B'}{\sin A' + \sin B'}.$$

But since $\frac{\sin}{\cos} = \frac{\tan}{R}$, and $\frac{\cos}{\sin} = \frac{R}{\tan}$, it follows that the two factors of the first member of this equation, are

$\frac{\tan\frac{1}{2}(A'-B')}{R}$, and $\frac{R}{\tan\frac{1}{2}(A'+B')}$, respectively; so that the equation manifestly becomes $\frac{\tan\frac{1}{2}(A'-B')}{\tan\frac{1}{2}(A'+B')} = \frac{\sin A' - \sin B'}{\sin A' + \sin B'}$. . . . . (XXII.)

This equation is readily converted into a very useful proportion, viz. *The sum of the sines of two arcs or angles, is to their difference, as the tangent of half the sum of those arcs or angles, is to the tangent of half their difference.*

26. Operating with the third and fourth formulæ of the preceding article, as we have already done with the first and second, we shall obtain

$$\frac{\tan\frac{1}{2}(A'+B') \cdot \tan\frac{1}{2}(A-B')}{R^2} = \frac{\cos B' - \cos A'}{\cos A' + \cos B'}.$$

In like manner, we have by division,

$$\frac{\sin A' + \sin B'}{\cos A' + \cos B'} = \frac{\sin\frac{1}{2}(A'+B')}{\cos\frac{1}{2}(A'+B')} = \tan\tfrac{1}{2}(A'+B'); \frac{\sin A' + \sin B'}{\cos B' - \cos A'} = \cot\tfrac{1}{2}(A'-B');$$

$$\frac{\sin A' - \sin B'}{\cos A' + \cos B'} = \tan\tfrac{1}{2}(A' - B') \ldots \frac{\sin A' - \sin B'}{\cos B' - \cos A'} = \cot\tfrac{1}{2}(A' + B'),$$
$$\frac{\cos A' + \cos B'}{\cos B' - \cos A'} = \frac{\cot\frac{1}{2}(A' + B')}{\tan\frac{1}{2}(A' - B')}.$$

Making $B = 0$, in one or other of these expressions, there results,

$$\left.\begin{aligned} \frac{\sin A'}{1 + \cos A'} &= \tan\tfrac{1}{2}A' = \frac{1}{\cot\frac{1}{2}A'}. \\ \frac{\sin A'}{1 - \cos A'} &= \cot\tfrac{1}{2}A' = \frac{1}{\tan\frac{1}{2}A'}. \\ \frac{1 + \cos A'}{1 - \cos A'} &= \frac{\cot\frac{1}{2}A'}{\tan\frac{1}{2}A'} = \cot^2\tfrac{1}{2}A' = \frac{1}{\tan^2\frac{1}{2}A'}. \end{aligned}\right\} \quad (xxii.)$$

These theorems will find their application in some of the investigations of spherical trigonometry.

27. Once more, dividing the expression for $\sin(A \pm B)$ by that for $\cos(A \pm B)$, there results

$$\frac{\sin(A \pm B)}{\cos(A \pm B)} = \frac{\sin A \cdot \cos B \pm \sin B \cdot \cos A}{\cos A \cdot \cos B \mp \sin A \cdot \sin B}:$$

then dividing both numerator and denominator of the second fraction, by $\cos A \cdot \cos B$, and recollecting that $\frac{\sin}{\cos} = \frac{\tan}{R}$, we shall thus obtain

$$\frac{\tan(A \pm B)}{R} = \frac{R(\tan A \pm \tan B)}{R^2 \mp \tan A \cdot \tan B};$$

or, lastly, $\tan(A \pm B) = \dfrac{R^2(\tan A \pm \tan B)}{R^2 \mp \tan A \cdot \tan B}$. . . . (XXIII.)

Also, since $\cot = \frac{R^2}{\tan}$, we shall have

$$\cot(A \pm B) = \frac{R^2}{\tan(A \pm B)} = \frac{R^2 \mp \tan A \cdot \tan B}{\tan A \pm \tan B};$$

which, after a little reduction, becomes

$$\cot(A \pm B) = \frac{\cot A \cdot \cot B \mp R^2}{\cot B \pm \cot A}. \quad \ldots \quad (XXIV.)$$

28. We might now, by making $A = B$, $A = 2B$, &c. proceed to deduce expressions for the tangents, cotangents, secants, &c. of multiple arcs; but we shall merely present a few for the tangents, as

$$\left.\begin{aligned} \tan 2A &= \frac{2\tan A}{1 - \tan^2 A} \; \ldots \; \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \\ \tan 4A &= \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A} \\ \tan 5A &= \frac{5\tan A - 10\tan^3 A + \tan^5 A}{1 - 10\tan^2 A + 5\tan^4 A} \end{aligned}\right\} \quad (xxiii.)$$

We might again from the obvious equation

$$\sec^2 A - \tan^2 A = \sec^2 B - \tan^2 B,$$

deduce the expression $\dfrac{\sec A + \tan A}{\sec B + \tan B} = \dfrac{\sec B - \tan B}{\sec A - \tan A}$;

and so, for many other analogies.

We might investigate also some of the usual formulæ of verification in the construction of tables, such as

$$\sin(54^\circ+A)+\sin(54^\circ-A)-\sin(18^\circ+A)-\sin(18^\circ-A)=\sin(90^\circ-A)$$
$$\sin A+\sin(36^\circ-A)+\sin(72^\circ+A)=\sin(36^\circ+A)+\sin(72^\circ-A).$$
&c. &c.

But, as these inquiries would extend this chapter to too great a length, we shall pass them by; and merely investigate a few properties where *more* than two arcs or angles are concerned, and which may be of use in some subsequent parts of this volume.

29. Let A, B, C, be any three arcs or angles, and suppose radius to be unity; then

$$\sin(B+C)=\frac{\sin A \,.\, \sin C+\sin B \,.\, \sin(A+B+C)}{\sin(A+B)}.$$

For, by equa. v, $\sin(A+B+C)=\sin A \,.\, \cos(B+C)+\cos A \,.\, \sin(B+C)$, which, (putting $\cos B \,.\, \cos C-\sin B \,.\, \sin C$ for $\cos(B+C)$), *is*$=\sin A \,.\, \cos B \,.\, \cos C-\sin A \,.\, \sin B \,.\, \sin C+\cos A \,.\, \sin(B+C)$; and, multiplying by $\sin B$, and adding $\sin A \,.\, \sin C$, there results $\sin A \,.\, \sin C+\sin B \,.\, \sin(A+B+C)$ $=\sin A \,.\, \cos B \,.\, \cos C \,.\, \sin B+\sin A \,.\, \sin C \,.\, \cos^2 B+\cos A \,.\, \sin B \,.\, \sin(B+C)=\sin A \,.\, \cos B \,.\, (\sin B \,.\, \cos C+\cos B \sin C)$ $+\cos A \,.\, \sin B \,.\, \sin(B+C)=\sin A \,.\, \cos B \,.\, \sin(B+C)+\cos A \,.\, \sin A \,.\, \sin(B+C)=(\sin A \,.\, \cos B+\cos A \,.\, \sin B)\times$ $\sin(B+C)=\sin(A+B) \,.\, \sin(B+C)$. Consequently, by dividing by $\sin(A+B)$, we obtain the expression above given.

In a similar manner it may be shown, that

$$\sin(B-C)=\frac{\sin A \,.\, \sin C-\sin B \,.\, \sin(A-B+C)}{\sin(A-B)}.$$

30. If A, B, C, D, represent four arcs or angles, then writing $C+D$ for C in the preceding investigation, there will result,

$$\sin(B+C+D)=\frac{\sin A \,.\, \sin(C+D)+\sin B \,.\, \sin(A+B+C+D)}{\sin(A+B)}.$$

A like process for five arcs or angles will give

$$\sin(B+C+D+E)=\frac{\sin A \,.\, \sin(C+D+E)+\sin B \,.\, \sin(A+B+C+D+E)}{\sin(A+B)}.$$

And for any number, A, B, C, &c. to L,

$$\sin(B+C+\ldots.L)=\frac{\sin A \,.\, \sin(C+D+\ldots L)+\sin B \,.\, \sin(A+B+C+\ldots L)}{\sin(A+B)}.$$

31. Taking again the three A, B, C, we have

$$\sin(B-C)=\sin B \,.\, \cos C-\sin C \,.\, \cos B,$$
$$\sin(C-A)=\sin C \,.\, \cos A-\sin A \,.\, \cos C,$$
$$\sin(A-B)=\sin A \,.\, \cos B-\sin B \,.\, \cos A.$$

Multiplying the first of these equations by $\sin A$, the second by $\sin B$, the third by $\sin C$; then adding together the equations thus transformed, and reducing; there will result,

$\sin A \cdot \sin(B-C)+\sin B \cdot \sin(C-A)+\sin C \cdot \sin(A-B)=0,$
$\cos A \cdot \sin(B-C)+\cos B \cdot \sin(C-A)+\cos C \cdot \sin(A-B)=0.$

These two equations obtaining for any three angles whatever, apply evidently to the three angles of any triangle.

32. Let the series of arcs or angles A, B, C, D . . . . L, be contemplated, then we have (art. 24),

$$\begin{aligned}
\sin(A+B) \cdot \sin(A-B) &= \sin^2 A-\sin^2 B,\\
\sin(B+C) \cdot \sin(B-C) &= \sin^2 B-\sin^2 C,\\
\sin(C+D) \cdot \sin(C-D) &= \sin^2 C-\sin^2 D,\\
&\text{\&c. \&c. \&c.}\\
\sin(L+A) \cdot \sin(L-A) &= \sin^2 L-\sin^2 A.
\end{aligned}$$

If all these equations be added together, the second member of the equation will vanish, and of consequence we shall have

$$\sin(A+B) \cdot \sin(A-B)+\sin(B+C) \cdot \sin(B-C)+\&c \ldots \ldots +\sin(L+A)+\sin(L-A)=0.$$

Proceeding in a similar manner with $\sin(A-B)$, $\cos(A+B)$, $\sin(B-C)$, $\cos(B+C)$, &c. there will at length be obtained

$$\cos(A+B) \cdot \sin(A-B)+\cos(B+C) \cdot \sin(B-C)+\&c. \ldots \ldots +\cos(L+A) \cdot \sin(L-A)=0.$$

33. If the arcs A, B, C, &c. . . . . L form an arithmetical progression, of which the first term is 0, the common difference D', and the last term L any number n of circumferences; then will $B-A=D'$, $C-B=D'$, &c. $A+B=2D'$, $B+C=3D'$, &c.: and dividing the whole by $\sin D'$, the preceding equations will become

$$\left.\begin{aligned}
\sin D' + \sin 3D' + \sin 5D' + \&c. = 0,\\
\cos D' + \cos 3D' + \cos 5D' + \&c. = 0.
\end{aligned}\right\} \quad \text{(XXV.)}$$

If E' were equal $2D'$, these equations would become

$\sin D'+\sin(D'+E')+\sin(D'+2E')+\sin(D'+3E')+\&c. = 0,$
$\cos D'+\cos(D'+E')+\cos(D'+2E')+\cos.(D'+3E')+\&c. = 0.$

34. The last equation, however, only shows the sums of sines and cosines of arcs or angles in arithmetical progression, when the common difference is to the first term in the ratio of 2 to 1. To investigate a *general* expression for an infinite series of this kind, let

$$s=\sin A+\sin(A+B)+\sin(A+2B)+\sin(A+3B)+\&c.$$

Then, since this series is a recurring series, whose scale of relation is $2\cos B-1$, it will arise from the developement of a fraction whose denominator is $1-2z \cdot \cos B+z^2$, making $z=1$.

Now this fraction will be $= \dfrac{\sin A+z\,[\sin(A+B)-2\sin A \cdot \cos B]}{1-2z \cdot \cos B+z^2}$.

Therefore, when $z=1$, we have

$s=\frac{\sin A+\sin(A+B)-2\sin A.\cos B}{2-2\cos B}$; and this, because $2\sin A.\cos B=\sin(A+B)+\sin(A-B)$ (art. 21), is equal to $\frac{\sin A-\sin(A-B)}{2(1-\cos B)}$. But, since $\sin A'-\sin B'=2\cos\frac{1}{2}(A'+B').\sin\frac{1}{2}(A'-B')$, by art. 25, it follows, that $\sin A-\sin(A-B)=2\cos(A-\frac{1}{2}B)\sin\frac{1}{2}B$; besides which, we have $1-\cos B=2\sin^2\frac{1}{2}B$. Consequently the preceding expression becomes $s=\sin A+\sin(A+B)+\sin(A+2B)+\sin(A+3B)+$&c. *ad infinitum* $=\frac{\cos(A-\frac{1}{2}B)}{2\sin\frac{1}{2}B}$ . . . . (XXVI.)

35. To find the sum of $n+1$ terms of this series we have simply to consider that the sum of the terms past the $(n+1)$th, that is, the sum of $\sin[A+(n+1)B]+\sin[A+(n+2)B]+\sin[A+(n+3)B]+$ &c. *ad infinitum*, is, by the preceding theorem, $=\frac{\cos[A+(n+\frac{1}{2})B]}{2\sin\frac{1}{2}B}$. Deducting this, therefore, from the former expression, there will remain, $\sin A+\sin(A+B)+\sin(A+2B)+\sin(A+3B)+\ldots\ldots\sin(A+nB)=\frac{\cos(A-\frac{1}{2}B)-\cos[A+(n+\frac{1}{2})B]}{2\sin\frac{1}{2}B}=\frac{\sin(A+\frac{1}{2}nB).\sin\frac{1}{2}(n+1)B}{\sin\frac{1}{2}B}$. (XXVII.)

By like means it will be found, that the sums of the cosines of arcs or angles in arithmetical progression, will be $\cos A+\cos(A+B)+\cos(A+2B)+\cos(A+3B)+$ &c. *ad infinitum* $=-\frac{\sin(A-\frac{1}{2}B)}{2\sin\frac{1}{2}B}$ . . . . (XXVIII.)

Also,

$\cos A+\cos(A+B)+\cos(A+2B)+\cos(A+3B+)\ldots\ldots\ldots(\cos A+nB)=-\frac{(A+\frac{1}{2}nB).\sin\frac{1}{2}(n+1)B}{\sin\frac{1}{2}B.}$ . . . (XXIX.)

36. With regard to the tangents of more than two arcs, the following property (the only one we shall here deduce) is a very curious one, which has not yet been inserted in works of Trigonometry, though it has been long known to mathematicians. Let the three arcs A, B, C, together make up the whole circumference, ○: then, since $\tan(A+B)=\frac{R^2(\tan A+\tan B)}{R^2-\tan A.\tan B}$ (by equa. XXIII), we have $R^2\times(\tan A+\tan B+\tan C)=R^2\times[\tan A+\tan B-\tan(A+B)]=R^2\times(\tan A+\tan B-\frac{R^2(\tan A+\tan B)}{R^2-\tan A.\tan B}=$, by actual multiplication and reduction, to $\tan A.\tan B.\tan C$, since $\tan C=\tan[○-(A+B)]=-\tan(A+B)=-\frac{R^2(\tan A+\tan B)}{R^2-\tan A.\tan B}$, by what has preceded in this article. The result therefore is, that *the*

sum of the tangents of any three arcs which together constitute a circle, multiplied by the square of the radius, is equal to the product of those tangents. . . . (XXX.)

Since both arcs in the second and fourth quadrants have their tangents considered negative, the above property will apply to arcs any way trisecting a semicircle ; and it will therefore apply to the angles of a plane triangle, which are, together, measured by arcs constituting a semicircle. So that, if radius be considered as unity, we shall find, that *the sum of the tangents of the three angles of any plane triangle, is equal to the continued product of those tangents.* (XXXI.)

37. Having thus given the chief properties of the sines, tangents, &c. of arcs, their sines, products, and powers, we shall merely subjoin investigations of theorems for the 2d and 3d cases in the solutions of plane triangles. Thus, with respect to the second case, where two sides and their included angles are given :

By equ. IV, $a : b :: \sin A : \sin B$.

By compos. and division. $\Big\}\ a + b : a - b :: \sin A + \sin B : \sin A - \sin B$;

but, eq. XXII, $\tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A - B) :: \sin A + \sin B : \sin A - \sin B$; whence, ex equal. $a+b : a-b :: \tan \frac{1}{2}(A+B) : \tan \frac{1}{2}(A-B)$. . . . (XXXII.)

Agreeing with the result of the geometrical investigation at pa. 387, vol. i.

38. If, instead of having the two sides a, b, given, we know their *logarithms*, as frequently happens in geodesic operations, $\tan \frac{1}{2}(A - B)$ may be readily determined without first finding the number corresponding to the logs. of a and b. For if a and b were considered as the sides of a right-angled triangle, in which φ denotes the angle opposite the side a, then would $\tan \varphi = \frac{a}{b}$. Now, since a is supposed greater than b, this angle will be greater than half a right angle, or it will be measured by an arc greater than $\frac{1}{8}$ of the circumference, or than $\frac{1}{8}\bigcirc$. Then, because $\tan (\varphi - \frac{1}{8}\bigcirc) = \frac{\tan \varphi - \tan \frac{1}{8}\bigcirc}{1 + \tan \varphi \tan \frac{1}{8}\bigcirc}$ and because $\tan \frac{1}{8}\bigcirc = R = 1$, we have

$$\tan (\varphi - \tfrac{1}{8}\bigcirc) = (\tfrac{a}{b} - 1) \div (1 + \tfrac{a}{b}) = \frac{a-b}{a+b}.$$

And, from the preceding article,

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(a+b)} = \frac{\tan \frac{1}{2}(A-B)}{\cot \frac{1}{2}C} : \text{ consequently,}$$

$\tan \frac{1}{2}(A-B) = \cot \frac{1}{2}C \,.\, \tan (\varphi - \frac{1}{8}\bigcirc)$. . . . (XXXIII.)

From this equation we have the following practical rule : Subtract the less from the greater of the given logs, the remainder will be the log tan of an angle : from this angle

take 45 degrees, and to the log tan of the remainder add the log cotan of half the given angle; the sum will be the log tan of half the *difference* of the other two angles of the plane triangle.

39. The remaining case is that in which the three sides of the triangle are known, and for which indeed we have already obtained expressions for the angle in arts. 6 and 8. But, as neither of these is best suited for logarithmic computation, (however well fitted they are for instruments of investigation), another may be deduced thus: In the equation for cos. A, (given equation II), viz. $\cos A = \frac{b^2+c^2-a^2}{2bc}$, if we substitute, instead of cos A, its value $1 - 2\sin^2 \frac{1}{2}A$, change the signs of all the terms, transpose the 1, and divide by 2, we shall have $\sin^2 \frac{1}{2}A = \frac{a^2-b^2-c^2+2bc}{4bc} = \frac{a^2-(b-c)^2}{4bc}$.

Here, the numerator of the second member being the product of the two factors $(a+b-c)$ and $(a-b+c)$, the equation will become $\sin^2 \frac{1}{2}A = \frac{\frac{1}{2}(a+b-c)\cdot\frac{1}{2}(a-b+c)}{bc}$. But, since $\frac{1}{2}(a+b-c) = \frac{1}{2}(a+b+c)-c$, and $\frac{1}{2}(a-b+c) = \frac{1}{2}(a+b+c)-b$; if we put $s = a+b+c$, and extract the square root, there will result,

$$\left.\begin{array}{l} \sin \frac{1}{2}A = \sqrt{\frac{(\frac{1}{2}s-b)\cdot(\frac{1}{2}s-c)}{bc}}. \\ \text{In like manner} \left\{ \begin{array}{l} \sin \frac{1}{2}B = \sqrt{\frac{(\frac{1}{2}s-a)\cdot(\frac{1}{2}s-c)}{ac}}. \\ \sin \frac{1}{2}C = \sqrt{\frac{(\frac{1}{2}s-a)\cdot(\frac{1}{2}s-b)}{ab}}. \end{array}\right. \end{array}\right\} \text{(XXXIV.)}$$

These expressions, besides their convenience for logarithmic computation, have the further advantage of being perfectly free from ambiguity, because the half of any angle of a plane triangle will always be *less* than a right angle.

40. The student will find it advantageous to collect into one place all those formulæ which relate to the computation of sines, tangents, &c.*; and, in another place, those which are of use in the solutions of plane triangles: the former of these are equations V, VIII, IX, X, XI, *x*, *xi*, XII, XIII, XIV, XV, XVI, XVII, XVIII, XIX, XX, XXII, *xxii*, XXIII, XXIV, XXVII; the latter are equa. II, III, IV, VII, XXXII, XXXIII, XXXIV.

* What is here given being only a brief sketch of an inexhaustible subject; the reader who wishes to pursue it further is referred to the copious Introduction to our Mathematical Tables, and the treatises on Trigonometry, by Emerson, Gregory, Bonnycastle, Woodhouse, Lardner, and many other modern writers on the same subject, where he will find his curiosity richly gratified.

To exemplify the use of some of these formulæ, the following exercises are subjoined.

EXERCISES.

Ex. 1. Find the sines and tangents of 15°, 30°, 45°, 60°; and 75°: and show how from thence to find the sines and tangents of several of their submultiples.

First, with regard to the arc of 45°, the sine and cosine are manifestly equal; or they form the perpendicular and base of a right-angled triangle whose hypothenuse is equal to the assumed radius. Thus, if radius be R, the sine and cosine of 45° will each be $= \sqrt{\frac{1}{2}R^2} = R\sqrt{\frac{1}{2}} = R\sqrt{2}$. If R be equal to 1, as is the case with the tables in use, then

$\sin 45° = \cos 45° = \frac{1}{2}\sqrt{2} = \cdot7071068$

$\tan 45° = \frac{\sin}{\cos} = 1\cdot = \frac{\cos}{\sin} =$ cotangent 45°.

Secondly, for the sines of 60° and of 30°: since each angle in an equilateral triangle contains 60°, if a perpendicular be demitted from any one angle of such a triangle on the opposite side, considered as a base, that perpendicular will be the sine of 60°, and the half base the sine of 30°, the side of the triangle being the assumed radius. Thus, if it be R, we shall have $\frac{1}{2}R$ for the sine of 30°, and $\sqrt{(R^2 - \frac{1}{4}R^2)} = \frac{1}{2}R\sqrt{3}$, for the sine of 60°. When $R = 1$, these become

$\sin 30° = \cdot5 \ldots\ldots \sin 60° = \cos 30 = \cdot8660254.$

Hence, $\tan 30° = \frac{\cdot5}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3} = \cdot5773503.$

$\tan 60° = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3} = \ldots\ldots 1\cdot7320508.$

Consequently, $\tan 60° = 3 \tan 30°$.

Thirdly, for the sines of 15° and 75°, the former arc is the half of 30°, and the latter is the complement of that half arc. Hence, substituting 1 for R and $\frac{1}{2}\sqrt{3}$ for cos A, in the expression $\sin \frac{1}{2}A = \pm \frac{1}{2}\sqrt{(2R^2 + 2R\cos A)} \ldots$ (equa. XII), it becomes $\sin 15° = \frac{1}{2}\sqrt{(2 - \sqrt{3})} = \cdot2588190$.

Hence, $\sin 75° = \cos 15° = \sqrt{[1 - \frac{1}{4}(2 - \sqrt{3})]} = \frac{1}{2}\sqrt{(2 + \sqrt{3})} = \frac{\sqrt{6} + \sqrt{2}}{4} = \cdot9659258.$

Consequently, $\tan 15° = \frac{\sin}{\cos} = \frac{\cdot2588190}{\cdot9659258} = \cdot2679492.$

And, $\tan 75° = \frac{\cdot9659258}{\cdot2588190} = 3\cdot7320508.$

Now, from the sine of 30°, those of 6°, 2°, and 1°, may easily be found. For, if $5A = 30°$, we shall have, from equation X, $\sin 5A = 5 \sin A - 20 \sin^3 A + 16 \sin^5 A$: or, if $\sin A = x$, this will become $16x^5 - 20x^3 + 5x = \cdot5$. This

equation solved by any of the approximating rules for such equations, will give $x = \cdot 1045285$, which is the sine of 6°.

Next, to find the sine of 2°, we have again, from equation x, $\sin 3A = 3 \sin A - 4 \sin^3 A$: that is, if x be put for $\sin 2°$, $3x - 4x^3 = \cdot 1045285$. This cubic solved, gives $x = \cdot 0348995 = \sin 2°$.

Then, if $s = \sin 1°$, we shall, from the second of the equations marked x, have $2s\sqrt{(1 - s^2)} = \cdot 0348995$; whence s is found $= \cdot 0174524 = \sin 1°$.

Had the expression for the sines of bisected arcs been applied successively from sin 15°, to sin 7°30′, sin 3°45′, sin 1°52½′, sin 56¼′, &c. a different series of values might have been obtained: or, if we had proceeded from the quinquisection of 45°, to the trisection of 9°, the bisection of 3°, and so on, a different series still would have been found. But what has been done above is sufficient to illustrate *this* method. The next example will exhibit a very simple and compendious way of ascending from the sines of smaller to those of larger arcs.

Ex. Given the sine of 1°, to find the sine of 2°, and then the sines of 3°, 4°, 5°, 6°, 7°, 8°, 9°, and 10°, each by a single proportion.

Here, taking first the expression for the sine of a double arc, equa. x, we have $\sin 2° = 2\sin 1° \sqrt{(1 - \sin^2 1°)} = \cdot 0348995$.

Then it follows from the rule in equa. xx, that

$$\begin{aligned}
\sin 1° &: \sin 2° - \sin 1° :: \sin 2° + \sin 1° : \sin 3° = \cdot 0523360 \\
\sin 2° &: \sin 3° - \sin 1° :: \sin 3° + \sin 1° : \sin 4° = \cdot 0697565 \\
\sin 3° &: \sin 4° - \sin 1° :: \sin 4° + \sin 1° : \sin 5° = \cdot 0871557 \\
\sin 4° &: \sin 5° - \sin 1° :: \sin 5° + \sin 1° : \sin 6° = \cdot 1045285 \\
\sin 5° &: \sin 6° - \sin 1° :: \sin 6° + \sin 1° : \sin 7° = \cdot 1218693 \\
\sin 6° &: \sin 7° - \sin 1° :: \sin 7° + \sin 1° : \sin 8° = \cdot 1391731 \\
\sin 7° &: \sin 8° - \sin 1° :: \sin 8° + \sin 1° : \sin 9° = \cdot 1564375 \\
\sin 8° &: \sin 9° - \sin 1° :: \sin 9° + \sin 1° : \sin 10° = \cdot 1736482
\end{aligned}$$

To check and verify operations like these, the proportions should be changed at certain stages. Thus,

$$\begin{aligned}
\sin 1° &: \sin 3° - \sin 2° :: \sin 3° + \sin 2° : \sin 5°, \\
\sin 1° &: \sin 4° - \sin 3° :: \sin 4° + \sin 3° : \sin 7°, \\
\sin 4° &: \sin 7° - \sin 3° :: \sin 7° + \sin 3° : \sin 10°.
\end{aligned}$$

The coincidence of the results of these operations with the analogous results in the preceding, will manifestly establish the correctness of both.

Cor. By the same method, knowing the sines of 5°, 10°, and 15°, the sines of 20°, 25°, 35°, 55°, 65°, &c. may be found, each by a single proportion. And the sines of 1°, 9°, and 10°, will lead to those of 19°, 29°, 39°, &c. So that the

sines may be computed to any arc; and the tangents and other trigonometrical lines, by means of the expressions in art. 4, &c.

Ex. 3. Find the sum of all the natural sines to every minute in the quadrant, radius = 1.

In this problem the actual addition of all the terms would be a most tiresome labour: but the solution by means of equation XXVII, is rendered very easy. Applying that theorem to the present case, we have $\sin(\text{A} + \frac{1}{2}n\text{B}) = \sin 45^\circ$, $\sin\frac{1}{2}(n+1)\text{B} = \sin 45^\circ 0' 30''$, and $\sin\frac{1}{2}\text{B} = \sin 30''$. Therefore $\frac{\sin 45^\circ \times \sin 45^\circ 0' 30''}{\sin 30''} = 3438 \cdot 2467465$ the sum required.

From another method, the investigation of which is omitted here, it appears that the same sum is equal to $\frac{1}{2}(\cot 30'' + 1)$.

Ex. 4. Explain the method of finding the *logarithmic* sines, cosines, tangents, secants, &c. the natural sines, cosines, &c. being known.

The *natural* sines and cosines being computed to the radius unity, are all proper fractions, or quantities less than unity, so that their logarithms would be negative. To avoid this, the tables of logarithmic sines, cosines, &c. are computed to a radius of 10000000000, or 10^{10}; in which case the logarithm of the radius is 10 times the log of 10, that is, it is 10.

Hence, if s represent any sine to radius 1, then $10^{10} \times s$ = sine of the same arc or angle to rad 10^{10}. And this, in logs is, $\log 10^{10} s = 10 \log 10 + \log s = 10 + \log s$.

The log cosines are found by the same process, since the cosines are the sines of the complements.

The logarithmic expressions for the tangents, &c. are deduced thus:

$$\text{Tan} = \text{rad}\,\frac{\sin}{\cos}. \quad \text{Theref. } \log\tan = \log\text{rad} + \log\sin - \log\cos = 10 + \log\sin - \log\cos.$$

$$\text{Cot} = \frac{\text{rad}^2}{\tan}. \quad \text{Theref. } \log\cot = 2\log\text{rad} - \log\tan = 20 - \log\tan.$$

$$\text{Sec} = \frac{\text{rad}^2}{\cos}. \quad \text{Theref. } \log\sec = 2\log\text{rad} - \log\cos = 20 - \log\cos.$$

$$\text{Cosec} = \frac{\text{rad}^2}{\sin}. \quad \text{Theref. l. } \text{cosec} = 2\log\text{rad} - \log\sin = 20 - \log\sin.$$

$$\text{Versed sine} = \frac{\text{chord}^2}{\text{diam}} = \frac{(2\sin\frac{1}{2}\text{arc})^2}{2\,\text{rad}} = \frac{2 \times \sin^2\frac{1}{2}\text{arc}}{\text{rad}}.$$

Therefore, $\log\text{vers}\sin = \log 2 + 2\log\sin\frac{1}{2}\text{arc} - 10$.

Ex. 5. Given the sum of the natural tangents of the angles A and B of a plane triangle = 3·1601988, the sum of the

tangents of the angles B and C = 3·8765577, and the continued product, tan A . tan B . tan C = 5·3047057 : to find the angles A, B, and C.

It has been demonstrated in art. 36, that when radius is unity, the sum of the natural tangents of the three angles of a plane triangle is equal to their continued product. Hence the process is this :

$$\begin{array}{ll}
\text{From } \tan A+\tan B+\tan C & = 5\cdot3047057 \\
\text{Take } \tan A+\tan B \;.\;.\;.\;. & = 3\cdot1601988 \\
\text{Remains } \tan C \;.\;.\;.\;.\;.\;. & = \overline{2\cdot1445069} = \tan 65^\circ. \\
\text{From } \tan A+\tan B+\tan C & = \overline{5\cdot3047057} \\
\text{Take } \tan B+\tan C \;.\;.\;.\;. & = 3\cdot8765577 \\
\text{Remains } \tan A \;.\;.\;.\;.\;.\;.\;. & = \overline{1\cdot4281480} = \tan 55^\circ.
\end{array}$$

Consequently, the three angles are 55°, 60°, and 65°.

Ex. 6. There is a plane triangle, whose sides are three consecutive terms in the natural series of integer numbers, and whose largest angle is just double the smallest. Required the sides and angles of that triangle ?

If A, B, C, be three angles of a plane triangle, a, b, c, the sides respectively opposite to A, B, C ; and $s = a + b + c$. Then from equa. III and XXXIV, we have

$$\sin A = \frac{2}{bc}\sqrt{[\tfrac{1}{2}s\,(\tfrac{1}{2}s-a)\,.\,(\tfrac{1}{2}s-b)\,.\,(\tfrac{1}{2}s-c)]},$$

$$\text{and } \sin \tfrac{1}{2}C = \sqrt{\frac{(\tfrac{1}{2}s-a)\;(\tfrac{1}{2}s-b)}{ab}}.$$

Let the three sides of the required triangle be represented by x, $x+1$, and $x+2$; the angle A being supposed opposite to the side x, and C opposite to the side $x+2$: then the preceding expressions will become

$$\sin A = \frac{2}{(x+1)\,.\,(x+2)}\sqrt{\frac{3x+3}{2}\,.\,\frac{x+3}{2}\,.\,\frac{x+1}{2}\,.\,\frac{x-1}{2}}.$$

$$\sin \tfrac{1}{2}C = \sqrt{\frac{(x+1)\,.\,(x+3)}{4x(x+1)}}.$$

Assuming these two expressions equal to each other, as they ought to be, by the question ; there results, after a little reduction, $\frac{1}{\sqrt{x}} = \frac{\sqrt{3(x-1)}}{x+2}$, or $3x\,(x-1) = (x+2)^2$, or $2x^2 - 7x = 4$, an equation whose root is 4 or $-\frac{1}{2}$. Hence 4, 5, and 6, are the sides of the triangle.

$$\sin A = \tfrac{2}{5}\cdot\tfrac{1}{6}\sqrt{(\tfrac{15}{2}\cdot\tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2})} = \tfrac{2}{5\cdot6}\sqrt{(\tfrac{15}{4}\,.\,\tfrac{15}{4}\,.\,7)} = \tfrac{2}{5}\cdot\tfrac{1}{6}\cdot\tfrac{15}{4}\sqrt{7} = \tfrac{1}{4}\sqrt{7}.$$

$$\sin B = \tfrac{5}{16}\sqrt{7}\,;\; \sin C = \tfrac{6}{16}\sqrt{7}\,;\; \sin \tfrac{1}{2}C = \sqrt{\tfrac{5\cdot7}{2\cdot2\cdot4\cdot5}} = \tfrac{1}{4}\sqrt{7}.$$

The angles are, $A = 40^\circ\cdot409603 = 41^\circ\,24'\,34''\,34'''$,

$B = 55^\circ\cdot771191 = 55\;\;46\;\;16\;\;18,$

$C = 82^\circ\cdot819206 = 82\;\;49\;\;\;9\;\;\;8.$

A geometrical construction of this example is given at p. 59, Gregory's Trigonometry.

Ex. 7. Demonstrate that sin $18° = \cos 72°$ is $= \frac{1}{4}$ R $(-1 + \sqrt{5})$, and sin $54° = \cos 36°$ is $= \frac{1}{4}$R $(1 + \sqrt{5})$.

Ex. 8. Demonstrate that the sum of the sines of two arcs which together make 60°, is equal to the sine of an arc which is greater than 60° by either of the two arcs: Ex. gr. $\sin 3' + \sin 59°57' = \sin 60°3'$; and thus that the tables may be continued by addition only.

Ex. 9. Show the truth of the following proportion: As the sine of half the difference of two arcs, which together make 60°, or 90°, respectively, is to the difference of their sines; so is 1 to $\sqrt{3}$, or $\sqrt{2}$, respectively.

Ex. 10. Demonstrate that the sum of the squares of the sine and versed sine of an arc, is equal to the square of double the sine of half the arc.

Ex. 11. Demonstrate that the sine of an arc is a mean proportional between half the radius and the versed sine of double the arc.

Ex. 12. Show that the secant of an arc is equal to the sum of its tangent and the tangent of half its complement.

Ex. 13. Prove that, in any plane triangle, the base is to the difference of the other two sides, as the sine of half the sum of the angles at the base, to the sine of half their difference: also, that the base is to the sum of the other two sides, as the cosine of half the sum of the angles at the base, to the cosine of half their difference.

Ex. 14. How must three trees A, B, C, be planted, so that the angle at A may be double the angle at B, the angle at B double that at C; and so that a line of 400 yards may just go round them?

Ex. 15. In a certain triangle, the sines of the three angles are as the numbers 17, 15, and 8, and the perimeter is 160. What are the sides and angles?

Ex. 16. The logarithms of two sides of a triangle are 2·2407293 and 2·5378191, and the included angle is 37°20′. It is required to determine the other angles, without first finding any of the sides?

Ex. 17. The sides of a triangle are to each other as the fractions, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$: what are the angles?

Ex. 18. Show that the secant of 60°, is double the tangent of 45°, and that the secant of 45° is a mean proportional between the tangent of 45° and the secant of 60°.

Ex. 19. Demonstrate that 4 times the rectangle of the

sines of two arcs, is equal to the difference of the squares of the chords of the sum and difference of those arcs.

Ex. 20. Convert the equations marked XXXIV into their equivalent logarithmic expressions; and by means of them and equa. IV, find the angles of a triangle whose sides are 5, 6, and 7.

Ex. 21. Find the arc whose tangent and cotangent shall together be equal to 4 times the radius.

Ex. 22. Find the arc whose sine added to its cosine shall be equal to a; and show the limits of possibility.

Ex. 23. Find the arc whose secant and cotangent shall be equal.

Ex. 24. If one angle A of a right-angled plane triangle A, B, C, be given, (B being be the right angle), and the area or surface be given $= s$. Demonstrate that

$$AB = \sqrt{(2s \cot A)} \ldots BC = \sqrt{(2s \tan A)},$$
$$\text{and } AC = 2\sqrt{(s \sec 2A)}$$

Ex. 25. Demonstrate,

1. That $\sin A = \dfrac{1}{\sqrt{(1+\cot^2 A)}} = \sqrt{(\frac{1}{2} - \frac{1}{2}\cos 2A)}$

$$= \frac{2}{\cot \frac{1}{2}A + \tan \frac{1}{2}A} = \frac{1}{\cot A + \tan \frac{1}{2}A}$$
$$= 2\sin^2(45^\circ + \tfrac{1}{2}A) - 1 = 1 - 2\sin^2(45^\circ - \tfrac{1}{2}A).$$

2. That $\tan A = \sqrt{(\dfrac{1}{\cos^2 A} - 1)} = \sqrt{\dfrac{1-\cos 2A}{1+\cos 2A}}$

$$= \frac{\sqrt{(1-\cos^2 A)}}{\cos A} = \frac{\sin A}{\sqrt{(1-\sin 2A)}}$$
$$= \frac{1-\cos 2A}{\sin 2A} = \frac{\sin 2A}{1+\cos 2A}$$
$$= \frac{2\cot \frac{1}{2}A}{\cot^2 \frac{1}{2}A - 1} = \frac{2\tan \frac{1}{2}A}{1-\tan^2 \frac{1}{2}A}$$
$$= \frac{2}{\cot \frac{1}{2}A - \tan \frac{1}{2}A} = \cot A - 2\cot 2A$$
$$= \tfrac{1}{2}[\tan(45^\circ + \tfrac{1}{2}A) - \tan(45^\circ - \tfrac{1}{2}A)].$$

SPHERICAL TRIGONOMETRY.

SECTION I.

General Properties of Spherical Triangles.

Art. 1. *Def.* 1. Any portion of a spherical surface bounded by three arcs of great circles, is called a *Spherical Triangle.*

Def. 2. Spherical Trigonometry is the art of computing the measures of the sides and angles of spherical triangles.

Def. 3. A *right-angled* spherical triangle has one right angle: the sides about the right angle are called *legs;* the side opposite to the right angle is called the *hypothenuse.*

Def. 4. A *quadrantal* spherical triangle has one side equal to 90° or a quarter of a great circle.

Def. 5. Two arcs or angles, when compared together, are said to be *alike*, or of the *same affection*, when both are less than 90°, or both are greater than 90°. But when one is greater and the other less than 90°, they are said to be *unlike*, or of *different affections.*

Art. 2. The small circles of the sphere do not fall under consideration in Spherical Trigonometry; but such only as have the same centre with the sphere itself. And hence it is that spherical trigonometry is of so much use in Practical Astronomy, the apparent heavens assuming the shape of a concave sphere, whose centre is the same as the centre of the earth.

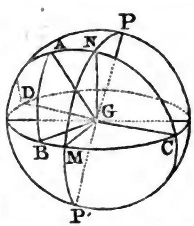
3. Every spherical triangle has three sides and three angles: and if any three of these six parts be given, the remaining three may be found, by some of the rules which will be investigated in this chapter.

4. In *plane* trigonometry, the knowledge of the three angles is not sufficient for ascertaining the sides: for in that case the *relations* only of the three sides can be obtained, and not their absolute values: whereas, in *spherical* trigonometry, where the sides are circular arcs, whose values depend on their proportion to the whole circle, that is, on the number of degrees they contain, the sides may always be determined when the three angles are known. Other remarkable differences between plane and spherical triangles are, 1st. That in the former, two angles always determine the third; while in the latter they never do. 2dly. The surface of a plane

riangle cannot be determined from a knowledge of the angles alone; while that of a spherical triangle always can.

5. The *sides* of a spherical triangle are all arcs of great circles, which, by their intersection on the surface of the sphere, constitute that triangle.

6. The *angle* which is contained between the arcs of two great circles, intersecting each other on the surface of the sphere, is called a spherical angle; and its measure is the same as the measure of the plane angle which is formed by two lines issuing from the same point of, and perpendicular to, the common section of the planes which determine the containing sides; that is to say, it is the same as the angle made by those planes. Or, it is equal to the plane angle formed by the tangents to those arcs at their point of intersection.



7. Hence it follows, that the surface of a spherical triangle BAC, and the three planes which determine it, form a kind of triangular pyramid, BCGA, of which the vertex G is at the centre of the sphere, the base ABC a portion of the spherical surface, and the faces AGC, AGB, BGC, sectors of the great circles whose intersections determine the side of the triangle.

Def. 6. *A line perpendicular to the plane of a great circle,* passing through the centre of the sphere, and terminated by two points, diametrically opposite, at its surface, is called the *axis* of such circle; and the extremities of the axis, or the points where it meets the surface, are called the *poles*, of that circle. Thus, PGP' is the axis, and P, P', are the poles, of the great circle CND.

If we conceive any number of less circles, each parallel to the said great circle, this axis will be perpendicular to them likewise; and the points P, P', will be their poles also.

8. Hence, each pole of a great circle is 90° distant from every point in its circumference; and all the arcs drawn from either pole of a little circle to its circumference, are equal to each other.

9. It likewise follows, that all the arcs of great circles drawn through the poles of another great circle, are perpendicular to it: for, since they are great circles by the supposition, they all pass through the centre of the sphere, and consequently through the axis of the said circle. The same thing may be affirmed with regard to small circles.

10. Hence, in order to find the *poles* of any circle, it is merely necessary to describe, upon the surface of the sphere,

two great circles perpendicular to the plane of the former; the points where these circles intersect each other will be the poles required.

11. It may be inferred also, from the preceding, that if it were proposed to draw, from any point assumed on the surface of the sphere, an arc of a circle which may measure the shortest distance from that point, to the circumference of any given circle; this arc must be so described, that its prolongation may pass through the poles of the given circle. And conversely, if an arc pass through the poles of a given circle, it will measure the shortest distance from any assumed point to the circumference of that circle.

12. Hence again, if upon the sides, AC and BC, (produced if necessary) of a spherical triangle BCA, we take the arcs CN, CM, each equal 90°, and through the radii GN, GM (figure to art. 7) draw the plane NGM, it is manifest that the point C will be the pole of the circle coinciding with the plane NGM: so that, as the lines GM, GN, are both perpendicular to the common section GC, of the planes AGC, BGC, they measure, by their inclination, the angle of these planes; or the arc NM measures that angle, and consequently the spherical angle BCA.

13. It is also evident that every arc of a little circle, described from the pole C as centre, and containing the same number of degrees as the arc MN, is equally proper for measuring the angle BCA; though it is customary to use only arcs of great circles for this purpose.

14. Lastly, we infer, that if a spherical angle be a right angle, the area of the great circles which form it, will pass mutually through the poles of each other: and that, if the planes of two great circles contain each the axis of the other, or pass through the poles of each other, the angle which they include is a right angle.

These obvious truths being premised and comprehended, the student may pass to the consideration of the following theorems.

THEOREM I.

Any two sides of a spherical triangle are together greater than the third.

This proposition is a necessary consequence of the truth, that the shortest distance between any two points, measured on the surface of the sphere, is the arc of a great circle passing through these points.

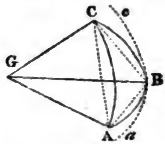
THEOREM II.

The sum of the three sides of any spherical triangle is less than 360 degrees.

For, let the sides AC, BC, (fig. to art. 7) containing any angle A, be produced till they meet again in D: then will the arcs DAC, DBC, be each 180°, because all great circles cut each other into two equal parts: consequently DAC + DBC = 360°. But (theorem 1) DA and DB are together greater than the third side AB of the triangle DAB; and therefore, since CA + CB + DA + DB = 360°, the sum CA + CB + AB is less than 360°. Q. E. D.

THEOREM III.

The sum of the three angles of any spherical triangle is always greater than two right angles, but less than six,



For, let ABC be a spherical triangle, G the centre of the sphere, and let the chords of the arcs AB, BC, AC, be drawn: these chords constitute a rectilinear triangle, the sum of whose three angles is equal to two right angles. But the angle at B made by the chords, AB, BC, is *less* than the angle *a*BC, *formed by the two tangents* B*a*, BC, or less than the angle of inclination of the two planes GBC, GBA, which (art. 6) is the spherical angle at B; consequently the spherical angle at B is *greater* than the angle at B made by the chords AB, CB. In like manner, the spherical angles at A and C, are greater than the respective angles made by the chords meeting at those points. Consequently, the sum of the three angles of the spherical triangle ABC, is greater than the sum of the three angles of the rectilinear triangle made by the chords AB, BC, AC, that is, greater than two right angles. Q. E. 1° D.

2. The angle of inclination of no two of the planes can be so great as two right angles; because, in that case, the two planes would become but one continued plane, and the arcs, instead of being arcs of distinct circles, would be joint arcs of one and the same circle. Therefore, each of the three spherical angles must be less than two right angles; and consequently their sum less than six right angles. Q. E. 2° D.

Cor. 1. Hence it follows, that a spherical triangle may have all its angles either right or obtuse; and therefore the knowledge of any two angles is not sufficient for the determination of the third.

Cor. 2. If the three angles of a spherical triangle be right

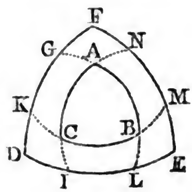
or obtuse, the three sides are likewise each equal to, or greater than 90°: and, if each of the angles be acute, each of the sides is also less than 90°; and conversely.

Scholium. From the preceding theorem the student may clearly perceive what is the essential difference between plane and spherical triangles, and how absurd it would be to apply the rules of plane trigonometry to the solution of cases in spherical trigonometry. Yet, though the difference between the two kinds of triangles be really so great, still there are various properties which are common to both, and which may be demonstrated exactly in the same manner. Thus, for example, it might be demonstrated here, (as well as with regard to plane triangles in the elements of Geometry, vol. 1) that two spherical triangles are equal to each other, 1st. When the three sides of the one are respectively equal to the three sides of the other. 2dly. When each of them has an equal angle contained between equal sides: and, 3dly. When they have each two equal angles at the extremities of equal bases. It might also be shown, that a spherical triangle is equilateral, isosceles, or scalene, according as it hath three equal, two equal, or three unequal angles: and again, that the greatest side is always opposite to the greatest angle, and the least side to the least angle. But the brevity that our plan requires compels us merely to *mention* these particulars. It may be added, however, that a spherical triangle may be at once *right-angled* and *equilateral*; which can never be the case with a plane triangle.

THEOREM IV.

If from the angles of a spherical triangle, as poles, there be described, on the surface of the sphere, three arcs of great circles, which by their intersections form another spherical triangle; each side of this new triangle will be the supplement to the measure of the angle which is at its pole, and the measure of each of its angles the supplement to that side of the primitive triangle to which it is opposite.

From B, A, and C, as poles, let the arcs DF, DE, FE, be described, and by their intersections form another spherical triangle DEF; either side, as DE, of this triangle, is the supplement of the measure of the angle A at its pole; and either angle, as D, has for its measure the supplement of the side AB.



Let the sides AB, AC, BC, of the primitive triangle, be produced till they meet those of the triangle DEF, in the points I, L, M, N, G, K: then, since the point A is the pole of the arc DILE, the distance of the points A and E (measured on an arc of a great circle) will be 90°; also, since C is the pole of the arc EF, the points C and E will be 90° distant: consequently (art. 8) the point E is the pole of the arc AC. In like manner it may be shown, that F is the pole of BC, and D that of AB.

This being premised, we shall have $DL = 90°$, and $IE = 90°$; whence $DL + IE = DL + EL + IL = DE + IL = 180°$. Therefore $DE = 180° - IL$: that is, since IL is the measure of the angle BAC, the arc DE is = the supplement of that measure. Thus also may it be demonstrated that EF is equal the supplement to MN, the measure of the angle BCA, and that DF is equal the supplement to GK, the measure of the angle ABC: which constitutes the first part of the proposition.

2dly. The respective measures of the angles of the triangle DEF are supplemental to the opposite sides of the triangle ABC. For, since the arcs AL and BG are each 90°, therefore is $AL + BG = GL + AB = 180°$; whence $GL = 180° - AB$; that is, the measure of the angle D is equal to the supplement to AB. So likewise may it be shown that AC, BC, are equal to the *supplements to the measures of the respectively opposite* angles E and F. Consequently, the measures of the angles of the triangle DEF are supplemental to the several opposite sides of the triangle ABC. Q. E. D.

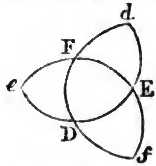
Cor. 1. Hence these two triangles are called *supplemental* or *polar* triangles.

Cor. 2. Since the three sides DE, EF, DF, are supplements to the measures of the three angles A, B, C; it results that $DE + EF + DF + A + B + C = 3 \times 180° = 540°$. But (th. 2), $DE + EF + DF < 360°$: consequently $A + B + C > 180°$. Thus the first part of theorem 3 is very compendiously demonstrated.

Cor. 3. This theorem suggests mutations that are sometimes of use in computation. Thus, if three angles of a spherical triangle are given, to find the sides: the student may subtract each of the angles from 180°, and the three remainders will be the three sides of a new triangle; the angles of this new triangle being found, if their measures be each taken from 180°, the three remainders will be the respective sides of the primitive triangle, whose angles were given.

Scholium. The invention of the preceding theorem is due to *Philip Langsberg*. Vide, Simon Stevin, liv. 3, de la Cosmographie, prop. 31, and Alb. Girard in loc. It is often how-

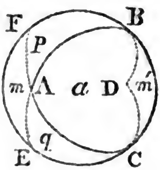
ever treated very loosely by authors on trigonometry: some of them speaking of sides as the supplements of angles, and scarcely any of them remarking which of the several triangles formed by the intersection of the arcs DE, EF, DF, is the one in question. Besides the triangle DEF, three others may be formed by the intersection of the semicircles, and if the *whole* circles be considered, there will be *seven* other triangles formed. But the proposition only obtains with regard to the central triangle (of each hemisphere), which is distinguished from the three others in this, that the two angles A and F are situated on the same side of BC, the two B and E on the same side of AC, and the two C and D on the same side of AB.



THEOREM V.

In every spherical triangle the following proportion obtains, viz. as four right angles (or 360°) to the surface of a hemisphere; or, as two right angles (or 180°) to a great circle of the sphere; so is the excess of the three angles of the triangle above two right angles, to the area of the triangle.

Let ABC be the spherical triangle. Complete one of its sides as BC into the circle BCEF, which may be supposed to bound the upper hemisphere. Prolong also, at both ends, the two sides AB, AC, until they form semicircles estimated from each angle, that is, until BAE = ABD = CAF = ACD = 180°. Then will CBF = 180° = BFE; and consequently the triangle AEF, on the anterior hemisphere, will be equal to the triangle BCD on the opposite hemisphere. Putting m, m', to represent the surface of these triangles, p for that of the triangle BAF, q for that of CAE, and a for that of the proposed triangle ABC. Then a and m' together (or their equal a and m togerther) make up the surface of a spheric lune comprehended between the two semicircles ACD, ABD, inclined in the angle A: a and p together make up the lune included between the semicircles CAF, CBF, making the angle C: a and q together make up the spheric lune included between the semicircles BCE, BAE, making the angle B. And the surface of each of these lunes, is to that of the hemisphere, as the angle made by the comprehending semicircles, to two



right angles. Therefore, putting $\frac{1}{2}s$ for the surface of the hemisphere, we have

$$180^\circ : A :: \tfrac{1}{2}s : a + m,$$
$$180^\circ : B :: \tfrac{1}{2}s : a + q,$$
$$180^\circ : C :: \tfrac{1}{2}s : a + p.$$

Whence, $180^\circ : A + B + C :: \frac{1}{2}s : 3a + m + p + q = 2a + \frac{1}{2}s$; and consequently, by division of proportion,

$$\text{as } 180^\circ : A + B + C - 180^\circ :: \tfrac{1}{2}s : 2a + \tfrac{1}{2}s - \tfrac{1}{2}s = 2a;$$

$$\text{or } 180^\circ : A + B + C - 180^\circ :: \tfrac{1}{4}s : a = \tfrac{1}{2}s \cdot \frac{A + B + C - 180^\circ}{360^\circ}.$$

Q. E. D.*

Cor. 1. Hence the excess of the three angles of any spherical triangle above two right angles, termed technically the *spherical excess*, furnishes a correct measure of the surface of that triangle.

Cor. 2. If $\pi = 3{\cdot}141593$, and d the diameter of the sphere, then is $\pi d^2 \cdot \frac{A + B + C - 180^\circ}{720^\circ}$ = the area of the spherical triangle.

Cor. 3. Since the length of the radius, in any circle, is equal to the length of 57·2957795 degrees, measured on the circumference of that circle; if the *spherical excess* be multiplied by 57·2957795, the product will express the surface of the triangle in square degrees†.

Cor. 4. When $a = 0$, then $A + B + C = 180^\circ$: and when $a = \frac{1}{2}s$, then $A + B + C = 540^\circ$. Consequently the sum of the three angles of a spherical triangle is always between 2 and 6 right angles; which is another confirmation of th. 3.

Cor. 5. When *two* of the angles of a spherical triangle are right angles, the surface of the triangle varies with its third angle. And when a spherical triangle has *three* right angles, its surface is one-eighth of the surface of the sphere.

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* This determination of the area of a spherical triangle is due to *Albert Girard* (who died about 1633). But the demonstration now commonly given of the rule was first published by Dr. Wallis. It was considered as a mere speculative truth, until General Roy, in 1787, employed it very judiciously in the great Trigonometrical Survey, to correct the errors of spherical angles. See Phil. Trans. vol. 80, and the next chapter of this volume.

† Excess in degrees, i. e. excess $^\circ$ = area $^\circ \div 57{\cdot}2957795$.

$$\text{Excess}'' = \frac{\text{area}^\circ \times 3600}{57{\cdot}2957795}$$
$$= \frac{\text{area in squ. feet} \times 3600}{57{\cdot}2957795 \times (365154{\cdot}6)^2}.$$

The log. of the factor to these square feet is 9·3267737, which is the log. employed in art. 5, *Schol.* Prob. 8, ch. v. following.

Remark. Some of the uses of the spherical excess, in the more extensive geodesic operations, will be shown in the following chapter. The mode of finding it, and thence the area when the three angles of a spherical triangle are given, is obvious enough; but it is often requisite to ascertain it by means of other data, as, when two sides and the included angle are given, or when all the three sides are given. In the former case, let a and b be the two sides, C the included angle, and E the spherical excess: then is $\cot \frac{1}{2}E = \frac{\cot \frac{1}{2}a \,.\, \cot \frac{1}{2}b + \cos C}{\sin C}$. When the three sides a, b, c, are given, the spherical excess may be found by the following very elegant theorem, discovered by Simon Lhuillier:

$$\tan \tfrac{1}{4}E = \sqrt{\left(\tan \frac{a+b+c}{4} \,.\, \tan \frac{a+b-c}{4} \,.\, \tan \frac{a-b+c}{4} \,.\, \tan \frac{-a+b+c}{4}\right)}.$$

The investigation of these theorems would occupy more space than can be allotted to them in the present volume.

THEOREM VI.

In every spherical polygon, or surface included by any number of intersecting great circles, the subjoined proportion obtains, viz. as four right angles, or 360°, to the surface of a hemisphere; or, as two right angles, or 180°, to a great circle of the sphere; so is the excess of the sum of the angles above the product of 180° and two less than the number of angles of the spherical polygon, to its area.

For, if the polygon be supposed to be divided into as many triangles as it has sides, by great circles drawn from all the angles through any point within it, forming at that point the vertical angles of all the triangles. Then, by th. 5, it will be as $360° : \frac{1}{2}S :: A+B+C-180°$: its area. Therefore, putting P for the sum of all the angles of the polygon, n for their number, and V for the sum of all the vertical angles of its constituent triangles, it will be, by composition,
as $360° : \frac{1}{2}S :: P+V-180° \,.\, n$: surface of the polygon. But V is manifestly equal to 360° or $180° \times 2$. Therefore, as $360° : \frac{1}{2}S :: P-(n-2)\,180° : \frac{1}{2}S \,.\, \frac{P-(n-2)\,180°}{360°}$, the area of the polygon. Q. E. D.

Cor. 1. If π and d represent the same quantities as in theor. 5, cor. 2, then the surface of the polygon will be expressed by $\pi d^2 \,.\, \frac{P-(n-2)\,180°}{720°}$.

Cor. 2. If $R^\circ = 57{\cdot}2957795$, then will the surface of the polygon in square degrees be $= R^\circ \,.\, (P - (n-2)\ 180^\circ)$.

Cor. 3. When the surface of the polygon is 0, then $P = (n-2)\ 180^\circ$; and when it is a maximum, that is, when it is equal to the surface of the hemisphere, then $P = (a-2)\ 180^\circ + 360^\circ = n \,.\, 180^\circ$: Consequently P, the sum of all the angles of any spheric polygon, is always *less* than $2n$ right angles, but *greater* than $(2n-4)$ right angles, n denoting the number of angles of the polygon.

GENERAL SCHOLIUM.

On the Nature and Measure of Solid Angles.

A *Solid angle* is defined by Euclid, that which is made by the meeting of more than two plane angles, which are not in the same plane, in one point.

Others define it the angular space comprised between several planes meeting in one point.

It may be defined still more generally, the *angular space* included between several plane surfaces, or one or more curved surfaces, meeting in the point which forms the summit of the angle.

According to this definition, solid angles bear just the same relation to the surfaces which comprise them, as plane angles do to the lines by which they are included: so that, as in the latter, it is not the magnitude of the lines, but their mutual inclination, which determines the angle; just so, in the former it is not the magnitude of the planes, but *their* mutual inclinations which determine the angles. And hence all those geometers, from the time of Euclid down to the present period, who have confined their attention principally to the magnitude of the plane angles, instead of their relative positions, have never been able to develope the properties of this class of geometrical quantities; but have affirmed that no solid angle can be said to be the half or the double of another, and have spoken of the bisection and trisection of solid angles, even in the simplest cases, as impossible problems.

But all this supposed difficulty vanishes, and the doctrine of solid angles becomes simple, satisfactory, and universal in its application, by* assuming *spherical surfaces* for their mea-

* Circular arcs are not merely assumed to be the measures of plane angles, they are demonstrated to be so. See Sim. Euclid, Prop. 33, Book VI. It ought also to be *demonstrated* that spherical surfaces are the measures of solid angles. *Ed.*

sure; just as circular arcs are assumed for the measures of plane angles *. Imagine, that from the summit of a solid angle (formed by the meeting of three planes) as a centre, any sphere be described, and that those planes are produced till they cut the surface of the sphere; then will the surface of the spherical triangle, included between those planes, be a proper measure of the solid angle made by the planes at their common point of meeting: for no change can be conceived in the relative position of those planes, that is, in the magnitude of the solid angle, without a corresponding and proportional mutation in the surface of the spherical triangle. If, in like manner, the three or more surfaces, which by their meeting constitute another solid angle, be produced till they cut the surface of the same or an equal sphere, whose centre coincides with the summit of the angle; the surface of the spheric triangle or polygon, included between the planes which determine the angle, will be a correct measure of *that* angle. And the ratio which subsists between the areas of the spheric triangles, polygons, or other surfaces thus formed, will be accurately the ratio which subsists between the solid angles, constituted by the meeting of the several planes or surfaces, at the centre of the sphere.

Hence, the comparison of solid angles becomes a matter of great ease and simplicity: for, since the areas of spherical triangles are measured by the excess of the sums of their angles each above two right angles (th. 5); and the areas of spherical polygons of n sides, by the excess of the sum of their angles above $2n-4$ right angles (th. 6); it follows, that the magnitude of a trilateral solid angle will be measured by the excess of the sum of the three angles, made respectively

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* It may be proper to anticipate here the only objection which *can* be made to this assumption; which is founded on the principle, *that quantities should always be measured by quantities of the same kind.* But this, often and positively as it is affirmed, is by no means necessary; nor in many cases is it possible. To measure is to *compare* mathematically; and if by comparing two quantities, whose ratio we know or can ascertain, with two other quantities whose ratio we wish to know, the point in question becomes determined; it signifies not at all whether the magnitudes which constitute one ratio, are like or unlike the magnitudes which constitute the other ratio. It is thus that mathematicians, with perfect safety and correctness, make use of space as a measure of velocity, mass as a measure of inertia, mass and velocity conjointly as a measure of force, space as a measure of time, weight as a measure of density, expansion as a measure of heat, a certain function of planetary velocity as a measure of distance from the central body, arcs of the same circle as measures of plane angles; and it is in conformity with this general procedure that we adopt surfaces, of the same sphere, as measures of solid angles.

by its bounding planes, above 2 right angles; and the magnitudes of solid angles formed by n bounding planes, by the excess of the sum of the angles of inclination of the several planes above $2n-4$ right angles.

As to solid angles limited by curve surfaces, such as the angles at the vertices of cones; they will manifestly be measured by the spheric surfaces cut off by the prolongation of their bounding surfaces, in the same manner as angles determined by planes are measured by the triangles or polygons, they mark out upon the same, or an equal sphere. In all cases, the maximum limit of solid angles will be the *plane* towards which the various planes determining such angles approach, as they diverge further from each other about the same summit: just as a right line is the maximum limit of plane angles, being formed by the two bounding lines when they make an angle of 180°. The maximum limit of solid angles is measured by the surface of a hemisphere, in like manner as the maximum limit of plane angles is measured by the arc of a semicircle. The solid right angle (either angle, for example, of a cube) is $\frac{1}{4}$ $(=\frac{1}{2}^2)$ of the maximum solid angle: while the plane right angle is half the maximum plane angle.

The analogy between plane and solid angles being thus traced, we may proceed to exemplify this theory by a few instances; assuming 1000 as the numeral measure of the *maximum solid angle = 4 times 90° solid = 360° solid.*

1. The solid angles of right prisms are compared with great facility. For, of the three angles made by the three planes which, by their meeting, constitute every such solid angle, two are right angles; and the third is the same as the corresponding plane angle of the polygonal base; on which, therefore, the measure of the solid angle depends. Thus, with respect to the right prism with an equilateral triangular base, each solid angle is formed by planes which respectively make angles of 90°, 90°, and 60°. Consequently $90°+90°+60°-180°=60°$, is the measure of such angle, compared with 360° the maximum angle. It is, therefore, one-sixth of the maximum angle. A right prism with a square base has, in like manner, each solid angle measured by $90°+90°+90°-180°=90°$, which is $\frac{1}{4}$ of the maximum angle. And thus it may be found, that each solid angle of a right prism, with an equilateral

triangular base	is $\frac{1}{6}$ max. angle	$=\frac{1}{6}$ .1000.
square base	is $\frac{1}{4}$	$=\frac{2}{8}$ .1000.
pentagonal base	is	$=\frac{3}{10}$.1000.
hexagonal	is $\frac{1}{3}$	$=\frac{4}{12}$.1000.
heptagonal	is	$=\frac{5}{14}$.1000.

octagonal base is $\frac{3}{8}$ . . . . . $= \frac{6}{16}.1000.$
nonagonal is . . . . . $= \frac{7}{18}.1000.$
decagonal is $\frac{4}{5}$ . . . . . $= \frac{8}{20}.1000.$
undecagonal is . . . . . $= \frac{9}{22}.1000.$
duodecagonal is $\frac{5}{12}$ . . . . . $= \frac{10}{24}.1000.$

m gonal is . . . . . $= \frac{m-2}{2m}.1000.$

Hence it may be deduced, that each solid angle of a regular prism, with triangular base, is *half* each solid angle of a prism with a regular hexagonal base. Each with regular
square base $= \frac{2}{3}$ of each, with regular octagonal base,
pentagonal $= \frac{3}{4}$ . . . . decagonal,
hexagonal $= \frac{4}{5}$ . . . . duodecagonal,

$\frac{1}{2}m$ gonal $= \frac{m-4}{m-2}$ . . . . . . m gonal base.

Hence again we may infer, that the sum of all the solid angles of any prism of triangular base, whether that base be regular or irregular, is *half* the sum of the solid angles of a prism of quadrangular base, regular or irregular. And the sum of the solid angles of any prism of
tetragonal base is $= \frac{2}{3}$ sum of angles in prism of pentag. base,
pentagonal . . . $= \frac{3}{4}$ . . . . hexagonal,
hexagonal . . . $= \frac{4}{5}$ . . . . heptagonal,

m gonal . . . . . $= \frac{m-2}{m-1}$ . . . . $(m+1)$ gonal.

2. Let us compare the solid angles of the five regular bodies. In these bodies, if m be the number of sides of each face; n the number of planes which meet at each solid angle; $\frac{1}{2}\bigcirc$ = half the circumference or 180°; and A the plane angle made by two adjacent faces: then we have $\sin\frac{1}{2}A = \dfrac{\cos\frac{1}{2n}\bigcirc}{\sin\frac{1}{2m}\bigcirc}$.

This theorem gives, for the plane angle formed by every two contiguous faces of the tetraëdron, 70°31′42″; of the hexaëdron, 90°; of the octaëdron, 109°28′18″; of the dodecaëdron, 116°33′54″; of the icosaëdron, 138°11′23″. But, in these polyedræ, the number of faces meeting about each solid angle, is 3, 3, 4, 3, 5 respectively. Consequently the several solid angles will be determined by the subjoined proportions:

	Solid Angle.	
360° : 3·70°31′42″ —180° : : 1000 :	87·73611	Tetraëdron.
360° : 3·90° —180° : : 1000 :	250·	Hexaëdron.
360° : 4·109°28′18″—360° : : 1000 :	216·35185	Octaëdron.
360° : 3·116°33′54″—180° : : 1000 :	471·375	Dodecaëdron.
360° : 5·138°11′23″—540° : : 1000 :	419·30169	Icosaëdron.

3. The solid angles at the vertices of cones, will be determined by means of the spheric segments cut off at the bases of those cones; that is, if right cones, instead of having plane bases, had bases formed of the segments of equal spheres, whose centres were the vertices of the cones, the surfaces of those segments would be measures of the solid angles at the respective vertices. Now, the surfaces of spheric segments, are to the surface of the hemisphere, as their altitudes, to the radius of the sphere; and therefore the solid angles at the vertices of right cones, will be to the maximum solid angle, as the excess of the slant side above the axis of the cone, to the slant side of the cone. Thus, if we wish to ascertain the solid angles at the vertices of the equilateral and the right-angled cones; the axis of the former is $\frac{1}{2}\sqrt{3}$, of the latter, $\frac{1}{2}\sqrt{2}$, the slant side of each being unity. Hence,

Angle at vertex.

$1 : 1 - \frac{1}{2}\sqrt{3} :: 1000 : 133{\cdot}97464$, equilateral cone,
$1 : 1 - \frac{1}{2}\sqrt{2} :: 1000 : 292{\cdot}89322$, right-angled cone.

4. From what has been said, the mode of determining the solid angles at the vertices of pyramids will be sufficiently obvious. If the pyramids be regular ones, if N be the number of faces meeting about the vertical angle in one, and A the angle of inclination of each two of its plane faces; if n be the number of planes meeting about the vertex of the other, and a the angle of inclination of each two of its faces: then will the vertical angle of the former, be to the vertical angle of the latter pyramid, as $NA-(N-2)\,180°$, to $na-(n-2)\,180°$.

If a cube be cut by diagonal planes, into 6 equal pyramids with square bases, their vertices all meeting at the centre of the circumscribing sphere; then each of the solid angles made by the four planes meeting at each vertex, will be $\frac{1}{3}$ of the maximum solid angle; and each of the solid angles at the base of the pyramids, will be $\frac{1}{12}$ of the maximum solid angle. Therefore, each solid angle at the base of such pyramid, is *one-fourth* of the solid angle at its vertex: and, if the angle at the vertex be bisected, as described below, either of the solid angles arising from the bisection will be double of either solid angle at the base. Hence also, and from the first subdivision of this scholium, each solid angle of a prism, with equilateral triangular base, will be *half* each vertical angle of these pyramids, and *double* each solid angle at their bases.

The angles made by one plane with another, must be ascertained, either by measurement or by computation, according to circumstances. But, the general theory being thus explained, and illustrated, the further application of it is left to the skill and ingenuity of geometers; the following simple example, merely, being added here.

Ex. Let the solid angle at the vertex of a square pyramid be bisected.

1st. Let a plane be drawn through the vertex and any two opposite angles of the base, that plane will bisect the solid angle at the vertex; forming two trilateral angles, each equal to half the original quadrilateral angle.

2dly. Bisect either diagonal of the base, and draw *any* plane to pass through the point of bisection and the vertex of the pyramid; such plane, if it do *not* coincide with the former, will divide the quadrilateral solid angle into two equal quadrilateral solid angles. For this plane, produced, will bisect the great circle diagonal of the spherical parallelogram cut off by the base of the pyramid; and any great circle bisecting such diagonal is known to bisect the spherical parallelogram, or square; the plane, therefore, bisects the solid angle.

Cor. Hence an indefinite number of planes may be drawn, each to bisect a given quadrilateral solid angle.

SECTION II.

Resolution of Spherical Triangles.

THE different cases of spherical trigonometry, like those in plane trigonometry, may be solved either geometrically or algebraically. We shall here adopt the analytical method, as well on account of its being more compatible with brevity, as because of its correspondence and connexion with the substance of the preceding chapter*. The whole doctrine may be comprehended in the subsequent problems and theorems.

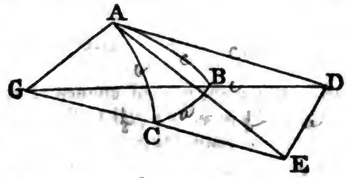
PROBLEM I.

To find equations, from which may be deduced the solution of all the cases of spherical triangles.

Let ABC be a spherical triangle; AD the tangent, and GD the secant, of the arc AB; AE the tangent, and GE the se-

* For the geometrical method, the reader may consult Simson's or Playfair's Euclid, or Bishop Horsley's Elementary Treatise on Practical Mathematics.

cant, of the arc AC; let the capital letters A, B, C, denote the angles of the triangle, and the small letters a, b, c, the opposite sides BC, AC, AB. Then the first equations in art. 6 Pl. Trig.



applied to the two triangles ADE, GDE, give, for the former, $DE^2 = \tan^2 b + \tan^2 c - 2 \tan b \cdot \tan c \cdot \cos A$; for the latter, $DE^2 = \sec^2 b + \sec^2 c - 2 \sec b \cdot \sec c \cdot \cos a$. Subtracting the first of these equations from the second, and observing that $\sec^2 b - \tan^2 b = R^2 = 1$, we shall have, after a little reduction, $1 + \frac{\sin b \cdot \sin c}{\cos b \cdot \cos c} \cos A - \frac{\cos a}{\cos b \cdot \cos c} = 0$. Whence the three following symmetrical equations are obtained:

$$\left.\begin{aligned} \cos a &= \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A \\ \cos b &= \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B \\ \cos c &= \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C \end{aligned}\right\} \quad (\text{I.})$$

THEOREM VII.

In every spherical triangle, the sines of the angles are proportional to the sines of their opposite sides.

If, from the first of the equations marked I, the value of cos A be drawn, and substituted for it in the equation $\sin^2 A = 1 - \cos^2 A$, we shall have

$$\sin^2 A = 1 - \frac{\cos^2 a + \cos^2 b \cdot \cos^2 c - 2 \cos a \cdot \cos b \cdot \cos c}{\sin^2 b \cdot \sin^2 c}$$

Reducing the terms of the second side of this equation to a common denominator, multiplying both numerator and denominator by $\sin^2 a$, and extracting the sq. root, there will result

$$\sin A = \sin a \cdot \frac{\sqrt{(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cdot \cos b \cdot \cos c)}}{\sin a \cdot \sin b \cdot \sin c}.$$

Here, if the whole fraction which multiplies $\sin a$, be denoted by K (see art. 8 chap. iii.), we may write $\sin A = K \cdot \sin a$. And, since the fractional factor, in the above equation, contains terms in which the sides a, b, c, are alike affected, we have similar equations for sin B, and sin C. That is to say, we have

$\sin A = K \cdot \sin a \ldots \sin B = K \cdot \sin b \ldots \sin C = K \cdot \sin c$.

Consequently, $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \ . \ . \ . \ (\text{II.})$ which is the algebraical expression of the theorem.

THEOREM VIII.

In every right-angled spherical triangle, the cosine of the hypothenuse, is equal to the product of the cosines of the sides including the right angle.

For if A be measured by $\frac{1}{4}\bigcirc$, its cosine becomes nothing, and the first of the equations I becomes $\cos a = \cos b \,.\, \cos c$. Q. E. D.

THEOREM IX.

In every right-angled spherical triangle, the cosine of either oblique angle, is equal to the quotient of the tangent of the adjacent side divided by the tangent of the hypothenuse.

If, in the second of the equations I, the preceding value of $\cos a$ be substituted for it, and for $\sin a$ its value $\tan a \,.\, \cos a = \tan a \,.\, \cos b \,.\, \cos c$; then, recollecting that $1 - \cos^2 c = \sin^2 c$, there will result, $\tan a \,.\, \cos c \,.\, \cos B = \sin c$: whence it follows that,

$$\tan a \,.\, \cos B = \tan c, \text{ or } \cos B = \frac{\tan c}{\tan a}.$$

$$\text{Thus also it is found that } \cos C = \frac{\tan b}{\tan a}.$$

THEOREM X.

In any right-angled spherical triangle, the cosine of one of the sides about the right angle, is equal to the quotient of the cosine of the opposite angle divided by the sine of the adjacent angle.

From th. 7, we have $\frac{\sin B}{\sin A} = \frac{\sin b}{\sin a}$: which, when A is a right angle, becomes simply $\sin B = \frac{\sin b}{\sin a}$. Again, from th. 9, we have $\cos C = \frac{\tan b}{\tan a}$. Hence, by division,

$$\frac{\cos C}{\sin B} = \frac{\tan b}{\sin b} \,.\, \frac{\sin a}{\tan a} = \frac{\cos a}{\cos b}.$$

Now, th. 8 gives $\frac{\cos a}{\cos b} = \cos c$. Therefore $\frac{\cos C}{\sin B} = \cos c$; and In like manner, $\frac{\cos B}{\sin C} = \cos b$. Q. E. D.

THEOREM XI.

In every right-angled spherical triangle, the tangent of either of the oblique angles, is equal to the quotient of the tangent of the opposite side, divided by the sine of the other side about the right angle.

For, since $\sin B = \frac{\sin b}{\sin a}$, and $\cos B = \frac{\tan c}{\tan a}$.

we have $\frac{\sin B}{\cos B} = \frac{\sin b}{\sin a} \cdot \frac{\tan a}{\tan c}$

Whence, because (th. 8) $\cos a = \cos b \,.\, \cos c$, and since $\sin a = \cos a \,.\, \tan a$, we have

$$\tan B = \frac{\sin b}{\cos a \,.\, \tan c} = \frac{\sin b}{\cos b \,.\, \cos c \,.\, \tan c} = \frac{\sin b}{\cos b} \cdot \frac{1}{\cos c \,.\, \tan c} = \frac{\tan b}{\sin c}.$$

In like manner, $\tan C = \frac{\tan c}{\sin b}$. Q. E. D.

THEOREM XII.

In every right-angled spherical triangle, the cosine of the hypothenuse, is equal to the quotient of the cotangent of one of the oblique angles, divided by the tangent of the other angle.

For, multiplying together the resulting equations of the preceding theorem, we have

$$\tan B \,.\, \tan C = \frac{\tan b}{\sin b} \cdot \frac{\tan c}{\sin c} = \frac{1}{\cos b \,.\, \cos c}.$$

But, by th. 8, $\cos b \,.\, \cos c = \cos a$.

Therefore $\tan B \,.\, \tan C = \frac{1}{\cos a}$, or $\cos a = \frac{\cot C}{\tan B}$. Q. E. D.

THEOREM XIII.

In every right-angled spherical triangle, the sine of the difference between the hypothenuse and base, is equal to the continued product of the sine of the perpendicular, cosine of the base, and tangent of half the angle opposite to the perpendicular; or equal to the continued product of the tangent of the perpendicular, cosine of the hypothenuse, and tangent of half the angle opposite to the perpendicular*.

* This theorem is due to M. Prony, who published it without demonstration in the *Connaissance des Temps* for the year 1808, and made use of it in the construction of a chart of the course of the Po.

Here retaining the same notation, since we have $\sin a = \frac{\sin b}{\sin B}$, and $\cos B = \frac{\tan c}{\tan a}$; if for the tangents there be substituted their values in sines and cosines, there will arise,

$$\sin c \,.\, \cos a = \cos B \,.\, \cos c \,.\, \sin a = \cos B \,.\, \cos c \,.\, \frac{\sin b}{\sin B}.$$

Then substituting for $\sin a$, and $\sin c \,.\, \cos a$, their values in the known formula (equ. v. chap. iii.) viz.

in $\sin (a - c) = \sin a \,.\, \cos c - \cos a \,.\, \sin c$,

and recollecting that $\frac{1 - \cos B}{\sin B} = \tan \frac{1}{2}B$,

it will become, $\sin (a - c) = \sin b \,.\, \cos c \,.\, \tan \frac{1}{2}B$;

which is the first part of the theorem: and, if in this result we introduce, instead of $\cos c$, its value $\frac{\cos a}{\cos b}$ (th. 8), it will be transformed into $\sin (a - c) = \tan b \,.\, \cos a \,.\, \tan \frac{1}{2}B$; which is the second part of the theorem. Q. E. D.

Cor. This theorem leads manifestly to an analogous one with regard to rectilinear triangles, which, if h, b, and p denote the hypothenuse, base, and perpendicular, and B, P, the angles respectively opposite to b, p; may be expressed thus:

$$h - b = p \,.\, \tan \tfrac{1}{2}P \;.\;.\;.\;.\;.\; h - p = b \,.\, \tan \tfrac{1}{2}B.$$

These theorems may be found useful in reducing inclined lines to the plane of the horizon.

PROBLEM II.

Given the three sides of a spherical triangle: it is required to find expressions for the determination of the angles.

Retaining the notation of prob. 1, in all its generality, we soon deduce from the equations marked I in that problem, the following; viz.

$$\left.\begin{aligned}\cos A &= \frac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c}\\ \cos B &= \frac{\cos b - \cos a \,.\, \cos c}{\sin a \,.\, \sin c}\\ \cos C &= \frac{\cos c - \cos a \,.\, \cos b}{\sin a \,.\, \sin b}\end{aligned}\right\}$$

As these equations, however, are not well suited for logarithmic computation; they must be so transformed that their second members will resolve into factors. In order to this, substitute in the known equation $1 - \cos A = 2 \sin^2 \frac{1}{2}A$, the preceding value of $\cos A$, and there will result

$$2 \sin^2 \tfrac{1}{2}A = \frac{\cos (b - c) - \cos a}{\sin b \,.\, \sin c}.$$

But, because $\cos B' - \cos A' = 2 \sin \frac{1}{2}(A'+B') \cdot \sin \frac{1}{2}(A'-B')$ (art. 25 ch. iii.), and consequently

$$\cos(b-c) - \cos a = 2 \sin \frac{a+b-c}{2} \cdot \sin \frac{a+c-b}{2}:$$

we have obviously,

$$\sin^2 \tfrac{1}{2}A = \frac{\sin \frac{1}{2}(a+b-c) \cdot \sin \frac{1}{2}(a+c-b)}{\sin b \cdot \sin c}.$$

Whence, making $s = a+b+c$, there results

$$\left.\begin{aligned} \sin \tfrac{1}{2}A &= \sqrt{\frac{\sin(\frac{1}{2}s-b) \cdot \sin(\frac{1}{2}s-c)}{\sin b \cdot \sin c}} \\ \text{So, also, } \sin \tfrac{1}{2}B &= \sqrt{\frac{\sin(\frac{1}{2}s-a) \cdot \sin(\frac{1}{2}s-c)}{\sin a \cdot \sin c}} \\ \text{And, } \sin \tfrac{1}{2}C &= \sqrt{\frac{\sin(\frac{1}{2}s-a) \cdot \sin(\frac{1}{2}s-b)}{\sin a \cdot \sin b}} \end{aligned}\right\} \quad \text{(III.)}$$

The expressions for the tangents of the half angles might have been deduced with equal facility; and we should have obtained, for example,

$$\tan \tfrac{1}{2}A = \sqrt{\frac{\sin(\frac{1}{2}s-b) \cdot \sin(\frac{1}{2}s-c)}{\sin \frac{1}{2}s \cdot \sin \frac{1}{2}(s-a)}} \quad (iii.)$$

Thus again, the expressions for the cosine and cotangent of half one of the angles, are

$$\cos \tfrac{1}{2}A = \sqrt{\frac{\sin \frac{1}{2}s \cdot \sin \frac{1}{2}(s-a)}{\sin b \cdot \sin c}}.$$

$$\cot \tfrac{1}{2}A = \sqrt{\frac{\sin \frac{1}{2}s \cdot \sin \frac{1}{2}(s-a)}{\sin(\frac{1}{2}s-b) \cdot \sin(\frac{1}{2}s-c)}}.$$

The three latter flowing naturally from the former, by means of the values $\tan = \frac{\sin}{\cos}$, $\cot = \frac{\cos}{\sin}$. (art. 4 ch. iii.)

Cor. 1. When two of the sides, as b and c, become equal, then the expression for $\sin \frac{1}{2}A$ becomes

$$\sin \tfrac{1}{2}A = \frac{\sin(\frac{1}{2}s-b)}{\sin b} = \frac{\sin \frac{1}{2}a}{\sin c}.$$

Cor. 2. When all the three sides are equal, or $a = b = c$, then $\sin \frac{1}{2}A = \frac{\sin \frac{1}{2}a}{\sin a}$.

Cor. 3. In this case, if $a = b = c = 90°$; then $\sin \frac{1}{2}A = \frac{\frac{1}{2}\sqrt{2}}{1} = \frac{1}{2}\sqrt{2} = \sin 45°$; and $A = B = C = 90°$.

Cor. 4. If $a=b=c=60°$: then $\sin \frac{1}{2}A = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{3}\sqrt{3} = \sin 35°15'51''$: and $A=B=C=70°31'42''$: the same as the angle between two contiguous planes of a tetraedron.

Cor. 5. If $a=b=c$ were assumed $= 120°$: then $\sin \frac{1}{2}A = \frac{\sin 60°}{\sin 120°} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}\sqrt{3}} = 1$; and $A = B = C = 180°$; which shows that no such triangle can be constructed (conformably to th. 2); but that the three sides would, in such case, form three continued arcs completing a great circle of the sphere.

PROBLEM III.

Given the three angles of a spherical triangle, to find expressions for the sides.

If from the first and third of the equations marked I (prob. 1), $\cos c$ be exterminated, there will result

$$\cos A \,.\, \sin c + \cos C \,.\, \sin a \,.\, \cos b = \cos a \,.\, \sin b.$$

But, it follows from th. 7, that $\sin c = \dfrac{\sin a \,.\, \sin C}{\sin A}$. Substituting for $\sin c$ this value of it, and for $\dfrac{\cos A}{\sin A}$, $\dfrac{\cos a}{\sin a}$, their equivalents $\cot A$, $\cot a$, we shall have,

$$\cot A \,.\, \sin C + \cos C \,.\, \cos b = \cot a \,.\, \sin b.$$

Now, $\cot a \,.\, \sin B = \dfrac{\cos a}{\sin a} \,.\, \sin b = \cos a \,.\, \dfrac{\sin b}{\sin a} = \cos a \,.\, \dfrac{\sin B}{\sin A}$, (th. 7). So that the preceding equation at length becomes,

$$\cos A \,.\, \sin C = \cos a \,.\, \sin B - \sin A \,.\, \cos C \,.\, \cos b.$$

In like manner, we have,

$$\cos B \,.\, \sin C = \cos b \,.\, \sin A - \sin B \,.\, \cos C \,.\, \cos a.$$

Exterminating $\cos b$ from these, there results

$$\left.\begin{aligned} \cos A &= \cos a \,.\, \sin B \,.\, \sin C - \cos B \,.\, \cos C. \\ \text{So likewise} \quad \cos B &= \cos b \,.\, \sin A \,.\, \sin C - \cos A \,.\, \cos C. \\ \cos C &= \cos c \,.\, \sin A \,.\, \sin B - \cos A \,.\, \cos B. \end{aligned}\right\} \text{(IV.)}$$

This system of equations is manifestly analogous to equation I; and if they be reduced in the manner adopted in the last problem, they will give

$$\left.\begin{aligned} \sin \tfrac{1}{2}a &= \sqrt{-\frac{\cos \frac{1}{2}(A+B+C) \,.\, \cos \frac{1}{2}(B+C-A)}{\sin B \,.\, \sin C}}. \\ \sin \tfrac{1}{2}b &= \sqrt{-\frac{\cos \frac{1}{2}(A+B+C) \,.\, \cos \frac{1}{2}(A+C-B)}{\sin A \,.\, \sin C}}. \\ \sin \tfrac{1}{2}c &= \sqrt{-\frac{\cos \frac{1}{2}(A+B+C) \,.\, \cos \frac{1}{2}(A+B-C)}{\sin A \,.\, \sin B}}. \end{aligned}\right\} \text{(V.)}$$

The expression for the tangent of half a side is

$$\tan \tfrac{1}{2}a = \sqrt{-\frac{\cos \frac{1}{2}(A+B+C) \,.\, \cos \frac{1}{2}(B+C-A)}{\cos \frac{1}{2}(A+C-B) \,.\, \cos \frac{1}{2}(A+B-C)}}.$$

The values of the cosines and cotangents are omitted, to save room; but are easily deduced by the student.

Cor. 1. When two of the angles, as B and C, become equal, then the value of $\cos \frac{1}{2}a$ becomes $\cos \frac{1}{2}a = \dfrac{\cos \frac{1}{2}A}{\sin B}$.

Cor. 2. When $A = B = C$; then $\cos \frac{1}{2}a = \dfrac{\cos \frac{1}{2}A}{\sin A}$.

Cor. 3. When $A = B = C = 90^\circ$, then $a = b = c = 90^\circ$.

Cor. 4. If A = B = C = 60°; then $\cos\frac{1}{2}a = \frac{\sin 60°}{\sin 60°} = 1$.

So that $a = b = c = 0$. Consequently no such triangle can be constructed: conformably to th. 3.

Cor. 5. If A = B = C = 120°; then $\cos\frac{1}{2}a = \frac{\cos 60°}{\sin 120°} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} =$ $\frac{1}{3}\sqrt{3} = \cos 54°44'9''$. Hence $a = b = c = 109°28'8''$.

Schol. If, in the preceding values of $\sin\frac{1}{2}a$, $\sin\frac{1}{2}b$, &c. the quantities under the radical were negative in reality, as they are in appearance, it would obviously be impossible to determine the value of $\sin\frac{1}{2}a$, &c. But this value is in fact always real. For, in general, $\sin(x - \frac{1}{4}\bigcirc) = -\cos x$: therefore, $\sin\left(\frac{A+B+C}{2} - \frac{1}{4}\bigcirc\right) = -\cos\frac{1}{2}(A+B+C)$; a quantity which is always positive, because, as $A + B + C$ is necessarily comprised between $\frac{1}{2}\bigcirc$ and $\frac{3}{2}\bigcirc$, we have $\frac{1}{2}(A + B + C) - \frac{1}{4}\bigcirc$ greater than nothing, and less than $\frac{1}{2}\bigcirc$. Further, any one side of a spherical triangle being smaller than the sum of the other two, we have, by the property of the polar triangle (theorem 4), $\frac{1}{2}\bigcirc - A$ less than $\frac{1}{2}\bigcirc - B + \frac{1}{2}\bigcirc - C$; whence $\frac{1}{2}(B + C - A)$ is less than $\frac{1}{4}\bigcirc$; and of course its cosine is positive.

PROBLEM. IV.

Given two sides of a spherical triangle, and the included angle; to obtain expressions for the other angles.

1. In the investigation of the last problem, we had

$$\cos A \,.\, \sin c = \cos a \,.\, \sin b - \cos c \,.\, \sin a \,.\, \cos b:$$

and by a simple permutation of letters, we have

$$\cos B \,.\, \sin c = \cos b \,.\, \sin a - \cos c \,.\, \sin b \,.\, \cos a:$$

adding together these two equations, and reducing, we have

$$\sin c\,(\cos A + \cos B) = (1 - \cos c)\,\sin(a + b).$$

Now, we have from theor. 7,

$$\frac{\sin a}{\sin A} = \frac{\sin c}{\sin C}, \text{ and } \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

Freeing these equations from their denominators, and respectively adding and subtracting them, there results

$$\sin c\,(\sin A + \sin B) = \sin C\,(\sin a + \sin b),$$

$$\text{and } \sin c\,(\sin A - \sin B) = \sin C\,(\sin a - \sin b).$$

Dividing each of these two equations by the preceding, there will be obtained

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin C}{1 - \cos c} \,.\, \frac{\sin a + \sin b}{\sin(a + b)}.$$

$$\frac{\sin A - \sin B}{\cos A + \cos B} = \frac{\sin C}{1 - \cos c} \,.\, \frac{\sin a - \sin b}{\sin(a + b)}.$$

Comparing these with the equations in arts. 25, 26, 27, ch. iii., there will at length result

$$\left.\begin{array}{l}\tan \tfrac{1}{2}(A+B) = \cot \tfrac{1}{2}C \cdot \dfrac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}. \\ \tan \tfrac{1}{2}(A-B) = \cot \tfrac{1}{2}C \cdot \dfrac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}.\end{array}\right\} \;.\;.\; \text{(VI.)}$$

Cor. When $a=b$, the first of the above equations becomes $\tan A = \tan B = \cot \frac{1}{2}C \cdot \sec a$.

And in this case it will be, as rad : $\sin \frac{1}{2}C$: : $\sin a$ or $\sin b$: $\sin \frac{1}{2}c$.

And, as rad : $\cos A$ or $\cos B$: : $\tan a$ or $\tan b$: $\tan \frac{1}{2}c$.

2. The preceding values of $\tan \frac{1}{2}(A+B)$ $\tan \frac{1}{2}(A-B)$ are very well fitted for logarithmic computation: it may, notwithstanding, be proper to investigate a theorem which will at once lead to one of the angles, by means of a subsidiary angle. In order to this, we deduce immediately from the second equation in the investigation of prob. 3,

$$\cot A = \frac{\cot a \cdot \sin b}{\sin C} - \cot C \cdot \cos b.$$

Then, choosing the subsidiary angle φ, so that

$$\tan \varphi = \tan a \cdot \cos C,$$

that is, finding the angle φ, whose tangent is equal to the product $\tan a \cdot \cos C$, which is equivalent to dividing the original triangle into two right-angled triangles, the preceding equation will become

$$\cot A = \cot C\,(\cot \varphi \cdot \sin b - \cos b) = \frac{\cot C}{\sin \varphi}\,(\cos \varphi \cdot \sin b - \sin \varphi \cdot \cos b).$$

And this, since $\sin(b-\varphi) = \cos \varphi \cdot \sin b - \sin \varphi \cdot \cos b$, becomes

$$\cot A = \frac{\cot C}{\sin \varphi} \cdot \sin (b-\varphi).$$

Which is a very simple and convenient expression.

PROBLEM V.

Given two angles of a spherical triangle, and the side comprehended between them; to find expressions for the other two sides.

1. Here, a similar analysis to that employed in the preceding problem, being pursued with respect to the equations IV, in prob. 3, will produce the following formulæ :

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{\sin c}{1 + \cos c} \cdot \frac{\sin A + \sin B}{\sin (A + B)},$$

$$\frac{\sin a - \sin b}{\cos a + \cos b} = \frac{\sin c}{1 + \cos c} \cdot \frac{\sin A - \sin B}{\sin (A + B)}.$$

Whence, as in prob. 4, we obtain

$$\left.\begin{array}{l}\tan\tfrac{1}{2}(a+b)=\tan\tfrac{1}{2}c\,.\,\dfrac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}.\\ \tan\tfrac{1}{2}(a-b)=\tan\tfrac{1}{2}c\,.\,\dfrac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}.\end{array}\right\}\ (\text{VII.})^*$$

2. If it be wished to obtain a side at once, by means of a subsidiary angle; then, find φ so that $\frac{\cot A}{\cos c}=\tan\varphi$; then will $\cot a=\frac{\cot c}{\cos\varphi}\,.\,\cos(B-\varphi)$.

PROBLEM VI.

Given two sides of a spherical triangle, and an angle opposite to one of them; to find the other opposite angle.

Suppose the sides given are a, b, and the given angle B: then from theor. 7, we have $\sin A=\frac{\sin a\,.\,\sin B}{\sin b}$; or, sin A, a fourth proportional to sin b, sin B, and sin a.

PROBLEM VII.

Given two angles of a spherical triangle, and a side opposite to one of them; to find the side opposite to the other.

Suppose the given angles are A, and B, and b the given side: then th. 7, gives $\sin a=\frac{\sin b\,.\,\sin A}{\sin B}$; or, sin a, a fourth proportional to sin B, sin b, and sin A.

Scholium.

In problems 2 and 3, if the circumstances of the question leave any doubt, whether the arcs or the angles sought are greater or less than a quadrant, or than a right angle, the difficulty will be entirely removed by means of the table of mutations of signs of trigonometrical quantities, in different quadrants, marked VII in chap. 3. In the 6th and 7th problems, the question proposed will often be susceptible of two solutions: by means of the subjoined table the student may always tell when this will or will not be the case.

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* The formulæ marked VI, and VII, converted into analogies, by making the denominator of the second member the first term, the other two factors the second and third terms, and the first member of the equation the fourth term of the proposition, as

$$\cos\tfrac{1}{2}(a+b):\cos\tfrac{1}{2}(a-b)::\cot\tfrac{1}{2}C:\tan\tfrac{1}{2}(A+B),$$
$$\sin\tfrac{1}{2}(a+b):\sin\tfrac{1}{2}(a-b)::\cot\tfrac{1}{2}C:\tan\tfrac{1}{2}(A-B),\ \&c.\ \&c.$$

are called *the Analogies of Napier*, being invented by that celebrated geometer. He likewise invented other rules for spherical trigonometry, known by the name of *Napier's Rules for the circular parts*: but these, notwithstanding their ingenuity, are not inserted here; because they are too artificial to be applied by a young computist, to every case that may occur without considerable danger of misapprehension and error.

IV. Hypothenuse, and one angle.	Adjacent leg.	Its tan = tan hyp × cos giv. ang.	If the things given be of like affection.
	Leg opp. to the given angle.	Its sin = sin hyp × sin giv. ang.	If the given angle be acute.
	Other angle.	Its tan = $\frac{\text{cot giv. angle}}{\text{cos hypothen.}}$.	If the things given be of like affection.
V. The two legs.	Hypothenuse.	Its cos = rectan. cos giv. legs.	If the given legs be of like affection.
	Either of the angles.	Its cos = $\frac{\text{tan oppos leg}}{\text{sin adjac. leg}}$.	If the opposite leg be less than 90°.
VI. The two angles.	Hypothenuse.	Its cos = rect. cot giv. angles.	If the angles be of like affection.
	Either of the legs.	Its cos = $\frac{\text{cos opposite angle}}{\text{sin adjacent angle}}$.	If the opposite angle be acute.

In working by the logarithms, the student must observe that when the resulting logarithm is the log of a quotient, 10 must be *added* to the index; when it is the log. of a product, 10 must be *subtracted* from the index. Thus when the two angles are given,

Log. cos hypothen. = log. cos one angle + log. cos other angle — 10:
Log. cos either leg = log. cos opp. angle — log. sin adjac. angle + 10.

In a quadrantal triangle, if the quadrantal side be called radius, the supplement of the angle opposite to that side be called hypothenuse, the other sides be called angles, and their opposite angles be called legs: then the solutions of all the cases will be as in this table; merely changing *like* for *unlike* in the determinations.

TABLE II.—*For the Solution of Oblique-Angled Spherical Triangles.*

An angle or a side being divided by a perpendicular, the first and second segments are denoted by 1 seg. and 2 seg.

Given.	Required.		Values of the Quantities required.
I. Two angles and a side opposite to one of them.	The side opp. to other angle.	By the common analogy.	Sines of angles are as sines of oppos. sides.
	Third side.	Let fall a per. on the side contained between the given angles.	Tan 1 seg. of this side = cos adj. angle × tan given side. Sin 2 seg. = $\frac{\text{sin 1 seg.} \times \text{tan ang. adj. given side}}{\text{tan ang. opp. given side}}$.
	Third angle.	Let fall a per. as before.	Cot 1 seg. of this ang. = cos giv. side × tan. adj. angle. Sin 2 seg. = $\frac{\text{sin 1 seg.} \times \text{cos. ang. opp. given side}}{\text{cos. ang. adj. given side}}$.
II. Two sides and an angle opposite to one of them.	The angle opp. to the other side.	By the common analogy.	Sines of sides are as sines of their opposite angles.
	Angle included between the given sides.	Let fall a perpendicular from the included angle.	Cot 1 seg. ang. req. = tan given ang. × cos adj. side. Cos 2 seg = $\frac{\text{cos 1 seg.} \times \text{tan giv. side adj. giv. angle}}{\text{tan side opp. given angle}}$.
	Third side.	Let fall a perpendicular as before.	Tan 1 seg. side req. = cos given ang. × tan adj. side. Cos 2 seg. = $\frac{\text{cos 1 seg.} \times \text{cos side opp. given angle}}{\text{cos side adj. given angle}}$.

III. Two sides and the included angle.	An angle oppos. to one of the given sides.	Let fall a perpen. from the third angle.	Tan 1 seg. of div. side = cos. giv. ang. × tan side opp. ang. sought. Tan ang. sought $= \frac{\text{tan giv. ang} \times \text{sin 1 seg.}}{\text{sin 2 seg. of div. side}}$.
	Third side.	Let fall a perpen. on one of the giv. sides.	Tan 1 seg. of div. side = cos giv. ang. × tan other given side. Cos side sought $= \frac{\text{cos side not div.} \times \text{cos 2 seg.}}{\text{cos 1 seg of side divided}}$.
IV. A side and the two adjacent angles.	A side opposite to one of the given angles.	Let fall a perpendicular on the third side.	Cot 1 seg. of div. ang. = cos giv. side × tan ang. opp. side sought. Tan side sought $= \frac{\text{tan giv. side} \times \text{cos 1 seg. div. ang.}}{\text{cos 2 seg. of divided angle}}$.
	Third angle.	Let fall a perpen. from one of the giv. angles.	Cot 1 seg. div. ang. = cos giv. side × tan other giv. angle. Cos angle sought $= \frac{\text{cos ang. not div.} \times \text{sin 2 seg.}}{\text{sin 1 seg. div. angle}}$.
V. The three sides.	An angle by the sine or cosine of its half.	Let a, b, c, be the sides; A, B, C, the angles, b and c including the angle sought, and $s = a + b + c$. Then, $\sin \frac{1}{2}A = \sqrt{\frac{\sin(\frac{1}{2}s - b) \cdot \sin(\frac{1}{2}s - c)}{\sin b \cdot \sin c}}$ .... $\cos \frac{1}{2}A = \sqrt{\frac{\sin \frac{1}{2}s \cdot \sin(\frac{1}{2}s - a)}{\sin b \cdot \sin c}}$.	
VI. The three angles.	A side by the sine or cosine of its half.	Let S be the sum of the angles A, B, and C; and let B and C be adjacent to a the side required. Then, $\sin \frac{1}{2}a = \sqrt{-\frac{\cos \frac{1}{2}S \cdot \cos(\frac{1}{2}S - A)}{\sin B \cdot \sin C}}$ ...... $\cos \frac{1}{2}a = \sqrt{-\frac{\sin(\frac{1}{2}S - B) \cdot \sin(\frac{1}{2}S - C)}{\sin B \cdot \sin C}}$.	

TABLE III.

For the Solution of all the Cases of Oblique-Angled Spherical Triangles, by the Analogies of Napier.

Given.	Required.	Values of the terms required.
I. Two angles and one of their opposite sides.	Side opp. to the other given angle.	By the common analogy, sines of angles as sines of opp. sides.
	Third side.	Tan of its half $= \frac{\text{tan } \frac{1}{2} \text{ diff. giv. sides} \times \text{sin } \frac{1}{2} \text{ sum opp. angles}}{\text{sin } \frac{1}{2} \text{ diff. of those angles}} = \frac{\text{tan } \frac{1}{2} \text{ sum giv. sides} \times \text{cos } \frac{1}{2} \text{ sum opp. angles}}{\text{cos } \frac{1}{2} \text{ diff. of those angles}}$.
	Third angle.	By the common analogy.
II. Two sides, and an opposite angle.	Angle opposite to the other known side.	By the common analogy.
	Third angle.	Cot of its half $= \frac{\text{tan } \frac{1}{2} \text{ diff. other two ang.} \times \text{sin } \frac{1}{2} \text{ sum giv. sides}}{\text{sin } \frac{1}{2} \text{ diff. those sides}} = \frac{\text{tan } \frac{1}{2} \text{ sum of other two ang.} \times \text{cos } \frac{1}{2} \text{ sum giv. sides}}{\text{cos } \frac{1}{2} \text{ diff. of those sides}}$.
	Third side.	By the common analogy.

III. Two sides, and the included angle.	The other two angles.	Tan $\frac{1}{2}$ their diff. $= \frac{\text{cot } \frac{1}{2} \text{ giv. ang.} \times \text{sin } \frac{1}{2} \text{ diff. giv. sides}}{\text{sin } \frac{1}{2} \text{ sum of those sides}}$. Tan $\frac{1}{2}$ their sum $= \frac{\text{cot } \frac{1}{2} \text{ giv. ang.} \times \text{cos } \frac{1}{2} \text{ diff. giv. sides}}{\text{cos } \frac{1}{2} \text{ sum of those sides}}$.
	Third side.	By the common analogy.
IV. Two angles, and the side between them.	The other two sides.	Tan $\frac{1}{2}$ their diff. $= \frac{\text{tan } \frac{1}{2} \text{ giv. side} \times \text{sin } \frac{1}{2} \text{ diff. giv. angles}}{\text{sin } \frac{1}{2} \text{ sum of those angles}}$. Tan $\frac{1}{2}$ their sum $= \frac{\text{tan } \frac{1}{2} \text{ giv. side} \times \text{cos } \frac{1}{2} \text{ diff. giv. angles}}{\text{cos } \frac{1}{2} \text{ sum of those angles}}$.
	Third angle.	By the common analogy.
V. The three sides.	Either of the angles.	Let fall a perpen. on the side adjacent to the angle sought. Tan $\frac{1}{2}$ sum or $\frac{1}{2}$ diff. of the seg. of the base $= \frac{\text{tan } \frac{1}{2} \text{ sum} \times \text{tan } \frac{1}{2} \text{ diff. of the sides}}{\text{tan } \frac{1}{2} \text{ base}}$. Cos angle sought = tan adj. seg. $\times$ cot adja. side.
VI. The three angles.	Either of the sides.	Will be obtained by finding its corresponding angle, in a triangle which has all its parts supplemental to those of the triangle whose three angles are given.

Questions for Exercise in Spherical Trigonometry.

Ex. 1. In the right-angled spherical triangle BAC, right-angled at A, the hypothenuse $a = 78°20'$, and one leg $c = 76°52'$, are given; to find the angles B, and C, and the other leg b.

$$\text{Here, by table I case 1, } \sin C = \frac{\sin c}{\sin a};$$

$$\cos B = \frac{\tan c}{\tan a}; \quad . \quad . \quad . \quad \cos b = \frac{\cos a}{\cos c}.$$

$$\text{Or, } \log \sin C = \log \sin c - \log \sin a + 10.$$
$$\log \cos B = \log \tan c - \log \tan a + 10.$$
$$\log \cos b = \log \cos a - \log \cos c + 10.$$

Hence,	$10 + \log \sin c =$	$10 + \log \sin 76°52' =$	19·9884894
	$\log \sin a =$	$\log \sin 78°20' =$	9·9909338
Remains,	$\log \sin C =$	$\log \sin 83°56' =$	9·9975556

Here C is acute, because the given leg is less than 90°.

Again,	$10 + \log \tan c =$	$10 + \log \tan 76°52' =$	20·6320468
	$\log \tan a =$	$\log \tan 78°20' =$	10·6851149
Remains,	$\log \cos B =$	$\log \cos 27°45' =$	9·9469319

B is here acute, because a and c are of like affection.

Lastly,	$10 + \log \cos a =$	$10 + \log \cos 78°20' =$	19·3058189
	$\log \cos c =$	$\log \cos 76°52' =$	9·3564426
Remains,	$\log \cos b =$	$\log \cos 27°\ 8' =$	9·9493763

where b is less than 90°, because a and c both are so.

Ex. 2. In a right-angled spherical triangle, denoted as above, are given $a = 78°20'$, B $= 27°45'$; to find the other sides and angle.

Ans. $b = 27°8'$, $c = 76°52'$, C $= 83°56'$.

Ex. 3. In a spherical triangle, with A a right angle, given $b = 117°34'$, C $= 31°51'$; to find the other parts.

Ans. $a = 113°55'$, $c = 28°51'$, B $= 104°8'$.

Ex. 4. Given $b = 27°6'$, $c = 76°52'$; to find the other parts. Ans. $a = 78°20'$, B $= 27°45'$, C $= 83°56'$.

Ex. 5. Given $b = 42°12'$, B $= 48°$; to find the other parts.

Ans. $a = 64°40'\frac{1}{2}$, or its supplement,
$c = 54°44'$, or its supplement,
C $= 64°35'$, or its supplement.

Ex. 6. Given B $= 48°$, C $= 64°35'$; required the other parts? Ans. $b = 42°12'$, $c = 54°44'$, $a = 64°40'\frac{1}{2}$.

Ex. 7. In the quadrantal triangle ABC, given the quadrantal side $a = 90°$, an adjacent angle C $= 42°12'$, and the opposite angle A $= 64°40'$; required the other parts of the triangle?

Ex. 8. In an oblique-angled spherical triangle are given the three sides, viz. $a = 56°40'$, $b = 83°13'$, $c = 114°30'$; to find the angles.

Here, by the fifth case of table 2, we have

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin(\frac{1}{2}s - b) \cdot \sin(\frac{1}{2}s - c)}{\sin b \cdot \sin c}}.$$

Or, 2 log sin $\frac{1}{2}$A = log sin$(\frac{1}{2}s - b)$ + log sin $(\frac{1}{2}s - c)$ + ar. comp. log sin b + ar. comp. log. sin c : where $s = a + b + c$.

log sin $(\frac{1}{2}s - b)$ =	log sin	$43°58'\frac{1}{2}$	= 9·8415749
log sin $(\frac{1}{2}s - c)$ =	log sin	$12°41'\frac{1}{2}$	= 9·3418385
A . C . log sin b =	A . C . log sin	$83°13'$	= 0·0030508
A . C . log sin c =	A . C . log sin	$114°30'$	= 0·0409771
	Sum of the four logs		19·2274413
Half sum = log sin $\frac{1}{2}$A =	log sin	$24°15'\frac{1}{2}$	= 9·6137206

Consequently the angle A is $48°31'$.

Then, by common analogy,

As, sin a . . .	sin	$56°40'$ . . .	log = 9·9219401
To, sin A . . .	sin	$48°31'$ . . .	log = 9·8745679
So is, sin b . . .	sin	$83°13'$ . . .	log = 9·9969492
To, sin B . . .	sin	$62°56'$ . . .	log = 9·9495770
And so is, sin c . . .	sin	$114°30'$ . . .	log = 9·9590229
To, sin C . . .	sin	$125°19'$ . . .	log = 9·9116507

So that the remaining angles are, B $= 62°56'$, and C $= 125°19'$.

2dly. By way of comparison of methods, let us find the angle A, by the analogies of Napier, according to case 5 table 3. In order to which, suppose a perpendicular demitted from the angle C on the opposite side c. Then shall we have tan $\frac{1}{2}$ diff. seg of $c = \dfrac{\tan\frac{1}{2}(b+a) \cdot \tan\frac{1}{2}(b-a)}{\tan\frac{1}{2}c}$.

This, in logarithms, is

log tan $\frac{1}{2}(b+a)$ =	log tan $69°56'\frac{1}{2}$	= 10·4375601
log tan $\frac{1}{2}(b-a)$ =	log tan $13°16'\frac{1}{2}$	= 9·3727819
	Their sum	= 19·8103420
Subtract log tan $\frac{1}{2}c$ =	log tan $57°15'$	= 10·1916394
Rem. log cos dif. seg =	log cos $22°34'$	= 9·6187026

Hence, the segments of the base are $79°49'$ and $34°41'$.

Therefore, since $\cos A = \tan 79°49' \times \cot b$:

$$\begin{aligned}
&\text{To log tan adja. seg.} = \log \tan 79°49' = 10·7456257 \\
&\text{Add log tan side } b = \log \tan 83°13' = 9·0753563 \\
&\left.\begin{array}{l}\text{The sum, rejecting 10 from the index} \\ = \log \cos A = \log \cos 48°32'\end{array}\right\} = \overline{\underline{9·8209820}}
\end{aligned}$$

The other two angles may be found as before. The preference is, in this case, manifestly due to the former method.

Ex. 9. In an oblique-angled spherical triangle, are given two sides, equal to 114°30′ and 56°40′ respectively, and the angle opposite the former equal to 125°20′; to find the other parts. Ans. Angles 48°30′ and 62°55′; side, 83°12′.

Ex. 10. Given, in a spherical triangle, two angles, equal to 48°30′ and 125°20′, and the side opposite the latter; to find the other parts.

Ans. Side opposite first angle, 56°40′; other side, 83°12′; third angle, 62°54′.

Ex. 11. Given two sides, equal 114°30′ and 56°40′; and their included angle 62°54: to find the rest.

Ex. 12. Given two angles, 125°20′ and 48°30′, and the side comprehended between them 83°12′: to find the other parts.

Ex. 13. In a spherical triangle, the angles are 48°31′, 62°56′, and 125°20′; required the sides?

Ex. 14. Given two angles, 50°12′, and 58°8′; and a side opposite the former, 62°42′; to find the other parts.

Ans. The third angle is either 130°54′33″ or 156°16′32″.
Side betw. giv. angles, either 119°3′32″ or 152°14′14″.
Side opp. 58°8′, either 72°12′13″ or 100°47′37″.

Ex. 15. The excess of the three angles of a triangle, measured on the earth's surface, above two right angles, is 1 second; what is its area, taking the earth's diameter at 7957¾ miles?

Ans. 76·75299, or nearly 76¾ square miles.

Ex. 16. Determine the solid angles of a regular pyramid with hexagonal base, the altitude of the pyramid being to each side of the base, as 2 to 1.

Ans. Plane angle between each two lateral faces 125°22′35″.
between the base and each face 66°35′12″.

Solid angle at the vertex	89·60648	The max. angle
Each ditto at the base	218·19367	being 1000.

ON GEODESIC OPERATIONS, AND THE FIGURE OF THE EARTH.

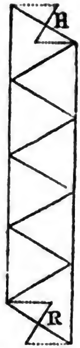
SECTION I.

General Account of this kind of Surveying.

Art. 1. In the treatise on Land Surveying in the first volume of this Course of Mathematics, the directions were restricted to the necessary operations for surveying fields, farms, lordships, or at most counties ; these being the only operations in which the generality of persons, who practise this kind of measurement, are likely to be engaged : but there are especial occasions when it is requisite to apply the principles of plane and spherical geometry, and the practices of surveying, to much more extensive portions of the earth's surface ; and when of course much care and judgment are called into exercise, both with regard to the direction of the practical operations, and the management of the computations. The extensive processes which we are now about to consider, and which are characterised by the terms *Geodesic Operations* and *Trigonometrical Surveying*, are usually undertaken for the accomplishment of one of these three objects. 1. The finding the difference of longitude, between two moderately distant and noted meridians ; as the meridians of the observatories at Greenwich and Oxford, or of those at Greenwich and Paris. 2. The accurate determination of the geographical positions of the principal places, whether on the coast or inland, in an island or kingdom ; with a view to give greater accuracy to maps, and to accommodate the navigator with the actual position, as to latitude and longitude, of the principal promontories, havens, and ports. These have, till lately, been desiderata, even in this country : the position of some important points, as the Lizard, not being known within seven minutes of a degree ; and, until the publication of the Board of Ordnance maps, the best county maps being so erroneous, as in some cases to exhibit *blunders of three miles in distances of less than twenty*. 3. The measurement of a degree in various situations ; and thence the determination of the figure and magnitude of the earth.

When objects so important as these are to be attained, it is manifest that, in order to ensure the desirable degree of correctness in the results, the instruments employed, the operations performed, and the computations required, must each have the greatest possible degree of accuracy. Of these, he first depend on the artist; the second on the surveyor, or engineer, who conducts them; and the latter on the theorist and calculator: they are these last which will chiefly engage our attention in the present chapter.

2. In the determination of distances of many miles, whether for the survey of a kingdom, or for the measurement of a degree, the whole line intervening between two extreme points is not *absolutely measured;* for this, on account of the inequalities of the earth's surface, would be always very difficult, and often impossible. But, a line of a few miles in length is very carefully measured on some plain, heath, or marsh, which is so nearly level as to facilitate the measurement of an actually horizontal line; and this line being assumed as the base of the operations, a variety of hills and elevated spots are selected, at which signals can be placed, suitably distant and visible one from another: the straight lines joining these points constitute a double series of triangles, of which the assumed base forms the first side; the angles of these, that is, the angles made at each station or signal staff, by two other signal staffs, are carefully measured by a theodolite, which is carried successively from one station to another. In such a series of triangles, care being always taken that one side is common to two of them, all the angles are known from the observations at the several stations; and a side of one of them being given, namely, that of the base measured, the sides of all the rest, as well as the distance from the first angle of the first triangle, to any part of the last triangle, may be found by the rules of trigonometry. And so, again, the bearing of any one of the sides, with respect to the meridian, being determined by observation, the bearings of any of the rest, with respect to the same meridian, will be known by computation. In these operations, it is always advisable, when circumstances will admit of it, to measure another base (called a base of verification) at or near the ulterior extremity of the series: for the length of this base, *computed* as one of the sides of the chain of triangles, compared with its length determined by *actual admeasurement*, will be a test of the accuracy of all the operations made in the series between the two bases.



3. Now, in every series of triangles, where such angle is to be ascertained with the same instrument, they should, as nearly as circumstances will permit, be equilateral. For, if it were possible to choose the stations in such manner, that each angle should be exactly 60 degrees; then, the half number of triangles in the series, multiplied into the length of one side of either triangle, would, as in the annexed figure, give at once the total distance; and then also, not only the sides of the scale or ladder, constituted by this series of triangles, would be perfectly parallel, but the diagonal steps, marking the progress from one extremity to the other, would be alternately parallel throughout the whole length. Here too, the first side might be found by a base crossing it perpendicularly of about half its length, as at B; and the last side verified by another such base, R, at the opposite extremity. If the respective sides of the series of triangles were 12 or 18 miles, these bases might advantageously be between 6 and 7, or between 9 and 10 miles respectively; according to circumstances. It may also be remarked (and the reason of it will be seen in the next section), that whenever only two angles of a triangle can be actually observed, each of them should be as nearly as possible 45°, or the sum of them about 90°: for the less the third computed angle differs from 90°, the less probability there will be of any considerable error. See prob. 1. sect. 2, of this chapter.

4. The student may obtain a general notion of the method employed in measuring an arc of the meridian, from the following brief sketch and introductory illustrations.

The earth, it is well known, is nearly spherical. It may be either an ellipsoid of revolution, that is, a body formed by the rotation of an ellipse, the ratio of whose axes is nearly that of equality, on one of those axes; or it may approach nearly to the form of such an ellipsoid or spheroid, while its deviations from that form, though small *relatively*, may still be sufficiently great in themselves, to prevent its being called a spheroid with much more propriety than it is called a sphere. One of the methods made use of to determine this point, is by means of extensive Geodesic operations.

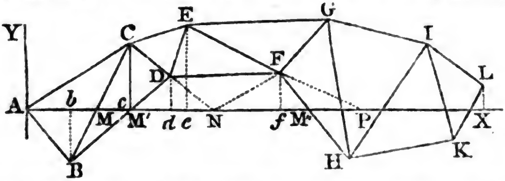
The earth, however, be its exact form what it may, is a planet, which not only revolves in an orbit, but turns upon an axis. Now, if we conceive a plane to pass through the axis of rotation of the earth, and through the zenith of any place on its surface, this plane, if prolonged to the limits of the apparent celestial sphere, would there trace the circum-

ference of a great circle, which would be the *meridian* of that place. All the points of the earth's surface, which have their zenith in that circumference, will be under the same celestial meridian, and will form the corresponding *terrestrial meridian*. If the earth be an irregular spheroid, this meridian will be a curve of double curvature; but if the earth be a solid of revolution, the terrestrial meridian will be a plane curve.

5. If the earth were a sphere, then every point upon a terrestrial meridian would be at an equal distance from the centre, and of consequence every degree upon that meridian would be of equal length. But if the earth be an ellipsoid of revolution slightly flattened at its poles, and protuberant at the equator; then, as will be shown soon, the degrees of the terrestrial meridian, in receding from the equator towards the poles, will be increased in the duplicate ratio of the right sine of the latitude; and the ratio of the earth's axes, as well as their actual magnitude, may be ascertained by comparing the lengths of a degree on the meridian in different latitudes. Hence appears the great importance of measuring a degree.

6. Now, instead of actually tracing a meridian on the surface of the earth,—a measure which is prevented by the interposition of mountains, woods, rivers, and seas,—a construction is employed which furnishes the same result. It consists in this.

Let ABCDEF, &c. be a series of triangles, carried on, as nearly as may be, in the direction of the meridian, according



to the observations in art. 3. These triangles are really *spherical* or spheroidal triangles; but as their curvature is extremely small, they are treated the same as rectilinear triangles, either by reducing them to the *chords* of the respective terrestrial arcs AC, AB, BC, &c. or by deducting *a third* of the excess, of the sum of the three angles of each triangle above two right angles, from each angle of that triangle, and working with the remainders, and the three sides, as the dimensions of a plane triangle; the proper reductions to the centre of the station, to the horizon, and to the level of the sea, having been previously made. These computations being made throughout the series, the sides of the successive triangles are contemplated as arcs of the terrestrial spheroid. Suppose

that we know, by observation, and the computations which will be explained in this chapter, the *azimuth*, or the inclination of the side AC to the first portion AM of the measured meridian, and that we find, by trigonometry, the point M where that curve will cut the side BC. The points A, B, C, being in the same horizontal plane, the line AM will also be in that plane: but, because of the curvature of the earth, the prolongation MM′ of that line, will be found *above* the plane of the second horizontal triangle BCD: if, therefore, without changing the angle CMM′, the line MM′ be brought down to coincide with the plane of this second triangle, by being turned about BC as an axis, the point M′ will describe an arc of a circle, which will be so very small, that it may be regarded as a right line perpendicular to the plane BCD: whence it follows, that the operation is reduced to bending down the side MM′ in the plane of the meridian, and calculating the distance AMM′, to find the position of the point M′. By bending down thus in imagination, one after another, the parts of the meridian on the corresponding horizontal triangles, we may obtain, by the aid of the computation, the direction and the length of such meridian, from one extremity of the series of triangles, to the other.

A line traced in the manner we have now been describing, or deduced from trigonometrical measures, by the means we have indicated, is called a *geodetic* or *geodesic line:* it has the property of being the shortest which can be drawn between its two extremities on the surface of the earth; and it is therefore the proper itinerary measure of the distance between those two points. Speaking rigorously, this curve differs a *little* from the terrestrial meridian, when the earth is not a solid of revolution: yet, in the real state of things, the difference between the two curves is so extremely minute, that it may safely be disregarded.

7. If now we conceive a circle perpendicular to the celestial meridian, and passing through the vertical of the place of the observer, it will represent the prime vertical of that place. The series of all the points of the earth's surface which have their zenith in the circumference of this circle will form the *perpendicular* to the meridian, which may be traced in like manner as the meridian itself.

In the sphere the perpendiculars to the meridian are great circles which all intersect mutually, on the equator, in two points diametrically opposite: but in the ellipsoid of revolution, and *a fortiori* in the irregular spheroid, these concurring perpendiculars are curves of double curvature. Whatever be the nature of the terrestrial spheroid, the parallels to the equator are curves of which all the points are at the same

latitude : on an ellipsoid of revolution, these curves are plane and circular.

8. The situation of a place is determined, when we know either the individual perpendicular to the meridian, or the individual parallel to the equator, on which it is found, and its position on such perpendicular, or on such parallel. Therefore, when all the triangles, which constitute such a series as we have spoken of, have been computed, according to the principles just sketched, the respective positions of their angular points, either by means of their longitudes and altitudes, or of their distances from the first meridian, and from the perpendicular to it. The following is the method of computing these distances.

Suppose that the triangles ABC, BCD, &c. (see the fig. to art. 6) make part of a chain of triangles, of which the sides are arcs of great circles of a sphere, whose radius is the distance from the level or surface of the sea to the centre of the earth ; and that we know by observation the angle CAX, which measures the *azimuth* of the side AC, or its inclination to the meridian AX. Then, having found the excess E, of the three angles of the triangle ACc (cc being perpendicular to the meridian) above two right angles, by reason of a theorem which will be demonstrated in prob. 8 of this chapter, subtract a third of this excess from each angle of the triangle, and thus by means of the following proportions find Ac, and cc.

$$\sin(90^\circ - \tfrac{1}{3}E) : \cos(cAC - \tfrac{2}{3}E) :: AC : Ac;$$
$$\sin(90^\circ - \tfrac{1}{3}E) : \sin(cAC - \tfrac{1}{3}E) :: AC : cc.$$

The azimuth of AB is known immediately, because BAX = CAB—CAX ; and if the spherical excess proper to the triangle ABM′ be computed, we shall have

$$AM'B = 180^\circ - M'AB - ABM' + E.$$

To determine the sides AM′, BM′, a third of E must be deducted from each of the angles of the triangle ABM′; and then these proportions will obtain : viz.

$$\sin(180^\circ - M'AB - ABM' + \tfrac{2}{3}E) : \sin(ABM' - \tfrac{1}{3}E) :: AB : AM',$$
$$\sin(108^\circ - M'AB - ABM' + \tfrac{2}{3}E) : \sin(M'AB - \tfrac{1}{3}E) :: AB : BM'.$$

In each of the right-angled triangles AbB, M′dD, are known two angles and the hypothenuse, which is all that is necessary to determine the sides Ab, bB, and M′d, dD. Therefore the distances of the points B, D, from the meridian and from the perpendicular, are known.

9. Proceeding in the same manner with the triangle ACN, or M′DN, to obtain AN and DN, the prolongation of CD ; and then with the triangle DNF to find the side NF and the angles DNF, DFN, it will be easy to calculate the rectangular co-ordinates of the point F.

The distance f F and the angles DEN, NFf, being thus known, we shall have (th. 6 cor. 3. Geom.)

$$f\text{FP} = 180° - \text{EFD} - \text{DFN} - \text{NF}f.$$

So that, in the right-angled triangle fFP, two angles and one side are known; and therefore the appropriate spherical excess may be computed, and thence the angle FPf and the sides fP, FP. Resolving next the right-angled triangle eEP, we shall in like manner obtain the position of the point E, with respect to the meridian AX, and to its perpendicular AY; that is to say, the distances Ee, and Ae = AP — eP. And thus may the computist proceed through the whole of the series. It is requisite however, previous to these calculations, to draw, by any suitable scale, the chain of triangles observed, in order to see whether any of the subsidiary triangles ACN, NFP, &c, formed to facilitate the computation of the distances from the meridian, and from the perpendicular to it, are too obtuse or too acute.

Such, in few words, is the method to be followed, when we have principally in view the finding the length of the portion of the meridian comprised between any two points, as A and X. It is obvious that, in the course of the computations, the azimuths of a great number of the sides of triangles in the series is determined; it will be easy therefore to check and verify the work in its process, by comparing the azimuths found by observation, with those resulting from the calculations. The amplitude of the whole arc of the meridian measured, is found by ascertaining the *latitude* at each of its extremities; that is, commonly by finding the differences of the zenith distances of some known fixed star, at both those extremities.

10. Some mathematicians, employed in this kind of operations, have adopted different means from the above. They draw through the summits of all the triangles, parallels to the meridian and to its perpendicular; by these means, the sides of the triangles become the hypothenuses of right-angled triangles, which they compute in order, proceeding from some known azimuth, and without regarding the spherical excess, considering all the triangles of the chain as described on a plain surface. This method, however, is manifestly defective in point of accuracy.

Others have computed the sides and angles of all the triangles, by the rules of spherical trigonometry. Others, again, reduce the observed angles to angles of the chords of the respective arches; and calculate by plane trigonometry, from such reduced angles and their chords. Either of these two methods is equally correct as that by means of the spherical excess: so that the principal reason for preferring one of these to the other must be derived from its relative facility.

As to the methods in which the several triangles are contemplated as spheroidal, they are abstruse and difficult, and may, happily, be safely disregarded: for M. Legendre has demonstrated in *Mémoires de la Classe des Sciences Physiques et Mathématiques de l'Institut*, 1806, p. 130, that the difference between spherical and spheroidal, angles is less than *one sixtieth* of a second, in the greatest of the triangles which occurred in the late measurement of an arc of a meridian between the parallels of Dunkirk and Barcelona.

11. Trigonometrical surveys for the purpose of measuring a degree of a meridian in different latitudes, and thence inferring the figure of the earth, have been undertaken by different philosophers, under the patronage of different governments. As by M. Maupertuis, Clairaut, &c. in Lapland, 1736; by M. Bouguer and Condamine, at the equator, 1736—1743; by Cassini, in lat. 45°, 1739—40; by Boscovich and Lemaire, lat. 43°, 1752; by Beccaria, lat. 44°44', 1768; by Mason and Dixon in America, 1764—8; by Colonel Lambton, in the East Indies, 1803; by Mechain, Delambre, &c. France, &c., 1790—1805; by Swanberg, Ofverbom, &c. in Lapland, 1802; and by General Roy, Colonel Williams, Mr. Dalby, General Mudge, and Colonel Colby, in England, from 1784 to the present time. The three last mentioned of these surveys are doubtless the most accurate and important.

The trigonometrical survey in England was *first* commenced, *in conjunction with similar operations in France*, in order to determine the difference of longitude between the meridians of the Greenwich and Paris observatories; for this purpose, three of the French Academicians, MM. Cassini, Mechain, and Legendre, met General Roy and Sir Charles Blagden, at Dover, to adjust their plans of operation. In the course of the survey, however, the English philosophers, selected from the Royal Artillery officers, expanded their views, and pursued their operations, under the patronage, and at the expense of the Honourable Board of Ordnance, in order to perfect the geography of England, and to determine the lengths of as many degrees on the meridian as fell within the compass of their labours.

12. It is not our province to enter into the history of these surveys: but it may be interesting and instructive to speak a little of the instruments employed, and of the extreme accuracy of some of the results obtained by them.

These instruments are, besides the signals, those for measuring distances, and those for measuring angles. The French philosophers used for the former purpose, in their measurement to determine the length of the *metre*, rulers of platina and of copper, forming metallic thermometers. The Swedish

mathematicians, Swanberg and Ofverbom, employed iron bars, covered towards each extremity with plates of silver. General Roy commenced his measurement of the base at Hounslow-Heath with *deal* rods, each of 20 feet in length. Though they, however, were made of the best seasoned timber, were perfectly straight, and were secured from bending in the most effectual manner; yet the changes in their lengths, occasioned by the variable moisture and dryness of the air, were so great, as to take away all confidence in the results deduced from them. Afterwards, in consequence of having found by experiments, that a solid bar of glass is more dilatable than a tube of the same matter, glass tubes were substituted for the deal rods. They were each 20 feet long, inclosed in wooden frames, so as to allow only of expansion or contraction in length, from heat or cold, according to a law ascertained by experiments. The base measured with these was found to be 27404·08, feet, or about 5·19 miles. Several years afterwards the same base was remeasured by General Mudge, with a steel-chain of 100 feet long, constructed by Ramsden, and jointed somewhat like a watch-chain. This chain was always stretched to the same tension, supported on troughs laid horizontally, and allowances were made for changes in its length by reason of variations of temperature, at the rate of ·0075 of an inch for each degree of heat from 62° of Fahrenheit: the result of the measurement by this chain was found not to differ more than $2\frac{3}{4}$ inches from General Roy's determination by means of the glass tubes: a minute difference in a distance of more than 5 miles; which, considering that the measurements were effected by different persons, and with different instruments, is a remarkable confirmation of the accuracy of both operations. And further, as steel chains can be used with more facility and convenience than glass rods, this remeasurement determines the question of the comparative fitness of these two kinds of instruments. Still greater improvements, however, in the construction of apparatus for the measurement of a base, are now ready for introduction into the survey, by its scientific and indefatigable conductor Colonel Colby.

13. For the determination of angles, the French and Swedish philosophers employed *repeating circles* of Borda's construction: instruments which are extremely portable, and with which, though they are not above 14 inches in diameter, the observers can take angles to within 1″ or 2″ of the truth. But this kind of instrument, however great its ingenuity in theory, has the accuracy of its observations necessarily limited by the imperfections of the *small* telescope which must be attached to it. Generals Roy and Mudge made use of a

very excellent theodolite constructed by Ramsden, which, having both an altitude and an azimuth circle, combines the powers of a theodolite, a quadrant, and a transit instrument, and is capable of measuring horizontal angles to fractions of a second. This instrument, besides, has a telescope of a much higher magnifying power than had ever before been applied to observations purely terrestrial; and this is one of the superiorities in its construction, to which is to be ascribed the extreme accuracy in the results of this trigonometrical survey.

Another circumstance which has augmented the accuracy of the English measures, arises from the mode of fixing and using this theodolite. In the method pursued by the Continental mathematicians, a reduction is necessary to the plane of the horizon, and another to bring the observed angles to the true angles at the centres of the signals: these reductions, of course, require formulæ of computation, the actual employment of which *may* lead to error. But, in the trigonometrical survey of England, great care has always been taken to place the centre of the theodolite exactly in the vertical line, previously or subsequently occupied by the centre of the signal: the theodolite is also placed in a perfectly horizontal position. Indeed, as was observed by professor Playfair, "In no other survey has the work in the field been conducted so much with a view to save that in the closet, and at the same time *to avoid all those causes of error, however minute, that are not essentially involved in the nature of the problem.* The French mathematicians trust to the *correction* of those errors; the English endeavour to *cut them off* entirely; and it can hardly be doubted that the latter, though perhaps the slower and more expensive, is by far the safest proceeding."

14. With a view to facilitate the observation of distant stations, many contrivances have been adopted; among which those recently (1826) invented by Lientenant Drummond, R. E. deserve peculiar notice: of these, one is applicable by day, the other by night. The first, which consists in employing the reflection of the sun from a plane mirror as a point of observation, was first suggested by Professor Gauss; and the result of the first trials made in the survey of Hanover proved very successful. Recourse was had to this method on some occasions that occurred in the Trigonometrical Survey of England, where, from peculiar local circumstances, much difficulty was experienced in discerning the usual signals.

Even as a temporary expedient, and under a rude form, viz. that of placing tin plates at the station to be observed in such a manner that the sun's reflection should be thrown

towards the observer at a particular time, the most essential service was derived from its use; and the consequence was, the invention of a more perfect instrument, of which a description is given, accompanied with a drawing.

The second method consists in the exhibition of a very brilliant light at night. At the commencement of the Survey of England, General Roy had recourse, on several occasions, and especially in carrying his triangles across the Channel, to the use of Bengal and white lights; for these, parabolic reflectors illuminated by Argand lamps were afterwards substituted as more convenient; but from want of power they appear in turn to have gradually fallen into disuse. With a view to remedy this defect, a series of experiments was undertaken by Lieutenant Drummond, the result of which was the production of a very intense light, varying between 60 and 90 times that of the brightest part of the flame of an Argand lamp.

This brilliant light is obtained from a small ball of lime about 3-8ths of an inch diameter, placed in the focus of the reflector, and exposed to a very intense heat by means of a simple apparatus, of which a description is given, in his account. A jet of oxygen gas directed through the flame of alcohol is employd as the source of heat. Zirconia, magnesia, and oxide of zinc were also tried; but the light emanating from them was much inferior to that from lime. Besides being easily procured, the lime admits of being turned in the lathe, so that any number of the small focal balls may be readily obtained, uniform in size, and perfect in figure. The chemical agency of this light is remarkable, causing the combination of chlorine and hydrogen, and blackening chloride of silver. Its application to the very important purpose of illuminating light-houses is suggested, especially in those situations where the lights are the first that are made by vessels arriving from distant voyages.

Both the methods now described, for accelerating geodesic operations, were resorted to with much success during the season of 1825 in Ireland; and on one occasion, where every attempt to discern a disant station had failed, the observations were effected by their means, the heliostat being seen during the day, when the outline of the hill ceased to be visible, and the light at night being seen with the naked eye, and appearing much brighter and larger at the distance of 66 miles, than a parabolic reflector, of equal size, illuminated by an Argand lamp, and placed nearly in the same direction, as an object of reference, at the distance of 15 miles.

15. In proof of the great correctness of the English sur-

vey, we shall state a very few particulars, besides what is already mentioned in art. 12.

General Roy, who first measured the base on Hounslow-Heath, measured another on the flat ground of Romney-Marsh in Kent, near the southern extremity of the first series of triangles, and at the distance of more than 60 miles from the first base. The length of this base of verification, as actually measured, compared with that resulting from the computation through the whole series of triangles, differed only by 28 inches.

General Mudge measured another base of verification on Salisbury-Plain. Its length was 36574·4 feet, or more than 7 miles; the measurement did not differ more than *one inch* from the computation carried through the series of triangles from Hounslow-Heath to Salisbury-Plain. A most remarkable proof of the accuracy with which all the angles, as well as the two bases, were measured!

The distance between Beachy-Head in Sussex, and Dunnose in the Isle of Wight, as deduced from a mean of four series of triangles, is 339397 feet, or more than $64\frac{1}{4}$ miles. The extremes of the four determinations do not differ more than 7 feet, which is less than $1\frac{2}{5}$ inches in a mile. Instances of this kind frequently occur in the English survey *. But we have not room to specify more. We must now proceed to discuss the most important problems connected with this subject; and refer those who are desirous to consider it more minutely to General Mudge and Colonel Colby's "Account of the Trigonometrical Survey;" Mechain and Delambre, "Base du Systême Métrique Décimal;" Swanberg, "Exposition des Opérations faites en Lapponie;" and Puissant's works entitled "Geodesie," and "Traite de Topographie, d'Arpentage, &c."

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* Puissant, in his "Geodésie," after quoting some of them, says, "Neanmoins, jusqu'à présent, rien n'égale en exactitude les opérations géodesiques qui ont servi de fondement à *notre* systême métrique." He, however, gives no instances. We have no wish to depreciate the labours of the French measures; but we cannot yield them the preference on mere assertion.

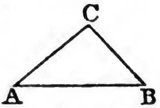
SECTION II.

Problems connected with the detail of Operations in Extensive Trigonometrical Surveys.

PROBLEM I.

It is required to determine the most advantageous conditions of triangles.

1. In any rectilinear triangle ABC, it is, from the proportionality of sides to the sines of their opposite angles, AB : BC : : sin C : sin A, and consequently AB . sin A = BC . sin C. Let AB be the base, which is supposed to be measured without perceptible error, and which therefore is assumed as constant; then finding the extremely small variation or fluxion of the equation on this hypothesis,



it is $\text{AB} \cdot \cos \text{A} \cdot \dot{\text{A}} = \sin \text{C} \cdot \dot{\text{BC}} + \text{BC} \cdot \cos \text{C} \cdot \dot{\text{C}}$. Here, since we are ignorant of the magnitude of the errors or variations expressed by $\dot{\text{A}}$ and $\dot{\text{C}}$, suppose them to be equal (a probable supposition, as they are both taken by the same instrument), and each denoted by v : then will

$$\dot{\text{BC}} = v \times \frac{\text{AB} \cos \text{A} - \text{BC} \cos \text{C}}{\sin \text{C}};$$

or, substituting $\frac{\text{BC}}{\sin \text{A}}$ for its equal $\frac{\text{AB}}{\sin \text{C}}$, the equation will become $\dot{\text{BC}} = v \times (\text{BC} \cdot \frac{\cos \text{A}}{\sin \text{A}} - \text{BC} \cdot \frac{\cos \text{C}}{\sin \text{C}})$;

or finally, $\dot{\text{BC}} = v \cdot \text{BC} (\cot \text{A} - \cot \text{C})$.

This equation (in the use of which it must be recollected that v taken in seconds should be divided by R″, that is, by the length of the radius expressed in seconds) gives the error $\dot{\text{BC}}$ in the estimation of BC occasioned by the errors in the angles A and C. Hence, that these errors, supposing them to be equal, may have no influence on the determination of BC, we must have A = C, for in that case the second member of the equation will vanish.

2. But, as the two errors, denoted by $\dot{\text{A}}$, and $\dot{\text{C}}$ which we have supposed to be of the same kind, or in the same direction, may be committed in different directions, when the equation will be $\dot{\text{BC}} = \pm\, v \cdot \text{BC} (\cot \text{A} + \cot \text{C})$; we must inquire what magnitude the angles A and C ought to have,

so that the sum of their cotangents shall have the least value possible; for in this state it is manifest that $\dot{\text{BC}}$ will have its least value. But, by the formulæ in chap. 3, we have

$$\cot A + \cot C = \frac{\sin (A + C)}{\sin A \cdot \sin C} = \frac{\sin (A + C)}{\frac{1}{2}\cos (A \backsim C) - \frac{1}{2}\cos (A + C)} = \frac{2 \sin B}{\cos (A \backsim C) + \cos B}.$$

Consequently, $\dot{\text{BC}} = \pm \dot{v} \cdot \text{BC} \cdot \frac{2 \sin B}{\cos (A \backsim C) + \cos B}$.

And hence, whatever be the magnitude of the angle B, the error in the value of BC will be the least when $\cos (A \backsim C)$ is the greatest possible, which is, when $A = C$.

We may therefore infer, for a general rule, that *the most advantageous state of a triangle, when we would determine one side only, is when the base is equal to the side sought.*

3. Since, by this rule, the base should be equal to the side sought, it is evident that *when we would determine two sides, the most advantageous condition of a triangle is that it be equilateral.*

4. It rarely happens, however, that a base can be commodiously measured which is as long as the sides sought. Supposing, therefore, that the length of the base is limited, but that its direction at least may be chosen at pleasure, we proceed to inquire what that direction should be, in the case where one only of the other two sides of the triangle is to be determined.

Let it be imagined, as before, that AB is the base of the triangle ABC, and BC the side required. It is proposed to find the least value of $\cot A \mp \cot C$, when we cannot have $A = C$.

Now, in the case where the negative sign obtains, we have

$$\cot A - \cot C = \frac{\text{AB} - \text{BC} \cdot \cos B}{\text{BC} \cdot \sin B} - \frac{\text{BC} - \text{AB} \cdot \cos B}{\text{AB} \cdot \sin B} = \frac{\text{AB}^2 - \text{BC}^2}{\text{AB} \cdot \text{BC} \cdot \sin B}.$$

This equation again manifestly indicates the equality of AB and BC, in circumstances where it is possible: but if AB and BC are constant, it is evident, from the form of the denominator of the last fraction, that the fraction itself will be the least, or $\cot A - \cot C$ the least, when $\sin B$ is a maximum, that is, when $B = 90°$.

5. When the positive sign obtains, we have $\cot A + \cot C =$

$$\cot A + \frac{\sqrt{(\text{BC}^2 - \text{AB}^2 \sin^2 A)}}{\text{AB} \sin A} = \cot A + \sqrt{\left(\frac{\text{BC}^2}{\text{AB}^2 \sin^2 A} - 1\right)}.$$

Here, the least value of the expression under the radical sign, is obviously when $A = 90°$. And in that case the first term, $\cot A$, would disappear. Therefore the least value of $\cot A + \cot C$, obtains when $A = 90°$; conformably to the rule given by M. Bouguer (*Fig. de la Terre*, p. 88). But we have

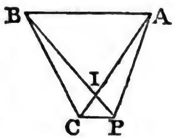
already seen that in the case of cot A—cot C, we must have B = 90 : whence we conclude, since the conditions A = 90°, B = 90°, cannot obtain simultaneously, that a medium result would give A = B.

If we apply to the side AC the same reasoning as to BC, similar results will be obtained : therefore in general, *when the base cannot be equal to one or to both the sides required, the most advantageous condition of the triangle is, that the base be the longest possible, and that the two angles at the base be equal.* These equal angles, however, should never, if possible, be less than 23 degrees.

PROBLEM II.

To deduce, from angles measured out of one of the stations, but near it, the true angles at the station.

When the centre of the instrument cannot be placed in the vertical line occupied by the axis of a signal, the angles observed must undergo a reduction, according to circumstances.



1. Let C be the centre of the station, P the place of the centre of the instrument, or the summit of the observed angle APB : it is required to find C, the measure of ACB, supposing there to be known APB = P, BPC = p, CP = d, BC = L, AC = R.

Since the exterior angle of a triangle is equal to the sum of the two interior opposite angles (th. 16 Geom.) we have, with respect to the triangle IAP, AIB = P + IAP ; and with regard to the triangle BIC, AIB = C + CBP. Making these two values of AIB equal, and transposing IAP, there results

$$C = P + IAP - CBP.$$

But the triangles CAP, CBP, give

$$\sin CAP = \sin IAP = \frac{CP}{AP} \sin APC = \frac{d \cdot \sin (P+p)}{R};$$

$$\sin CBP = \frac{CP}{BC} \cdot \sin BPC = \frac{d \cdot \sin p}{L}.$$

And, as the angles CAP, CBP, are, by the hypothesis of the problem, always very small, their sines may be substituted for their arcs or measures : therefore

$$C - P = \frac{d \cdot \sin (P+p)}{R} - \frac{d \cdot \sin p}{L}.$$

Or, to have the reduction in seconds,

$$C - P = \frac{d}{\sin 1''} \cdot \left(\frac{\sin (P+p)}{R} - \frac{\sin p}{L}\right).$$

The use of this formula cannot in any case be embarrassing, provided the signs of sin p, and sin $(P + p)$ be attended to. Thus, the first term of the correction will be positive, if the angle $(P + p)$ is comprised between 0 and 180° ; and it will become negative, if that angle surpass 180°. The contrary will obtain in the same circumstances with regard to the second term, which answers to the angle of direction p. The letter R denotes the distance of the object A to the right, L the distance of the object B situated to the left, and p the angle at the place of observation, between the centre of the station and the object to the left.

2. An approximate reduction to the centre may indeed be obtained by a single term; but it is not quite so correct as the form above. For, by reducing the two fractions in the second member of the last equation but one to a common denominator, the correction becomes

$$C - P = \frac{dL \cdot \sin(P+p) - dR \cdot \sin p}{LR}.$$

But the triangle ABC gives $L = \frac{R \cdot \sin A}{\sin B} = \frac{R \cdot \sin A}{\sin(A+C)}$.

And because P is always very nearly equal to C, the sine of $A + P$ will differ extremely little from sin $(A + C)$, and may therefore be substituted for it, making $L = \frac{R \sin A}{\sin(A+P)}$.

Hence we manifestly have

$$C - P = \frac{d \cdot \sin A \cdot \sin(P+p) - d \cdot \sin p \cdot \sin(A+P)}{R \cdot \sin A};$$

Which by taking the expanded expressions for sin $(P + p)$, and sin $(A + P)$, and reducing to seconds, gives

$$C - P = \frac{d}{\sin 1''} \cdot \frac{\sin P \cdot \sin(A-p)}{R \cdot \sin A}.$$

3. When either of the distances R, L, becomes infinite, with respect to d, the corresponding term in the expression art. 1 of this problem, vanishes, and we have accordingly

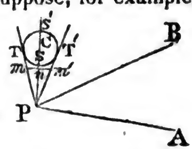
$$C - P = -\frac{d \cdot \sin p}{L \cdot \sin 1''}, \text{ or } C - P = \frac{d \cdot \sin(P+p)}{R \cdot \sin 1''}.$$

The first of these will apply when the object A is a heavenly body, the second when B is one. When both A and B are such, then $C - P = 0$.

But without supposing either A or B infinite, we may have $C - P = 0$, or $C = P$ in innumerable instances: that is, in all cases in which the centre P of the instrument is placed in the circumference of the circle that passes through the three points A, B, C ; or when the angle BPC is equal to the angle BAC, or to BAC + 180°. Whence though C should be inaccessible, the angle ACB may commonly be obtained by ob-

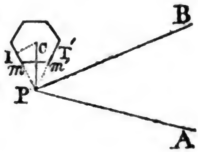
servation, without any computation. It may further be observed, that when P falls in the circumference of the circle passing through the three points A, B, C, the angles A, B, C, may be determined solely by measuring the angles APB and BPC. For the opposite angles ABC, APC, of the quadrangle inscribed in a circle, are (theor. 54 Geom.) $= 180°$. Consequently, ABC $= 180° -$ APC, and BAC $= 180° - ($ABC $+$ ACB$)$ $= 180° - ($ABC$+$APB$)$.

4. If one of the objects, viewed from a further station, be a vane or staff in the centre of a steeple, it will frequently happen that such object, when the observer comes near it, is both invisible and inaccessible. Still there are various methods of finding the exact angle at C. Suppose, for example, the signal-staff be in the centre of a circular tower, and that the angle APB was taken at P near its base. Let the tangents PT, PT′, be marked, and on them two equal and arbitrary distances Pm, Pm', be measured. Bisect mm' at the point n: and, placing there a signal staff, measure the angle nPB, which (since Pn prolonged obviously passes through C the centre) will be the angle p of the preceding investigation. Also, the distance Ps added to the radius Cs of the tower, will give PC $= d$ in the former investigation.



If the circumference of the tower cannot be measured, and the radius thence inferred, proceed thus: Measure the angles BPT, BPT′, then will BPC $= \frac{1}{2}($BPT $+$ BPT′$) = p$; and CPT $=$ BPT $-$ BPC : Measure PT, then PC $=$ PT . sec CPT$=d$. With the values of p and d, thus obtained, proceed as before.

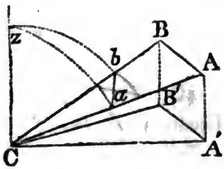
5. If the base of the tower be polygonal, and *regular*, as most commonly happens: assume P in the point of intersection of two of the sides prolonged, and BPC′$= \frac{1}{2}$ (BPT $+$ BPT′) as before, PT $=$ the distance from P to the middle of one of the sides whose prolongation passes through P; and hence PC is found, as above. If the figure be a regular hexagon, then the triangle Pmm' is equilateral, and PC$=$ $m'm\sqrt{3}$.



PROBLEM III.

To reduce angles measured in a plane inclined to the horizon, the corresponding angles in the horizontal plane.

Let BCA be an angle measured in a plane inclined to the horizon, and let B′CA′ be the corresponding angle in the horizontal plane. Let d and d' be the zenith distances, or the complements of the angles of elevation ACA′, BCB′. Then from z the zenith of the observer, or of the angle C, draw the arcs za zb, of vertical circles, measuring the zenith distances d, d', and draw the arc ab of another great circle to measure the angle C. It follows from this construction, that the angle z, of the spherical triangle zab, is equal to the horizontal angle A′C′B; and that, to find it, the three sides $za = d$, $zb = d'$, $ab =$ C, are given. Call the sum of these s; then the resulting formula of prob. 2, ch. iv, applied to the present instance, becomes



$$\sin \tfrac{1}{2}z = \sin \tfrac{1}{2}\text{C} = \sqrt{\frac{\sin \frac{1}{2}(s-d) \,.\, \sin \frac{1}{2}(s-d')}{\sin d \,.\, \sin d'}}.$$

If h and h' represent the angles of altitude ACA′, BCB′, the preceding expression will become

$$\sin \tfrac{1}{2}\text{C} = \sqrt{\frac{\sin \frac{1}{2}(\text{C}+h-h') \,.\, \sin \frac{1}{2}(\text{C}+h'-h)}{\cos h \,.\, \cos h'}}.$$

Or, in logarithms,

$$\log \sin \tfrac{1}{2}\text{C} = \tfrac{1}{2}(20 + \log \sin \tfrac{1}{2}(\text{C} + h - h') + \log \sin \tfrac{1}{2}(\text{C} + h' - h) - \log \cos h - \log \cos h'.$$

Cor. 1. If $h=h'$, then is $\sin \frac{1}{2}\text{C} = \frac{\sin \frac{1}{2}\text{ACB}}{\cos h}$; and

$$\log \sin \tfrac{1}{2}\text{A}'\text{CB}' = 10 + \log \sin \tfrac{1}{2}\text{ACB} - \log \cos h.$$

Cor. 2. If the angles h and h' be very small, and nearly equal; then, since the cosines of small angles vary extremely slowly, we may, without sensible error, take

$$\log \sin \tfrac{1}{2}\text{A}'\text{CB}' = 10 + \log \sin \tfrac{1}{2}\text{ACB} - \log \cos \tfrac{1}{2}(h+h').$$

Cor. 3. In this case the correction $x =$ A′CB′ — ACB, may be found by the expression

$$x = \sin 1'' \left(\tan \tfrac{1}{2}\text{C}(\tfrac{1}{4}\bigcirc - \tfrac{d+d'}{2})^2 - \cot \tfrac{1}{2}\text{C}\,(\tfrac{d-d'}{2})^2\right).$$

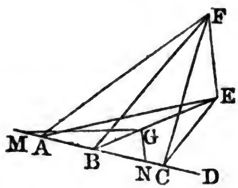
And in this formula, as well as the first given for $\frac{1}{2}$C, d and d' may be either one or both greater or less than a quadrant; that is, the equations will obtain whether ACA′ and BCB′ be each an elevation or a depression.

Scholium. By means of this problem, if the altitude of a hill be found barometrically, according to the method described in the 1st volume, or geometrically, according to some of those described in heights and distances, or that given in the following problem ; then, finding the angles formed at the place of observation, by any objects in the country below, and their respective angles of depression, their horizontal angles, and thence their distances, may be found, and their relative places fixed in a map of the country ; taking care to have a sufficient number of angles between intersecting lines, to verify the operations.

PROBLEM IV.

Given the angles of elevation of any distant object, taken at three places in a horizontal right line, which does not pass through the point directly below the object ; and the respective distances between the stations ; to find the height of the object, and its distance from either station.

Let AED be the horizontal plane : FE the perpendicular height of the object F above that plane ; A, B, C, the three places of observation ; FAE, FBE, FCE, the respective angles of elevation, and AB, BC, the given distances. Then, since the triangles AEF, BEF, CEF, are all right angled at E, the distances AE, BE, CE, will manifestly be as the cotangents of the angles of elevation at A, B, and C : and we have to determine the point E, so that those lines may have that ratio. To effect this geometrically, use the following



Construction. Take BM, on AC produced, equal to BC, BN equal to AB ; and make

MG : BM (= BC) :: cot A : cot B,

and BN (= AB) : NG :: cot B : cot C.

With the lines MN, MG, NG, constitute the triangle MNG ; and join BG. Draw AE so, that the angle EAB may be equal to MGB ; this line will meet BG produced in E, the point in the horizontal plane falling perpendicularly below F.

Demonstration. By the similar triangles AEB, GMB, we have AE : BE : : MG : MB : : cot A : cot B,

and BE : BA (= BN) : : BM : BG.

Therefore the triangles BEC, BGN, are similar; consequently BE : EC : : BN : NG : : cot B : cot C. Whence it is obvious that AE, BE, CE, are respectively as cot A, cot B, cot C.

Calculation. In the triangle MGN, all the sides are given, to find the angle GMN = angle AEB. Then, in the triangle MGB, two sides and the included angle are given, to find the angle MGB = angle EAB. Hence, in the triangle AEB, are known AB and all the angles, to find AE, and BE. And then EF = AE . tan A = BE . tan B.

Otherwise, independent of the construction, thus.

Put AB = D, BC = d, EF = x; and then express algebraically the follwing theorem, given at p. 128 Simpson's Select Exercises:

$$AE^2 . BC + CE^2 . AB = BE^2 . AC + AC . AB . BC,$$

the line EB being drawn from the vertex E of the triangle ACE, to any point B in the base. The equation thence originating is

$$dx^2 . \cot^2 A + Dx^2 . \cot^2 C = (D+d)x^2 . \cot^2 B + (D+d)Dd.$$

And from this, by transposing all the unknown terms to one side, and extracting the root, there results

$$x = \sqrt{\frac{(D+d)\,Dd}{d . \cot^2 A + D . \cot^2 C - (D+d)\cot^2 B}}.$$

Whence EF is known, and the distances AE, BE, CE, are readily found.

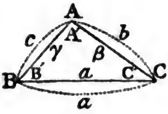
Cor. When D = d, or D + d = 2D = 2d, the expression becomes better suited for logarithmic computation, being then

$$x = d \div \sqrt{(\tfrac{1}{2}\cot^2 A + \tfrac{1}{2}\cot^2 C - \cot^2 B)}.$$

In this case, therefore, the rule is as follows: Double the log. cotangents of the angles of elevation of the extreme stations, find the natural numbers answering thereto, and take half their sum; from which subtract the natural number answering to twice the log. cotangent of the middle angle of elevation: then half the log. of this remainder subtracted from the log. of the measured distance between the 1st and 2d, or the 2d and 3d stations, will be the log. of the height of the object.

PROBLEM V.

In any spherical triangle, knowing two sides and the included angle; it is required to find the angle comprehended by the chords of those two sides.



Let the angles of the spherical triangle be A, B, C, the corresponding angles included by the chords A′, B′, C′; the spherical sides opposite the former a, b, c, the chords respectively opposite the latter, α, β, γ; then, there are given b, c, and A, to find A′.

Here, from prob. 1, equa. I, chap. iv, we have

$$\cos a = \sin b \,.\, \sin \,.\, c \,.\, \cos \mathrm{A} + \cos b \,.\, \cos c.$$

But $\cos c = \cos(\frac{1}{2}c + \frac{1}{2}c) = \cos^2 \frac{1}{2}c - \sin^2 \frac{1}{2}c$ (by equa. v, ch. iii) $= (1 - \sin^2 \frac{1}{2}c) - \sin^2 \frac{1}{2}c = 1 - 2\sin^2 \frac{1}{2}c$. And in like manner $\cos a = 1 - 2\sin^2 \frac{1}{2}a$, and $\cos b = 1 - 2\sin^2 \frac{1}{2}b$. Therefore the preceding equation becomes 1X,A

$$1 - 2\sin^2 \tfrac{1}{2}a = 4 \sin \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}b \,.\, \sin \tfrac{1}{2}c \,.\, \cos \tfrac{1}{2}c \,.\, \cos \mathrm{A} + (1 - 2\sin^2 \tfrac{1}{2}b) \,.\, (1 - 2\sin^2 \tfrac{1}{2}c).$$

But $\sin \frac{1}{2}a = \frac{1}{2}\alpha$, $\sin \frac{1}{2}b = \frac{1}{2}\beta$, $\sin \frac{1}{2}c = \frac{1}{2}\gamma$: which values substituted in the equation, we obtain, after a little reduction,

$$2 \times \frac{\beta^2 + \gamma^2 - \alpha^2}{4} = \beta\gamma \,.\, \cos \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}c \,.\, \cos \mathrm{A} + \tfrac{1}{4}\beta^2\gamma^2.$$ 11

Now, (equa. II, ch. iii), $\cos \mathrm{A}' = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma}$. Therefore, by substitution,

$$\beta\gamma \,.\, \cos \mathrm{A}' = \beta\gamma \,.\, \cos \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}c \,.\, \cos \mathrm{A} + \tfrac{1}{4}\beta^2\gamma^2;$$

whence, dividing by $\beta\gamma$, there results

$$\cos \mathrm{A}' = \cos \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}c \,.\, \cos \mathrm{A} + \tfrac{1}{2}\beta \,.\, \tfrac{1}{2}\gamma;$$

or, lastly, by restoring the values of $\frac{1}{2}\beta$, $\frac{1}{2}\gamma$, we have

$$\cos \mathrm{A}' = \cos \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}c \,.\, \cos \mathrm{A} + \sin \tfrac{1}{2}b \,.\, \sin \tfrac{1}{2}c \ldots \text{(I.)}$$

Cor. 1. It follows evidently from this formula, that when the spherical angle is right or obtuse, it is always *greater* than the corresponding angle of the chords.

Cor. 2. The spherical angle, if acute, is *less* than the corresponding angle of the chords, when we have cos A greater than $\frac{\sin \frac{1}{2}b \,.\, \sin \frac{1}{2}c}{1 - \cos \frac{1}{2}b \,.\, \cos \frac{1}{2}c}$.

PROBLEM VI.

Knowing two sides and the included angle of a rectilinear triangle, it is required to find the spherical angle of the two arcs of which those two sides are the chords.

Here β, γ, and the angle A′ are given, to find A. Now, since in all cases, $\cos = \sqrt{(1 - \sin^2)}$, we have

$$\cos \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}c = \sqrt{[(1 - \sin^2 \tfrac{1}{2}b) \,.\, (1 - \sin^2 \tfrac{1}{2}c)]};$$

we have also, as above, $\sin \frac{1}{2}b = \frac{1}{2}\beta$, and $\sin \frac{1}{2}c = \frac{1}{2}\gamma$. Substituting these values in the equation I of the preceding problem, there will result, by reduction,

$$\cos A = \frac{\cos A' - \frac{1}{4}\beta\gamma}{\sqrt{(1-\frac{1}{2}\beta)\cdot(1+\frac{1}{2}\beta)\cdot(1-\frac{1}{2}\gamma)\cdot(1+\frac{1}{2}\gamma)}} \ldots . \text{(II.)}$$

To compute by this formula, the values of the sides β, γ, must be reduced to the corresponding values of the chords of a circle whose radius is unity. This is easily effected by dividing the values of the sides given in feet, or toises, &c. by such a power of 10, that neither of the sides shall exceed 2, the value of the greatest chord, when radius is equal to unity.

From this investigation, and that of the preceding problem, the following corollaries may be drawn.

Cor. 1. If $c = b$, and of consequence $\gamma = \beta$, then will

$\cos A' = \cos A \cdot \cos^2 \frac{1}{2}c + \sin^2 \frac{1}{2}c$; and thence

$1 - 2\sin^2 \frac{1}{2}A' = (1 - 2\sin^2 \frac{1}{2}A)\cos^2 \frac{1}{2}c + (1 - \cos^2 \frac{1}{2}c)$;

from which may be deduced

$$\sin \tfrac{1}{2}A' = \sin \tfrac{1}{2}A \cdot \cos \tfrac{1}{2}c. \ldots \text{(III)}.$$

Cor. 2 Also, since $\cos \frac{1}{2}c = \sqrt{(1 - \sin^2 \frac{1}{2}c)} = \sqrt{(1 - \frac{1}{4}\gamma^2)}$, equa. II, will, in this case, reduce to

$$\sin \tfrac{1}{2}A = \frac{\sin \frac{1}{2}A'}{\sqrt{(1-\frac{1}{2}\gamma)\cdot(1+\frac{1}{2}\gamma)}} \ldots \text{(IV.)}$$

Cor. 3. From the equation III, it appears that the vertical angle of an isosceles spherical triangle is always *greater* than the corresponding angle of the chords.

Cor. 4. If $A = 90°$, the formulæ I, II, give

$$\cos A' = \sin \tfrac{1}{2}b \cdot \sin \tfrac{1}{2}c = \tfrac{1}{4}\beta\gamma. \ldots \text{(V.)}$$

These five formulæ are strict and rigorous, whatever be the magnitude of the triangle. But if the triangles be small, the arcs may be put instead of the sines in equa. V, then

Cor. 5. As $\cos A' = \sin(90° - A') =$ in this case, $90° A'$; the small excess of the spherical right angle over the corresponding rectilinear angle, will, supposing the arcs b, c, taken in seconds, be given in seconds by the following expression,

$$90° - A' = \frac{\frac{1}{4}bc}{R''} = \frac{bc}{4R''} \ldots \text{(VI.)}$$

The error in this formula will not amount to a second, when $b+c$ is less than 10°, or than 700 miles measured on the earth's surface.

Cor. 6. If the hypothenuse does not exceed $1\frac{1}{2}°$, we may substitute $a \sin C$ instead of c, and $a \cos C$ instead of b; this will give $bc = a^2 \cdot \sin C \cdot \cos C = \frac{1}{2}a^2 \cdot \sin 2(90° - B) = \frac{1}{2}a^2 \cdot \sin 2B$: whence

$$(90° - A') = \frac{a^2 \cdot \sin 2C}{8R''} = \frac{a^2 \cdot \sin 2B}{8R''} \ldots \text{(VII.)}$$

If $a = 1\frac{1}{2}°$, and $B = C = 45°$ nearly; then will $90° - A' = 17''7$.

Cor. 7. Retaining the same hypothesis of $A = 90°$, and $a =$ or $< 1\frac{1}{2}°$, we have

$$B - B' = \frac{b^2 \cdot \cot B}{8R''} = \frac{bc}{8R''} \ldots \text{(VIII.)}$$

Also $c - c' = \frac{bc}{8R''}$. . . . . (IX.)

Cor. 8. Comparing formulæ VIII, IX, with VI, we have $B - B' = C - C' = \frac{1}{2}(90^\circ - A')$. Whence it appears that the sum of the two excesses of the oblique spherical angles, over the corresponding angles of the chords, in a small right-angled triangle, is equal to the excess of the right angle over the corresponding angle of the chords. So that either of the formulæ VI, VII, VIII, IX, will suffice to determine the difference of each of the three angles of a small right-angled spherical triangle, from the corresponding angles of the chords. And hence *this* method may be applied to the measuring an arc of the meridian by means of a series of triangles. See arts. 8, 9, sect. 1 of this chapter *.

PROBLEM VII.

In a spherical triangle ABC, right angled at A, knowing the hypothenuse BC (*less than* 4°) and the angle B, it is required to find the error e committed through finding by plane trigonometry, the opposite side AC.

Referring still to the diagram of prob. 5, where we now suppose the spherical angle A to be right, we have (theor. 10 chap. IV) $\sin b = \sin a \cdot \sin B$. But it has been remarked at pa. 382 vol. i, that the sine of any arc A is equal to the sum of the following series;

$$\sin A = A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} - \frac{A^7}{2.3.4.5.6.7} + \&c.$$

$$\text{or, } \sin A = A - \frac{A^3}{6} + \frac{A^5}{120} - \frac{A^7}{5040} + \&c.$$

And, in the present inquiry, all the terms after the second may be neglected, because the 5th power of an arc of 4° divided by 120, gives a quotient not exceeding 0″·01. Consequently, we may assume $\sin b = b - \frac{1}{6}b^3$, $\sin a = a - \frac{1}{6}a^3$; and thus the preceding equation will become,

$$b - \frac{1}{6}b^3 = \sin B\,(a - \frac{1}{6}a^3)$$

$$\text{or, } b = a : \sin B - \frac{1}{6}\,(a^3 \cdot \sin B - b^3).$$

Now, if the triangle were considered as rectilinear, we should have $b = a \cdot \sin B$: a theorem which manifestly gives the side b or AC too great by $\frac{1}{6}(a^3 \cdot \sin B - b^3)$. But, neglecting quantities of the fifth order, for the reason already assigned, the last equation but one gives $b^3 = a^3 \cdot \sin^3 B$. Therefore,

* On this subject some elegant investigations by Captain Everest, of the Bengal Artillery, are inserted in the *Memoirs of the Astronomical Society of London*, vol. ii. p. 37, &c.

by substitution, $e = -\frac{1}{6}a^3 \cdot \sin B(1-\sin^2 B)$: or, to have this error in seconds, take R″ = the radius expressed in seconds, so shall $e = -a \cdot \sin B \cdot \frac{a^2 \cdot \cos^2 B}{6R''R''}$.

Cor. 1. If $a = 4°$, and $B = 35°16'$, in which case the value of $\sin B \cdot \cos^2 B$ is a maximum, we shall find $e = -4\frac{1}{4}''$.

Cor. 2. If, with the same data, the correction be applied, to find the side c adjacent to the given angle, we should have

$$e' = a \cdot \cos B \cdot \frac{a^2 \cdot \sin^2 B}{3R'R''}.$$

So that this error exists in a contrary sense to the other; the one being subtractive, the other additive.

Cor. 3. The data being the same, if we have to find the angle C, the error to be corrected will be

$$e'' = a^2 \cdot \frac{\sin 2B}{4R''}.$$

As to the excess of the arc over its chord, it is easy to find it correctly from the expressions in prob. 5: but for arcs that are very small, compared with the radius, a near approximation to that excess will be found in the same measures as the radius of the earth, by taking $\frac{1}{24}$ of the quotient of the cube of the length of the arc divided by the square of the radius.

PROBLEM VIII.

It is required to investigate a theorem, by means of which, spherical triangles, whose sides are small compared with the radius, may be solved by the rules for plane trigonometry, without considering the chords of the respective arcs or sides.

Let a, b, c, be the sides, and A, B, C, the angles of a spherical triangle, on the surface of a sphere whose radius is r: then a similar triangle on the surface of a sphere whose radius $= 1$, will have for its sides $\frac{a}{r}$, $\frac{b}{r}$, $\frac{c}{r}$; which, for the sake of brevity, we represent by α, β, γ, respectively: then by equa. I, chap. IV, we have $\cos A = \frac{\cos \alpha - \cos \beta \cdot \cos \gamma}{\sin \beta \cdot \sin \gamma}$.

Now, r being very great with respect to the sides a, b, c, we may, as in the investigation of the last problem, omit all the terms containing higher than 4th powers, in the series for the sine and consine of an arc, given at pa. 382, vol. i: so shall we have, without perceptible error,

$$\cos \alpha = 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{2 \cdot 3 \cdot 4} \ldots \sin \beta = \beta - \frac{\beta^3}{2 \cdot 3}.$$

And similar expressions may be adopted for cos β, cos γ, sin γ. Thus, the preceding equation will become

$$\cos A = \frac{\frac{1}{2}(\beta^2+\gamma^2-\alpha^2)+\frac{1}{24}(\alpha^4-\beta^4-\gamma^4)-\frac{1}{4}\beta^2\gamma^2}{\beta\gamma(1-\frac{1}{6}\beta^2-\frac{1}{6}\gamma^2)}.$$

Multiplying both terms of this fraction by $1+\frac{1}{6}(\beta^2+\gamma^2)$, to simplify the denominator, and reducing, there will result,

$$\cos A = \frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}+\frac{\alpha^4+\beta^4+\gamma^4-2\alpha^2\beta^2-2\alpha^2\gamma^2-2\beta^2\gamma^2}{24\beta\gamma}.$$

Here, restoring the values of α, β, γ, the second member of the equation will be entirely constituted of like combinations of the letters, and therefore the whole may be represented by

$$\cos A = \frac{M}{2bc}+\frac{N}{24bcr^2}. \quad . \quad . \quad . \quad (1.)$$

Let, now, A' represent the angle opposite to the side a, in the rectilinear triangle whose sides are equal in length to the arcs a, b, c; and we shall have

$$\cos A' = \frac{b^2+c^2-a^2}{2bc} = \frac{M}{2bc}.$$

Squaring this, and substituting for $\cos^2 A'$ its value $1-\sin^2 A'$, there will result

$$-4b^2c^2\sin^2 A' = a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2 = N.$$

So that, equa. I reduces to the form

$$\cos A = \cos A' - \frac{bc}{6r^2}\sin^2 A'.$$

Let $A = A'+x$, then, as x is necessarily very small, its second power may be rejected, and we may assume $\cos A = \cos A' - x \cdot \sin A'$; whence, substituting for cos A this value of it, we shall have $x = \frac{bc}{6r^2}\sin A'$.

It hence appears that x is of the second order, with respect to $\frac{b}{r}$ and $\frac{c}{r}$; and of course that the result is exact to quantities within the fourth order. Therefore, because $A = A'+x$,

$$A = A' + \frac{bc}{6r^2} \cdot \sin A'.$$

But, by prob. 2 rule 2, Mensuration of Planes, $\frac{1}{2}bc \sin A'$ is the area of the rectilinear triangle, whose sides are a, b, and c.

$$\text{Therefore } A = A' + \frac{\text{area}}{3r^2};$$

$$\text{or } A' = A - \frac{\text{area}}{3r^2}.$$

$$\text{In like manner} \begin{cases} B' = B - \dfrac{\text{area}}{3r^2}. \\ C' = C - \dfrac{\text{area}}{3r^2}. \end{cases}$$

$$\text{And } A' + B' + C' = 180° = A + B + C - \frac{\text{area}}{r^2}:$$

$$\text{or, } \frac{\text{area}}{r^2} = A + B + C - 180°.$$

Whence, since the spherical excess is a measure of the area (th. 5, ch. iv), we have this theorem: viz.

A spherical triangle being proposed, of which the sides are very small, compared with the radius of the sphere; if from each of its angles one third of the excess of the sum of its three angles above two right angles be subtracted, the angles so diminished may be taken for the angles of a rectilinear triangle, whose sides are equal in length to those of the proposed spherical triangle.*

Scholium.

We have already given, at th. 5, chap. iv, expressions for finding the spherical excess, in the two cases, where two sides and the included angle of a triangle are known, and where the three sides are known. A few additional rules may with propriety be presented here.

1. The spherical excess E, may be found in seconds, by the expression $E = \frac{R''s}{r}$; where s is the surface of the triangle $= \frac{1}{2}bc \cdot \sin A = \frac{1}{2}ab \cdot \sin C = \frac{1}{2}ac \cdot \sin B = \frac{1}{2}a^2 \cdot \frac{\sin B \cdot \sin C}{\sin (B + C)}$, r is the radius of the earth, in the same measures as a, b, and c, and $R'' = 206264''\cdot8$, the seconds in an arc equal in length to the radius.

If this formula be applied logarithmically; then $\log R'' = \log \frac{1}{\text{arc } 1''} = 5\cdot3144251$.

2. From the logarithm of the area of the triangle, taken as a plane one, in feet, subtract the constant log 9·3267737, then the remainder is the logarithm of the excess above 180°, in seconds nearly†.

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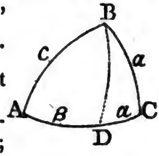
* This curious theorem was first announced by M. Legendre, in the Memoirs of the Paris Academy, for 1787. Legendre's investigation is nearly the same as the above: a shorter investigation is given by Swanberg, at p. 40, of his "Exposition des Opérations faites en Lapponie;" but it is defective in point of perspicuity.

† This is commonly called "General Roy's rule," and given by him in the Philosophical Transactions, for 1790, p. 171; it is, however, due to the late Mr. Isaac Dalby, who was then General Roy's assistant in the Trigonometrical Survey, and for several years the entire conductor of the mathematical department.

3. Since $s = \frac{1}{2}bc \,.\, \sin A$, we shall manifestly have $E = \frac{R''}{2r^2} bc \,.\, \sin A$. Hence, if from the vertical angle B we demit the perpendicular BD upon the base AC, dividing it into the two segments α, β, we shall have $b = \alpha + \beta$, and thence $E = \frac{R''}{2r^2} c\ (\alpha + \beta) \sin A = \frac{R''}{2r^2} ac \,.\, \sin A + \frac{R''}{2r^2} \beta c \,.\, \sin A$. But the two right angled triangles ABD, CBD, being nearly rectilinear, give $\alpha = A \,.\, \cos C$, and $\beta = c \,.\, \cos A$; whence we have



$$E = \frac{R''}{2r^2} ac \,.\, \sin A \,.\, \cos C + \frac{R''}{2r^2} c^2 \,.\, \sin A \,.\, \cos A.$$

In like manner, the triangle ABC, which itself is so small as to differ but little from a plane triangle, gives $c \,.\, \sin A = a \,.\, \sin C$. Also, $\sin A \,.\, \cos A = \frac{1}{2} \sin 2A$, and $\sin C \,.\, \cos C = \frac{1}{2} \sin 2C$ XY. A (equa. xv, ch. iii). Therefore, finally,

$$E = \frac{R''}{4r^2} a^2 \,.\, \sin 2C + \frac{R''}{4r^2} c^2 \,.\, \sin 2A.$$

From this theorem a table may be formed, from which the spherical excess may be found; entering the table with each of the sides above the base and its adjacent angle, as arguments.

4. If the base b, and height h, of the triangle are given, then we have evidently $E = \frac{1}{2} bh \frac{R''}{r^2}$. Hence results the following simple logarithmic rule: Add the logarithm of the base of the triangle, taken in feet, to the logarithm of the perpendicular, taken in the same measure; deduct from the sum the logarithm 9·6278037; the remainder will be the common logarithm of the spherical excess in seconds and decimals.

5. Lastly, when the three sides of the triangle are given in feet; add to the logarithm of half their sum, the logs. of the three differences of those sides and that half sum, divide the total of these 4 logs. by 2, and from the quotient subtract the log. 9·3267737; the remainder will be the logarithm of the spherical excess in seconds, &c. as before.

One or other of these rules will apply to all cases in which the spherical excess will be required.

PROBLEM IX.

Given the measure of a base on any elevated level; to find its measure when reduced to the level of the sea.

Let r represent the radius of the earth, or the distance from its centre to the surface of the sea, $r+h$ the radius referred to the level of the base measured, the altitude h being determined by the rule for the measurement of such altitudes by the barometer and thermometer, (in this volume); let B be the length of the base measured at the elevation h, and b that of the base referred to the level of the sea. Then because the measured base is all along reduced to the horizontal plane, the two, B and b, will be concentric and similar arcs, to the respective radii $r+h$ and r. Therefore, since similar arcs, whether of spheres or spheroids, are as their radii of curvature, we have

$$r+h : r :: \text{B} : b = \frac{r\text{B}}{r+h}.$$

Hence, also $\text{B} - b = \text{B} - \frac{r\text{B}}{r+h} = \frac{\text{B}h}{r+h}$; or, by actually dividing Bh by $r+h$, we shall have

$$\text{B} - b = \text{B} \times \left(\frac{h}{r} - \frac{h^2}{r^2} + \frac{h^3}{r^3} - \frac{h^4}{r^4} + \&c.\right)$$

Which is an *accurate* expression for the excess of B above b.

But the mean radius of the earth being more than 21 million feet, if h the difference of level were 50 feet, the second and all succeeding terms of the series could never exceed the fraction $\frac{1}{1760004000000}$; and may therefore safely be neglected; so that for all practical purposes we may assume $\text{B} - b = \frac{\text{B}h}{r}$. Or, in logarithms, add the logarithm of the measured base in feet, to the logarithm of its height above the level of the sea, subtract from the sum the logarithm Radius 7·3223947, the remainder will be the logarithm of a number, which taken from the measured base, will leave the reduced base required.

PROBLEM X.

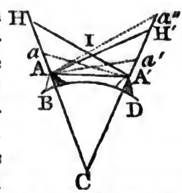
To determine the horizontal refraction.

1. Particles of light, in passing from any object through the atmosphere, or part of it, to the eye, do not proceed in a right line; but the atmosphere being composed of an infinitude of strata (if we may so call them) whose density increases

= 1 : 21,000 000 = 7.

as they are posited nearer the earth, the luminous rays which pass through it are acted on as if they passed successively through media of increasing density, and are therefore inflected more and more towards the earth as the density augments. In consequence of this it is, that rays from objects, whether celestial or terrestrial, proceed in curves which are *concave* towards the earth; and thus it happens, since the eye always refers the place of objects to the direction in which the rays reach the eye, that is, to the direction of the tangent to the curve at that point, that the apparent, or observed elevations of objects, are always *greater* than the true ones. The difference of these elevations, which is, in fact, the *effect* of refraction, is, for the sake of brevity, called *refraction:* and it is distinguished into two kinds, *horizontal* or *terrestrial* refraction, being that which affects the altitudes of hills, towers, and other objects on the earth's surface; and *astronomical* refraction, or that which is observed with regard to the altitudes of heavenly bodies. Refraction is found to vary with the state of the atmosphere, in regard to heat or cold, humidity or dryness, &c.: so that, determinations obtained for one state of the atmosphere, will not answer correctly for another, without modification. Tables commonly exhibit the refraction at different altitudes, for some assumed mean state.

2. With regard to the *horizontal* refraction, the following method of determining it has been successfully practised in the English Trigonometrical Survey.



Let A, A′, be two elevated stations on the surface of the earth, BD the intercepted arc of the earth's surface, C the earth's centre, AH′, A′H, the horizontal lines at A, A′, produced to meet the opposite vertical lines CH′, CH. Let a, a', represent the apparent places of the objects A, A′, then is a'AA′ the refraction observed at A, and aA′A the refraction observed at A′; and half the sum of those angles will be the horizontal refraction, if we assume it equal at each station.

Now, an instrument being placed at each of the stations A, A′, the reciprocal observations are made at the same instant of time, which is determined by means of signals or watches previously regulated for that purpose: that is, the observer at A takes the apparent depression of A′, at the same moment that the other observer takes the apparent depression of A.

In the quadrilateral ACA′I, the two angles A, A′ are right angles, and therefore the angles I and C are together equal to

two right angles: but the three angles of the triangle IAA′ are together equal to two right angles; and consequently the angles A and A′ are together equal to the angle C, which is measured by the arc BD. If therefore the sum of the two depressions HA′a, H′Aa', be taken from the sum of the angles HA′A, H′AA′, or, which is equivalent, from the angle C, (which is known, because its measure BD is known); the remainder is the sum of both refractions, or angles aA′A, a'AA′. Hence this rule, *take the sum of the two depressions from the measure of the intercepted terrestrial arc, half the remainder is the refraction.*

3. If, by reason of the minuteness of the contained arc BD, one of the objects, instead of being depressed, appears elevated, as suppose A′ to a'': then the sum of the angles a''AA′ and aA′A will be greater than the sum IAA′, + IA′A, or than C, by the angle of elevation a''AA′; but if from the former sum there be taken the depression HA′A, there will remain the sum of the two refractions. So that in this case the rule becomes as follows: *take the depression from the sum of the contained arc and elevation, half the remainder is the refraction.*

4. The quantity of this terrestrial refraction is estimated by Dr. Maskelyne at one-tenth of the distance of the object observed, expressed in degrees of a great circle. So, if the distance be 10000 fathoms, its 10th part, 1000 fathoms, is the 60th part of a degree of a great circle on the earth, or 1′, which therefore is the refraction in the altitude of the object at that distance.

But M. Legendre is induced, he says, by several experiments, to allow only $\frac{1}{14}$th part of the distance for the refraction in altitude. So that, on the distance of 10000 fathoms, the 14th part of which is 714 fathoms, he allows only 44″ of terrestrial refraction, so many being contained in the 714 fathoms. See his Memoir concerning the Trigonometrical Operations, &c.

Again, M. Delambre, an eminent French astronomer, makes the quantity of the terrestrial refraction to be the 11th part of the arch of distance. But the English measurers, especially Gen. Mudge, from a multitude of exact observations, determine the quantity of the medium refraction to be the 12th part of the said distance.

The quantity of this refraction, however, is found to vary considerably, with the different states of the weather and atmosphere, from the $\frac{1}{5}$th to the $\frac{1}{15}$th of the contained arc. See Trigonometrical Survey, vol. 1. pa. 160, 355.

Scholium.

Having given the mean results of observations on the terrestrial refraction, it may not be amiss, though we cannot enter at large into the investigation, to present here a correct table of mean astronomical refractions. The table which has been most commonly given in books of astronomy is Dr. Bradley's, computed from the rule $r = 57'' \times \cot(a + 3r)$, where a is the altitude, r the refraction, and $r = 2'35''$ when $a = 20°$. But it has been found by numerous observations, that the refractions thus computed are rather too *small.*—Laplace, in his Mecanique Celeste (tome iv. pa. 27) deduces a formula which is strictly similar to Bradley's; for it is $r = m \times \tan(z - nr)$, where z is the zenith distance, and m and n are two constant quantities to be determined from observation. The only advantage of the formula given by the French philosopher, over that given by the English astronomer, is, that Laplace and his colleagues have found more correct coefficients than Bradley had.

Now, if $\text{R} = 57°\cdot2957795$, the arc equal to the radius, if we make $m = \frac{k\text{R}}{n}$, (where k is a constant coefficient which, as well as n, is an abstract number), the preceding equation will become $\frac{nr}{\text{R}} = k \times \tan(z - nr)$. Here, as the refraction r is always very small, as well as the correction nr, the trigonometrical tangent of the arc nr may be substituted for $\frac{nr}{\text{R}}$; thus we shall have $\tan nr = k \cdot \tan(z - nr)$.

But $nr = \frac{1}{2}z - (\frac{1}{2}z - nr) \ldots \; z - nr = \frac{1}{2}z + (\frac{1}{2}z - nr)$;

$$\text{Conseq.}\ \frac{\tan nr}{\tan(z-nr)} = \frac{\tan\left(\frac{z}{2} - \frac{z-2nr}{2}\right)}{\tan\left(\frac{z}{2} + \frac{z-2nr}{2}\right)} = \frac{\sin z - \sin(z - 2nr)}{\sin z + \sin(z - 2nr)} = k.$$

$$\text{Hence, } \sin(z - 2nr) = \frac{1-k}{1+k} \cdot \sin z.$$

This formula is easy to use, when the coefficients n and $\frac{1-k}{1+k}$ are known: and it has been ascertained, by a mean of many observations, that these are 4 and $\cdot99765175$ respectively. Thus Laplace's equation becomes

$$\sin(z - 8r) = \cdot99765175 \sin z:$$

and from this the following table has been computed. Besides the refractions, the differences of refraction, for every 10 minutes of altitude, are given; an addition which will render the table more extensively useful in all cases where great accuracy is required.

Table of Refractions.

Barom. 29·92 *inc. Fah. Thermom.* 54°.

Alt. app.	Refrac.	Diff. on 10'	Alt. app.	Refr.	Diff. 10'.	Alt. app.	Refr.	Diff. 10'.	Alt app	Refr.	Di 10'.
D. M.	M. S.	S.	D. M.	M. S.	S.	D.	M. S.	S	D	S.	S.
0 0	33 46·3	112·0	7 0	7 24·8	9·5	14	3 49·8	2·58	56	39·3	0·25
10	31 54·3	105·0	10	7 15·3	9·0	15	3 34·3	2·28	57	37·8	0·24
20	30 9·3	97·3	20	7 6·3	8·6	16	3 20·6	2·02	58	36·4	0·24
30	28 32·1	89·8	30	6 57·7	8·1	17	3 8·5	1·82	59	35·0	0·23
40	27 2·2	83·6	40	6 49·6	7·7	18	2 57·6	1·65	60	33·6	0·22
50	25 38·6	77·4	50	6 41·9	7·5	19	2 47·7	1·48	61	32·3	0·22
1 0	24 21·2	71·6	8 0	6 34·4	7·3	20	2 38·8	1·37	62	31·0	0·21
10	23 9·6	66·2	10	6 27·1	7·1	21	2 30·6	1·24	63	29·7	0·21
20	22 3·4	61·5	20	6 20·0	6·9	22	2 23·2	1·11	64	28·4	0·20
30	21 1·9	57·1	30	6 13·1	6·7	23	2 16·5	1·05	65	27·2	0·20
40	20 4·8	53·3	40	6 6·4	6·5	24	2 10·2	0·98	66	25·9	0·20
50	19 11·5	49·3	50	5 59·9	6·3	25	2 4·3	0·90	67	24·7	0·20
2 0	18 22·2	45·9	9 0	5 53·6	6·2	26	1 58·9	0·83	68	23·5	0·20
10	17 36·3	43·1	10	5 47·4	5·9	27	1 53·9	0·78	69	22·4	0·20
20	16 53·2	39·8	20	5 41·5	5·7	28	1 49·2	0·73	70	21·2	0·20
30	16 13·4	37·4	30	5 35·8	5·5	29	1 44·8	0·70	71	20·0	0·19
40	15 36·0	35·1	40	5 30·3	5·3	30	1 40·6	0·65	72	18·9	0·18
50	15 0·9	32·8	50	5 25·0	5·2	31	1 36·7	0·60	73	17·8	0·18
3 0	14 28·1	30·8	10 0	5 19·8	5·1	32	1 33·1	0·58	74	16·7	0·18
10	13 57·3	28·8	10	5 14·7	5·0	33	1 29·6	0·56	75	15·6	0·18
20	13 28·5	27·2	20	5 9·7	4·8	34	1 26·2	0·53	76	14·5	0·17
30	13 1·3	25·7	30	5 4·9	4·6	35	1 23·1	0·50	77	13·5	0·17
40	12 35·6	24·3	40	5 0·3	4·4	36	1 20·1	0·48	78	12·4	0·17
50	12 11·3	23·0	50	4 55·9	4·2	37	1 17·2	0·47	79	11·3	0·17
4 0	11 48·3	21·7	11 0	4 51·7	4·1	38	1 14·4	0·45	80	10·3	0·17
10	11 26·6	20·5	10	4 47·6	4·0	39	1 11·8	0·42	81	9·2	0·17
20	11 6·1	19·4	20	4 43·6	4·0	40	1 9·3	0·40	82	8·2	0·17
30	10 46·7	18·4	30	4 39·6	3·9	41	1 6·9	0·38	83	7·2	0·17
40	10 28·3	17·4	40	4 35·7	3·9	42	1 4·6	0·37	84	6·1	0·17
50	10 10·9	16·6	50	4 31·8	3·8	43	1 2·4	0·35	85	5·1	0·17
5 0	9 54·3	15·9	12 0	4 28·0	3·7	44	1 0·3	0·34	86	4·1	0·17
10	9 38·4	15·0	10	4 24·3	3·6	45	0 58·2	0·33	87	3·1	0·17
20	9 23·4	14·4	20	4 20·7	3·5	46	0 56·2	0·32	88	2·0	0·17
30	9 9·0	13·7	30	4 17·2	3·4	47	0 54·3	0·31	89	1·0	0·17
40	8 55·3	13·0	40	4 13·8	3·2	48	0 52·4	0·30	90	0·0	
50	8 42·3	12·4	50	4 10·6	3·1	49	0 50·6	0·29			
6 0	8 29·9	11·8	13 0	4 7·5	3·1	50	0 48·9	0·28			
10	8 18·1	11·5	10	4 4·4	3·0	51	0 47·2	0·27			
20	8 6·6	11·0	20	4 1·4	3·0	52	0 45·5	0·26			
30	7 55·6	10·6	30	3 58·4	2·9	53	0 43·9	0·26			
40	7 45·0	10·3	40	3 55·5	2·9	54	0 42·3	0·25			
50	7 34·7	9·9	50	3 52·6	2·8	55	0 40·8	0·25			
7 0	7 24·8		14 0	3 49·8		56	0 39·3				

For refraction under different temperatures,

Ref. $= \frac{a}{29 \cdot 6} \times \tan(z - 3r) \times 57'' \times \frac{400}{350 + h}$, according to Dr. Maskelyne.

Ref. $= \frac{a}{29 \cdot 6} \times \tan(z - 3 \cdot 2r) \times 56 \cdot 9'' \times \frac{500}{450 + h}$, according to Dr. Brinkley.

Where $a =$ alt. barometer in inches, $z =$ zenith distance, $r = 57''$ tan. z, $h =$ height of Fahrenheit's thermometer, and 29·6 is assumed for the mean height of the barometer.

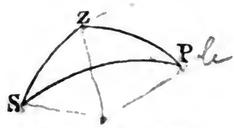
On the general subject of astronomical and terrestrial refractions, the reader may advantageously consult an elaborate paper by Mr. H. Atkinson of Newcastle, in Mem. Astron. Soc. London, vol. ii.

PROBLEM XI.

To find the angle made by a given line with the meridian.

1. The easiest method of finding the angular distance of a given line from the meridian, is to measure the greatest and the least angular distance of the vertical plane in which is the star marked α in Ursa minor (commonly called the *pole star*), from the said line: for half the sum of these two measures will manifestly be the angle required.

2. Another method is to observe when the sun is on the given line; to measure the altitude of his centre at that time, and correct it for refraction and parallax. Then, in the spherical triangle ZPS, where Z is the zenith of the place of observation, P the elevated pole, and S the centre of the sun, there are supposed given ZS the zenith distance, or co-altitude of the sun, PS the co-declination of that luminary, PZ the co-latitude of the place of observation, and ZPS the hour angle, measured at the rate of 15° to an hour, to find the angle SZP between the meridian PZ and the vertical ZS, on which the sun is at the given time. And here, as three sides and one angle are known, the required angle is readily found, by saying, as sine ZS : sine ZPS : : sine PS : sine PZS ; that is, as the consine of the sun's altitude, is to the sine of the hour angle from noon ; so is the consine of the sun's declination, to the sine of the angle made by the given vertical and the meridian.

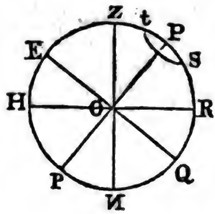


Note. Many other methods are given in books of Astronomy ; but the above are sufficient for our present purpose. The first is independent of the latitude of the place ; the second requires it.

PROBLEM XII.

To find the latitude of a place.

The latitude of a place may be found by observing the greatest and least altitude of a circumpolar star, and then applying to each the correction for refraction ; so shall half the sum of the altitudes, thus corrected, be the altitude of the pole, or the latitude.



For, if P be the elevated pole, st the circle described by the star, PR = EZ the latitude: then since Ps = Pt, PR must be $= \frac{1}{2}(Rt + Rs)$.

This method is obviously independent of the declination of the star: it is therefore most commonly adopted in trigonometrical surveys, in which the telescopes employed are of such power as to enable the observer to see stars in the day-time: the pole-star being here also made use of.

Numerous other methods of solving this problem likewise are given in books of Astronomy; but they need not be detailed here.

Corol. If the mean altitude of a circumpolar star be thus measured, at the two extremities of any arc of a meridian, the difference of the altitudes will be the measure of that arc: and if it be a small arc, one for example not exceeding a degree of the terrestrial meridian, since such small arcs differ extremely little from arcs of the circle of curvature at their middle points, we may, by a simple proportion, infer the length of a degree whose middle point is the middle of that arc.

Scholium.

Though it is not consistent with the purpose of this chapter to enter largely into the doctrine of astronomical spherical problems; yet it may be here added, for the sake of the young student, that if a = right ascension, d = declination, l = latitude, λ = longitude, p = angle of position (or, the angle at a heavenly body formed by two great circles, one passing through the pole of the equator and the other through the pole of the ecliptic), i = inclination or obliquity of the ecliptic, then the following equations, most of which are new, obtain generally, for all the stars and heavenly bodies.

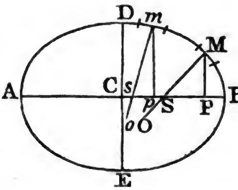
1. $\tan a = \tan \lambda \cdot \cos i - \tan l \cdot \sec \lambda \cdot \sin i.$
2. $\sin d = \sin \lambda \cdot \cos l \cdot \sin i + \sin l \cdot \cos i.$
3. $\tan \lambda = \sin i \cdot \tan d \cdot \sec a + \tan a \cdot \cos i.$
4. $\sin l = \sin d \cdot \cos i - \sin a \cdot \cos d \cdot \sin i.$
5. $\text{cotan}\, p = \cos d \cdot \sec a \cdot \cot i + \sin d \cdot \tan. a.$
6. $\text{cotan}\, p = \cos l \cdot \sec \lambda \cdot \cot i - \sin l \cdot \tan \lambda.$
7. $\cos a \cdot \cos d = \cos l \cdot \cos \lambda.$
8. $\sin p \cdot \cos d = \sin i \cdot \cos \lambda.$
9. $\sin p \cdot \cos \lambda = \sin i \cdot \cos a.$

$$\left.\begin{array}{ll} 10. & \tan a = \tan \lambda \cdot \cos i. \\ 11. & \cos \lambda = \cos a \cdot \cos d. \end{array}\right\} \text{when } l = 0, \text{ as is always the case within the sun.}$$

The investigation of these equations, which is omitted for the sake of brevity, depends on the resolution of the spherical triangle whose angles are the poles of the ecliptic and equator, and the given star, or luminary.

PROBLEM XIII.

To determine the ratio of the earth's axes, and their actual magnitude, from the measure of a degree or smaller portion of a meridian in two given latitudes; the earth being supposed a spheroid generated by the rotation of an ellipse upon its minor axis.



Let ADBE represent a meridian of the earth, DE its minor axis, AB a diameter of the equator, M, m, arcs of the same number of degrees, or the same parts of a degree, of which the lengths are measured, and which are so small, compared with the magnitude of the earth, that they may be considered as coinciding with arcs of the osculatory circles at their respective middle points; let MO, mo, the radii of curvature of those middle points, be $= R$ and r respectively; MP, mp, ordinates perpendicular to AB: suppose further $CD = c$; $CB = d$; $d^2 - c^2 = e^2$; $CP = x$; $cp = u$; the radius or sine total $= 1$; the known angle BSM, or the latitude of the middle point M, $= L$; the known angle Bsm, or the latitude of the point m, $= l$; the measured lengths of the arcs M and m being denoted by those letters respectively.

Now the similar sectors whose arcs are M, m, and radii of curvature R, r, give $R : r :: M : m$; and consequently $Rm = rM$. The central equation to the ellipse investigated at p. 536 of the 1st vol. gives $PM = \frac{c}{d}\sqrt{(d^2 - x^2)}$; $pm = \frac{c}{d}\sqrt{(d^2 - u^2;)}$ also $SP = \frac{c^2x}{d^2}$; $sp = \frac{c^2u}{d^2}$ (by th. 17 Ellipse). And the method of finding the radius of curvature (Flux. art. 74, 75), applied to the central equations above, gives

$R = \frac{(d^4 - e^2x^2)^{\frac{3}{2}}}{c^4 d}$; and $r = \frac{(d^4 - e^2u^2)^{\frac{3}{2}}}{c^4 d}$. On the other hand, the triangle SPM gives $SP : PM :: \cos L : \sin L$; that is, $\frac{c^2x}{d^2} : \frac{c}{d}\sqrt{(d^2 - x^2)} :: \cos L : \sin L$; whence $x^2 = \frac{d^4 \cos^2 L}{d^2 - e^2 \sin^2 L}$.

And from a like process there results, $u^2 = \frac{d^4 \cos^2 l}{d^2 - e \cdot \sin^2 l}$.
Substituting in the equation $\text{R}m = r\text{M}$, for R, and r their values, for x^2 and u^2 their values just found, and observing that $\sin^2 \text{L} + \cos^2 \text{L} = 1$, and $\sin^2 l + \cos^2 l = 1$, we shall find

$$\frac{m}{(d^2 - e^2 \sin^2 \text{L})^{\frac{3}{2}}} = \frac{\text{M}}{(d^2 - e^2 \sin^2 l)^{\frac{3}{2}}},$$

$$\text{or } m(d^2 - e^2 \sin^2 l)^{\frac{3}{2}} = \text{M}(d^2 - e^2 \sin^2 \text{L})^{\frac{3}{2}},$$

$$\text{or } m^{\frac{2}{3}}(d^2 - e^2 \sin^2 l) = \text{M}^{\frac{2}{3}}(d^2 - e^2 \sin^2 \text{L}).$$

From this there arises $e^2 = d^2 - c^2$ (by hyp.) $=$

$$\frac{d^2(\text{M}^{\frac{2}{3}} - m^{\frac{2}{3}})}{\text{M}^{\frac{2}{3}} \sin^2 \text{L} - m^{\frac{2}{3}} \sin^2 l}. \quad \text{But } \frac{c^2}{d^2} = 1 - \frac{d^2 - c^2}{d^2};$$

and consequently the reciprocal of this fraction, or

$$\frac{d^2}{c^2} = \frac{\text{M}^{\frac{2}{3}} \sin^2 \text{L} - m^{\frac{2}{3}} \sin^2 l}{m^{\frac{2}{3}} \cos^2 l - \text{M}^{\frac{2}{3}} \cos^2 \text{L}} = \frac{(\text{M}^{\frac{1}{3}} \sin \text{L} + m^{\frac{1}{3}} \sin l) \cdot (\text{M}^{\frac{1}{3}} \sin \text{L} - m^{\frac{1}{3}} \sin l)}{(m^{\frac{1}{3}} \cos l + \text{M}^{\frac{1}{3}} \cos \text{L}) \cdot (m^{\frac{1}{3}} \cos l - \text{M}^{\frac{1}{3}} \cos \text{L})}.$$

Whence, by extracting the root, there results finally

$$\frac{d}{c} = \sqrt{\frac{(\text{M}^{\frac{1}{3}} \sin \text{L} + m^{\frac{1}{3}} \sin l) \cdot (\text{M}^{\frac{1}{3}} \sin \text{L} - m^{\frac{1}{3}} \sin l)}{(m^{\frac{1}{3}} \cos l + \text{M}^{\frac{1}{3}} \cos \text{L}) \cdot (m^{\frac{1}{3}} \cos l - \text{M}^{\frac{1}{3}} \cos \text{L})}}.$$

This expression, which is simple and symmetrical, has been obtained without any developememt into series, without any omission of terms on the supposition that they are indefinitely small, or any possible deviation from correctness, except what may arise from the want of coincidence of the circles of curvature at the middle points of the arcs measured, with the arcs themselves; and this source of error may be diminished at pleasure, by diminishing the magnitude of the arcs measured: though it must be acknowledged that such a procedure may give rise to errors in the practice, which may more than counterbalance the small one to which we have just adverted.

Cor. Knowing the number of degrees, or the parts of degrees, in the measured arcs M, m, and their lengths, which are here regarded as the lengths of arcs to the circles which have R, r, for radii, those radii evidently become known in magnitude. At the same time there are given the algebraic values of R and r: thus, taking R for example, and extermi- nating e^2 and x^2, there results $\text{R} = \frac{d^5}{c(d^2 - (d^2 - c^2) \sin^2 \text{L})^{\frac{3}{2}}}$. Therefore, by putting in this equation the known ratio of d to c, there will remain only one unknown quantity d or c, which may of course be easily determined by the reduction of the last equation; and thus all the dimensions of the terrestrial spheroid will become known.

General Scholium and Remarks.

1. The value $\frac{d}{c} - 1, = \frac{d-c}{c}$, is called the *compression* of the terrestrial spheroid, and it manifestly becomes known when the ratio $\frac{d}{c}$ is determined. But the measurements of philosophers, however carefully conducted, furnish resulting compressions, in which the discrepancies are much greater than might be wished. General Roy has recorded several of these in the Phil. Trans. vol. 77, and later measurers have deduced others. Thus, the degree measured at the equator by Bouguer, compared with that of France measured by Mechain and Delambre, gives for the compression $\frac{1}{334}$, also $d = 3271208$ toises, $c = 3261443$ toises, $d-c = 9765$ toises. General Roy's sixth spheroid, from the degrees at the equator and in latitude 45°, gives $\frac{1}{309{\cdot}3}$. Mr. Dalby makes $d = 3489932$ fathoms, $c = 3473656$, Gen. Mudge $d = 3491420$, $c = 3468007$, or 7935 and 7882 miles. The degree measured at Quito, compared with that measured in Lapland by Swanberg, gives compression $= \frac{1}{309{\cdot}4}$. Swanberg's observations, compared with Bouguer's, give $\frac{1}{329{\cdot}25}$. Swanberg's compared with the degree of Delambre and Mechain $\frac{1}{307{\cdot}4}$. Compared with Major Lambton's degree $\frac{1}{307{\cdot}17}$. A minimum of errors in Lapland, France, and Peru gives $\frac{1}{323{\cdot}4}$. Laplace, from the lunar motions, finds compression $= \frac{1}{314}$. From the theory of gravity as applied to the latest observations of Burg, Maskelyne, &c. $\frac{1}{309{\cdot}05}$. From the variation of the pendulum in different latitudes $\frac{1}{335{\cdot}78}$. Dr. Robison, assuming the variation of gravity at $\frac{1}{180}$, makes the compression $\frac{1}{319}$. The most accurately computed results from Capt. Sabine's experiments on the pendulum in different latitudes, give $\frac{1}{300}$.* Others give results varying from $\frac{1}{178{\cdot}4}$ to $\frac{1}{577}$: but far the greater number of observations differ but little from $\frac{1}{304}$,

* See Ivory in Phil. Mag. July 1826.

which the computation from the phenomena of the precession of the equinoxes and the nutation of the earth's axis, gives for the maximum limit of the compression.

2. From the various results of careful admeasurements it happens, as Gen. Roy has remarked, "that philosophers are not yet agreed in opinion with regard to the exact figure of the earth; some contending that it has no regular figure, that is, not such as would be generated by the revolution of a curve around its axis. Others have supposed it to be an ellipsoid; regular, if both polar sides should have the same degree of flatness; but irregular if one should be flatter than the other. And lastly, some suppose it to be a spheroid differing from the ellipsoid, but yet such as would be formed by the revolution of a curve around its axis." According to the theory of gravity, however, the earth must of necessity have its axes approaching nearly to either the ratio of 1 to 680 or of 303 to 304; and as the former ratio obviously does not obtain, the figure of the earth *must* be such as to correspond nearly with the latter ratio.

3. Besides the method above described, others have been proposed for determining the figure of the earth, by measurement. Thus, that figure might be ascertained by the measurement of a degree in two parallels of latitude; but not so accurately as by meridional arcs, 1st. Because, when the distance of the two stations, in the same parallel, is measured, the celestial arc is not that of a parallel circle, but is nearly the arc of a great circle, and always exceeds the arc that corresponds truly with the terrestrial arc. 2dly. The interval of the meridian's passing through the two stations must be determined by a time-keeper, a very small error in the going of which will produce a very considerable error in the computation. Other methods which have been proposed, are, by comparing a degree of the meridian in any latitude, with a degree of the curve perpendicular to the meridian in the same latitude; by comparing the measures of degrees of the curves perpendicular to the meridian in different latitudes; and by comparing an arc of a meridian with an arc of the parallel of latitude that crosses it. The theorems connected with these and some other methods are investigated by Professor Playfair in the Edinburgh Transactions, vol. v. to which, together with the books mentioned at the end of the 1st section of this chapter, the reader is referred for much useful information on this highly interesting subject.

Having thus solved the chief problems connected with Trigonometrical Surveying, the student is now presented with the following examples by way of exercise.

Ex. 1. The angle subtended by two distant objects at a third object is $66°30'39''$; one of those objects appeared under an elevation of $25'47''$, the other under a depression of $1''$. Required the reduced horizontal angle. Ans. $66°30'36\frac{1}{2}''$.

Ex. 2. Going along a straight and horizontal road which passed by a tower, I wished to find its height, and for this purpose measured two equal distances each of 84 feet, and at the extremities of those distances took three angles of elevation of the top of the tower, viz. $36°50'$, $21°24'$, and $14°$. What is the height of the tower? Ans. 53·96 feet.

Ex. 3. Investigate General Roy's rule for the spherical excess, given in the scholium to prob. 8.

Ex. 4. The three sides of a triangle measured on the earth's surface (and reduced to the level of the sea) are 17, 18, and 10 miles: what is the spherical excess? Ans. $1''·096$.

Ex. 5. The base and perpendicular of another triangle are 24 and 15 miles. Required the spherical excess.
Ans. $2''21'''52\frac{1}{2}^{iv}$.

Ex. 6. In a triangle two sides are 18 and 23 miles, and they include an angle of $50°24'36''$. What is the spherical excess? Ans. $2''·31639$.

Ex. 7. The length of a base measured at an elevation of 38 feet above the level of the sea is 34286 feet: required the length when reduced to that level? Ans. 34285·9379.

Ex. 8. Given the latitude of a place $48°51'$N, the sun's declination $18°30'$N, and the sun's apparent altitude at $10^h\ 11^m\ 26^s$AM, $52°35'$; to find the angle that the vertical on which the sum is, makes with the meridian. Ans. $45°23'2''\frac{5}{6}$.

Ex. 9. When the sun's longitude is $29°13'43''$, what is his right ascension? The obliquity of the ecliptic being $23°27'40''$. Ans. $27°10'13''\frac{5}{6}$.

Ex. 10. Required the longitude of the sun, when his right ascension and declination are $32°46'52''\frac{1}{2}$, and $13°13'27''$N respectively. See the theorems in the scholium to prob. 12.

Ex. 11. The right ascension of the star α Ursæ majoris is $162°50'34''$, and the declination $62°50'$N: what are the longitude and latitude? The obliquity of the ecliptic being as above.

Ex. 12. Given the measure of a degree on the meridian in N. lat. $49°3'$, 60833 fathoms, and of another in N. lat. $12°32'$, 60494 fathoms: to find the ratio of the earth's axes.

Ex. 13. Demonstrate that, if the earth's figure be that of an oblate spheroid, a degree of the earth's equator is the first of two mean proportionals between the last and first degrees of latitude.

Ex. 14. Demonstrate that the degrees of the terrestrial meridian, in receding from the equator towards the poles, are increased very nearly in the duplicate ratio of the sine of the latitude.

Ex. 15. If p be the measure of a degree of a great circle perpendicular to a meridian at a certain point, m that of the corresponding degree on the meridian itself, and d the length of a degree on an oblique arc, that arc making an angle a with the meridian, then is $d = \frac{pm}{p + (m-p) \sin^2 a}$. Required a demonstration of this theorem.

ON THE NATURE AND SOLUTION OF EQUATIONS IN GENERAL.

1. In order to investigate the general properties of the higher equations, let there be assumed between an unknown quantity x, and given quantities a, b, c, d, an equation constituted of the continued product of uniform factors: thus

$$(x-a) \times (x-b) \times (x-c) \times (x-d) = 0.$$

This, by performing the multiplications, and arranging the final product according to the powers or *dimensions* of x, becomes

$$x^4 \left.\begin{matrix}-a\\-b\\-c\\-d\end{matrix}\right\} x^3 \left.\begin{matrix}+ab\\+ac\\+ad\\+bc\\+bd\\+cd\end{matrix}\right\} x^2 \left.\begin{matrix}-abc\\-abd\\-acd\\-bcd\end{matrix}\right\} x + abcd = 0 \quad . \; . \; . \; (A)$$

Now it is obvious that the assemblage of terms which compose the first side of this equation may become equal to nothing in four different ways; namely, by supposing either $x = a$, or $x = b$, or $x = c$, or $x = d$; for in either case one or other of the factors $x - a$, $x - b$, $x - c$, $x - d$, will be equal to nothing, and nothing multiplied by any quantity whatever will give *nothing* for the product. If any other value e be put for x, then none of the factors $e-a$, $e-b$, $e-c$, $e-d$, being equal to nothing, their continued product cannot be equal to nothing. There are therefore, in the proposed equation, four roots or values of x; and that which characterises these roots is, that on substituting each of them successively instead of x, the aggregate of the terms of the equation vanishes, by the opposition of the signs $+$ and $-$.

The preceding equation is only of the fourth power or degree; but it is manifest that the above remark applies to equations of higher or lower dimensions: viz. that in general an equation of any degree whatever has as many roots as there are units in the exponent of the highest power of the unknown quantity, and that each root has the property of rendering, by its substitution in place of the unknown quantity, the aggregate of all the terms of the equation equal to nothing.

It must be observed that we cannot have all at once $x = a$, $x = b$, $x = c$, &c. for the roots of the equation; but that the particular equations $x - a = 0$, $x - b = 0$, $x - c = 0$, &c. obtain only in a *disjunctive* sense. They exist as factors in the same equation, because algebra gives, by one and the same formula, not only the solution of the particular problem from which that formula may have originated, but also the solution of all problems which have similar conditions. The different roots of the equation satisfy the respective conditions; and those roots may differ from one another, by their *quantity*, and by their *mode* of existence.

It is true, we say frequently that the roots of an equation are $x = a$, $x = b$, $x = c$, &c. as though those values of x existed conjunctively; but this manner of speaking is an abbreviation, which it is necessary to understand in the sense explained above.

2. In the equation A, all the roots are positive; but if the factors which constitute the equation had been $x + a$, $x + b$, $x + c$, $x + d$, the roots would have been negative or subtractive. Thus

$$x^4\left.\begin{matrix}+a\\+b\\+c\\+d\end{matrix}\right\} x^3\left.\begin{matrix}+ab\\+ac\\+ad\\+bc\\+bd\\+cd\end{matrix}\right\} x^2\left.\begin{matrix}+abc\\+abd\\+acd\\+bcd\end{matrix}\right\} x + abcd = 0 \quad . \; . \; . \; . \quad \text{(B)}$$

has negative roots, those roots being $x = -a$, $x = -b$, $x = -c$, $x = -d$: and here again we are apt to apply them disjunctively.

3. Some equations have their roots in part positive, in part negative. Such is the following:

$$x^3\left.\begin{matrix}-a\\-b\\+c\end{matrix}\right\} x^2\left.\begin{matrix}+ab\\-ac\\-bc\end{matrix}\right\} x + abc = 0 \quad . \; . \; . \; . \; . \; . \; . \quad \text{(C)}$$

Here are the two positive roots, viz. $x = a$, $x = b$; and one negative root, viz. $x = -c$: the equation being constituted of the continued product of the three factors, $x - a = 0$, $x - b = 0$, $x + c = 0$.

From an inspection of the equations A, B, C, it may be inferred, that a complete equation consists of a number of terms exceeding by *unity* the number of its roots.

4. The preceding equations have been considered as formed from equations of the first degree, and then each of them contains so many of those constituent equations as there are units in the exponent of its degree. But an equation which exceeds the second dimension may be considered as composed of one or more equations of the second degree, or of the third, &c, combined, if it be necessary, with equations of the first degree, in such manner, that the product of all those constituent equations shall form the proposed equation. Indeed, when an equation is formed by the successive multiplication of several simple equations, quadratic equations, cubic equations, &c. are formed ; which of course may be regarded as factors of the resulting equation.

5. It sometimes happens that an equation contains imaginary roots ; and then they will be fouud also in its constituent equations. This class of roots always enters an equation by pairs ; because they may be considered as containing, in their expression at least, one *even* radical placed before a negative quantity, and because an *even* radical is necessarily preceded by the double sign $\pm$. Let, for example, the equation be $x^4 - (2a - 2c)x^3 + (a^2 + b^2 - 4ac + c^2 + d^2)x^2 + (2a^2c + 2b^2c - 2ac^2 - 2ad^2)x + (a^2 + b^2) . (c^2 + d^2) = 0$. This may be regarded as constituted of the two subjoined quadratic equations, $x^2 - 2ax + a^2 + b^2 = 0$, $x^2 + 2cx + c^2 + d^2 = 0$: and each of these quadratics contains two imaginary roots; the first givng $x = a \pm b\sqrt{-1}$, and the second $x = -c \pm d\sqrt{-1}$.

In the equation resulting from the product of these two quadratics, the coefficients of the powers of the unknown quantity, and of the last term of the equation, are real quantities, though the constituent equations contain imaginary quantities ; the reason is, that these latter disappear by means of addition and multiplication.

The same will take place in the equation $(x - a) . (x + b) . (x^2 + 2cx + c^2 + d^2) = 0$, which is formed of two equations of the first degree, and one equation of the second whose roots are imaginary.

These remarks being premised, the subsequent general theorems will be easily established.

THEOREM I.

Whatever be the species of the roots of an equation, when the equation is arranged according to the powers of the unknown quantity, if the first term be positive, and have unity for its coefficient, the following properties may be traced :

I. The first term of the equation is the unknown quantity raised to the power denoted by the number of roots.

II. The second term contains the unknown quantity raised to a power less than the former by unity, with a coefficient equal to the sum of the roots taken with contrary signs.

III. The third term contains the unknown quantity raised to a power less by 2 than that of the first term, with a coefficient equal to the sum of all the products which can be formed by multiplying all the roots two and two.

IV. The fourth term contains the unknown quantity raised to a power less by 3 than that of the first term, with a coefficient equal to the sum of all the products which can be made by multiplying any three of the roots with contrary signs.

V. And so on to the last term, which is the continued product of all the roots taken with contrary signs.

All this is evident from inspection of the equations exhibited in arts. 1, 2, 3, 5.

Cor. 1. Therefore an equation having all its roots real, but some positive, the others negative, will want its second term when the sum of the positive roots is equal to the sum of the negative roots. Thus, for example, the equation c will want its second term, if $a + b = c$.

Cor. 2. An equation whose roots are all imaginary will want the second term, if the sum of the real quantities which enter into the expression of the roots, is partly positive, partly negative, and has the result reduced to nothing, the imaginary parts mutually destroying each other by addition in each pair of roots. Thus, the first equation of art. 5 will want the second term if $-2a + 2c = 0$, or $a = c$. The second equation of the same article, which has its roots partly real, partly imaginary, will want the second term if $b - a + 2c = 0$, or $a - b = 2c$.

Cor. 3. An equation will want its third term, if the sum of the products of the roots taken two and two, is partly positive, partly negative, and these mutually destroy each other.

Remark. An *incomplete* equation may be thrown into the form of *complete* equations, by introducing, with the coefficient *a cypher*, the absent powers of the unknown quantity : thus for the equation $x^3 + r = 0$, may be written $x^3 + 0\,x^2 + 0\,x + r = 0$. This in some cases will be useful.

Cor. 4. An equation with positive roots may be transformed into another which shall have negative roots of the same value, and reciprocally. In order to this, it is only necessary to change the signs of the alternate terms, beginning with the second. Thus, for example, if instead of the equation $x^3 - 8x^2 + 17x - 10=0$, which has three positive roots 1, 2, and 5, we write $x^3 + 8x^2 + 17x + 10 = 0$, this latter equation will have three negative roots $x = -1$, $x = -2$, $x = -5$. In like manner, if instead of the equation $x^3 + 2x^2 - 13x + 10=0$, which has two positive roots $x=1$, $x=2$, and one negative root $x = -5$, there be taken $x^3 - 2x^2 - 13x - 10 = 0$, this latter equation will have two negative roots, $x = -1$, $x = -2$, and one positive root $x = 5$.

In general, if there be taken the two equations, $(x-a) \times (x-b) \times (x-c) \times (x-d) \times \&c. =0$, and $(x+a) \times (x+b) \times (x+c) \times (x+d) \times \&c. =0$, of which the roots are the same in magnitude, but with different signs : if these equations be developed by actual multiplication, and the terms arranged according to the powers of x, as in arts. 1, 2 ; it will be seen that the second terms of the two equations will be affected with different signs, the third terms with like signs, the fourth terms with different signs, &c.

When an equation has not all its terms, the deficient terms must be supplied by cyphers, before the preceding rule can be applied,

Cor. 5. The sum of the roots of an equation, the sum of their squares, the sum of their cubes, &c. may be found without knowing the roots themselves. For, let an equation of any degree or dimension, m, be $x^m + fx^{m-1} + gx^{m-2} + hx^{m-3} + \&c. = 0$, its roots being a, b, c, d, &c. Then we shall have,

1st. The sum of the first powers of the roots, that is, of the roots themselves, or $a + b + c + \&c. = -f$; since the coefficient of the unknown quantity in the second term, is equal to the sum of the roots taken with different signs.

2dly. The sum of the squares of the roots, is equal to the square of the coefficient of the second term made less yy twice the coefficient of the third term : viz. $a^2 + b^2 + c^2 + \&c. = f^2 - 2g$. For, if the polynomial $a + b + c + \&c.$ be squared, it will be found that the square contains the sum of the squares of the terms, a, b, c, &c. *plus* twice the sum of the products formed by multiplying two and two all the roots a, b, c, &c. That is, $(a + b + c + \&c.)^2 = a^2+b^2+c^2 + \&c. + 2(ab + ac + bc + \&c)$. But it is obvious, from equa. A, B, that $(a + b + c + \&c.)^2=f^2$, and $(ab + ac + bc + \&c)=g$. Thus we have $f^2 = (a^2 + b^2 + c^2 + \&c.) + 2g$; and consequently $a^2 + b^2 + c^2 + \&c. = f^2 - 2g$.

3dly. The sum of the cubes of the roots, is equal to 3 times the rectangle of the coefficient of the second and third terms, made less by the cube of the coefficient of the second term, and 3 times the coefficient of the fourth term: viz. $a^3 + b^3 + c^3 + \&c. = -f^3 + 3fg - 3h$. For we shall by actual involution, have $(a + b + c + \&c.)^3 = a^3 + b^3 + c^3 + \&c. + 3(a+b+c) \times (ab + ac + bc) - 3abc$. But $(a+b+c+\&c.)^3 = -f^3$, $(a+b+c+\&c.) \times (ab + ac + bc + \&c.) = -fg$, $abc = -h$. Hence therefore, $-f^3 = a^3 + b^3 + c^3 + \&c. - 3fg + 3h$; and consequently, $a^3 + b^3 + c^3 + \&c. = -f^3 + 3fg - 3h$. And so on, for other powers of the roots.

THEOREM II.

In every equation, which contains only real roots:

I. If all the roots are positive, the terms of the equation will be + and — alternately.

II. If all the roots are negative, all the terms will have the sign +.

III. If the roots are partly positive, partly negative, there will be as many positive roots as there are *variations* of signs, and as many negative roots as there are *permanencies* of signs; these variations and permanencies being observed from one term to the following through the whole extent of the equation.

In all these, either the equations are complete in their terms, or they are made so.

The first part of this theorem is evident from the examination of equation A; and the second from equation B.

To demonstrate the third, we revert to the equation C (art. 3), which has two positive roots, and one negative. It may happen that either $c > a + b$, or $c < a + b$.

In the first case, the second term is positive, and the third is negative; because, having $c > a + b$, we shall have $ac + bc > (a + b)^2 > ab$. And, as the last term is positive, we see that from the first to the second there is a permanence of signs; from the second to the third a variation of signs; and from the third to the fourth another variation of signs. Thus there are two variations and one permanence of signs; that is, as many variations as there are positive roots, and as many permanencies as there are negative roots.

In the second case, the second term of the equation is negative, and the third may be either positive or negative. If that term is positive, there will be from the first to the second a variation of signs; from the second to the third another variation; from the third to the fourth a permanence; making in all two variations and one permanence of signs. If the third term be negative, there will be one variation of signs

from the first to the second; one permanence from the second to the third; and one variation from the third to the fourth: thus making again two variations and one permanence. The number of variations of signs therefore, in this case as well as in the former, is the same as that of the positive roots; and the number of permanencies, the same as that of the negative roots.

Corol. Whence it follows, that if it be known by any means whatever, that an equation contains only real roots, it is also known how many of them are positive, and how many negative. Suppose, for example, it be known that, in the equation $x^5 + 3x^4 - 23x^3 - 27x^2 + 166x - 120 = 0$, all the roots are real: it may immediately be concluded that there are *three* positive and *two* negative roots. In fact this equation has the three positive roots $x = 1$, $x = 2$, $x = 3$; and two negative roots, $x = -4$, $x = -5$.

If the equation were incomplete, the absent terms must be supplied by adopting cyphers for coefficients, and those terms must be marked with the ambiguous sign $\pm$. Thus, if the equation were

$$x^5 - 20x^3 + 30x^2 + 19x - 30 = 0,$$

all the roots being real, and the second term wanting, it must be written thus:

$$x^5 \pm 0x^4 - 20x^3 + 30x^2 + 19x - 30 = 0.$$

Then it will be seen that, whether the second term be positive or negative, there will be 3 variations and 2 permanencies of signs: and consequently the equation has 3 positive and 2 negative roots. The roots in fact are, 1, 2, 3, -1, -5.

This rule only obtains with regard to equations whose roots are real. If, for example, it were inferred that, because the equation $x^2 + 2x + 5 = 0$ had two permanencies of signs, it had two negative roots, the conclusion would be erroneous: for both the roots of this equation are imaginary.

THEOREM III.

Every equation may be transformed into another whose roots shall be greater or less by a given quantity.

In any equation whatever, of which x is unknown, (the equations A, B, C, for example) make $x = z + m$, z being a new unknown quantity, m any given quantity, positive or negative: then substituting, instead of x and its powers, their values resulting from the hypothesis that $x = z + m$; so shall there arise an equation, whose roots shall be greater or less than the roots of the primitive equation, by the assumed quantity m.

Corol. The principal use of this transformation is, to take away any term out of an equation. Thus, to transform an equation into one which shall want the *second* term, let m be so assumed that $nm - a = 0$, or $m = \frac{a}{n}$, n being the index of the highest power of the unknown quantity, and a the coefficient of the second term of the equation, with its sign changed : then if the roots of the transformed equation can be found, the roots of the original equation may also be found, because $x = z + \frac{a}{n}$.

THEOREM IV.

Every equation may be transformed into another, whose roots shall be equal to the roots of the first multiplied or divided by a given quantity.

1. Let the equation be $z^3 + az^2 + bz + c = 0$: if we put $fz = x$, or $z = \frac{x}{f}$, the transformed equation will be $x^3 + fax^2 + f^2bx + f^3c = 0$, of which the roots are the respective products of the roots of the primitive equation multiplied into the quantity f.

By means of this transformation, an equation with fractional quantities, may be changed into another which shall be free from them. Suppose the equation were $z^3 + \frac{az^2}{g} + \frac{bz}{h} + \frac{d}{k} = 0$: multiplying the whole by the product of the denominators, there would arise $ghkz^3 + hkaz^2 + gkbz + ghd = 0$: then assuming $ghkz = x$, or $z = \frac{x}{ghk}$, the transformed equa. would be $x^3 + hkax^2 + g^2k^2hbx + g^3k^3h^3d = 0$.

The same transformation may be adopted, to exterminate the radical quantities which affect certain terms of an equation. Thus, let there be given the equation $z^3 + az^2 \surd k + bz + c \surd k$: make $z \surd k = x$; then will the transformed equation be $x^3 + akx^2 + bkx + ck^2 = 0$, in which there are no radical quantities.

2. Take, for one more example, the equation $z^3 + az^2 + bz + c = 0$. Make $\frac{z}{f} = x$; then will the equation be transformed to $x^3 + \frac{ax^2}{f} + \frac{bx}{f^2} + \frac{c}{f^3} = 0$, in which the roots

are equal to the quotients of those of the primitive equations divided by f.

It is obvious that, by analogous methods, an equation may be transformed into another, the roots of which shall be to those of the proposed equation, in any required ratio. But the subject need not be enlarged on here. The preceding succinct view will suffice for the usual purposes, so far as relates to the nature and chief properties of equations. We shall therefore conclude this chapter with a summary of the most useful rules for the solution of equations of different degrees, besides those already given in the first volume.

I. *Rules for the Solution of Quadratics by Tables of Sines and Tangents.*

1. If the equation be of the form $x^2+px=q$:

Make $\tan A = \frac{2}{p}\sqrt{q}$; then will the two roots be,

$x=+\tan \frac{1}{2}A\sqrt{q} \ldots\ldots x=-\cot \frac{1}{2}A\sqrt{q}$.

2. For quadratics of the form $x^2-px=q$.

Make, as before, $\tan A = \frac{2}{p}\sqrt{q}$: then will

$x=-\tan \frac{1}{2}A\sqrt{q} \ldots\ldots x=+\cot \frac{1}{2}A\sqrt{q}$.

3. For quadratics of the form $x^2+px=-q$.

Make $\sin A = \frac{2}{p}\sqrt{q}$: then will

$x=-\tan \frac{1}{2}A\sqrt{q} \ldots\ldots x=-\cot \frac{1}{2}A\sqrt{q}$.

4. For quadratics of the form $x^2-px=-q$.

Make $\sin A = \frac{2}{p}\sqrt{q}$: then will

$x=+\tan \frac{1}{2}A\sqrt{q} \ldots\ldots x=+\cot \frac{1}{2}A\sqrt{q}$.

In the last two cases, if $\frac{2}{p}\sqrt{q}$ exceed unity, $\sin A$ is imaginary, and consequently the values of x.

The logarithmic application of these formulæ is very simple. Thus, in case 1st. Find A by making

$$10+\log 2+\tfrac{1}{2}\log q-\log p=\log \tan A.$$

$$\text{Then } \log x = \begin{cases} +\log \tan \frac{1}{2}A+\frac{1}{2}\log q-10. \\ -(\log \cot \frac{1}{2}A+\frac{1}{2}\log q-10). \end{cases}$$

Note. This method of solving quadratics, is chiefly of use when the quantities p and q are large integers, or complex fractions.

II. *Rules for the Solution of Cubic Equations by Tables of Sines, Tangents, and Secants.*

1. For cubics of the form $x^3+px\pm q=0$.

Make $\tan B = \dfrac{\frac{1}{3}p}{q} \cdot 2\sqrt{\tfrac{1}{3}p}$ . . . . . $\tan A = \sqrt[3]{\tan \tfrac{1}{2}B}$.

$$\text{Then } x = \mp \cot 2A \cdot 2\sqrt{p}.$$

2. For cubics of the form $x^3-px\pm q=0$.

Make $\sin B = \dfrac{\frac{1}{3}p}{q} \cdot 2\sqrt{\tfrac{1}{3}p}$ . . . . . $\tan A = \sqrt[3]{\tan \tfrac{1}{2}B}$.

$$\text{Then } x = \mp \operatorname{cosec} 2A \cdot 2\sqrt{\tfrac{1}{3}p}.$$

Here, if the value of sin B should exceed unity, B would be imaginary, and the equation would fall in what is called the *irreducible case* of cubics. In that case we must make $\operatorname{cosec} 3a = \dfrac{\frac{1}{3}p}{q} \cdot 2\sqrt{\tfrac{1}{3}p}$: and then the three roots would be

$$x = \pm \sin A \cdot 2\sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin (60^\circ - A) \cdot 2\sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin (60^\circ + A) \cdot 2\sqrt{\tfrac{1}{3}p}.$$

If the value af sin B were 1, we should have $B=90^\circ$, $\tan A = 1$; therefore $A = 45^\circ$, and $x = \mp 2\sqrt{\tfrac{1}{3}p}$. But this would not be the only root. The second solution would give $\operatorname{cosec} 3A=1$: therefore $A = \dfrac{90^\circ}{3}$; and then

$$x = \pm \sin 30^\circ \cdot 2\sqrt{\tfrac{1}{3}p} = \pm \sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin 30^\circ \cdot 2\sqrt{\tfrac{1}{3}p} = \pm \sqrt{\tfrac{1}{3}p}.$$
$$x = \mp \sin 30^\circ \cdot 2\sqrt{\tfrac{1}{3}p} = \mp 2\sqrt{\tfrac{1}{3}p}.$$

Here it is obvious that the first two roots are equal, that their sum is equal to the third with a contrary sign, and that this third is the one which is produced from the first solution*.

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* The tables of sines, tangents, &c. besides their use in trigonometry, and in the solution of the equations, are also very useful in finding the value of algebraic expressions where extraction of roots would be otherwise required. Thus if a and b be any two quantities, of which a is the greater. Find x, z, &c. so, that $\tan x = \sqrt{\dfrac{b}{a}}$, $\sin z = \sqrt{\dfrac{b}{a}}$, $\sec y = \dfrac{a}{b}$, $\tan u = \dfrac{b}{a}$, and $\sin t = \dfrac{b}{a}$: then will

$$\log\sqrt{(a^2-b^2)} = \log a + \log \sin y = \log b + \log \tan y.$$
$$\log\sqrt{(a^2-b^2)} = \tfrac{1}{2}[\log (a+b) + \log (a-b)].$$
$$\log\sqrt{(a^2+b^2)} = \log a + \log \sec u = \log b + \log \operatorname{cosec} u.$$
$$\log\sqrt{(a+b)} = \tfrac{1}{2}\log a + \log \sec x = \tfrac{1}{2}\log a + \tfrac{1}{2}\log 2 + \log \cos \tfrac{1}{2}y.$$
$$\log\sqrt{(a-b)} = \tfrac{1}{2}\log a + \log \cos z = \tfrac{1}{2}\log a + \tfrac{1}{2}\log 2 + \log \sin \tfrac{1}{2}y.$$

$$\log (a\pm b)^{\frac{m}{n}} = \frac{m}{n}\,[\log a + \log \cos t + \log \tan 45^\circ \pm \tfrac{1}{2}t].$$

The first three of these formulæ will often be usefcl, when two sides of a right-angled triangle are given, to find the third.

In these solutions, the double signs in the value of x, relate to the double signs in the value of q.

N. B. Cardan's Rule for the solution of Cubics is given in the first volume of this course.

III. *Solution of Biquadratic Equations.*

Let the proposed biquadratic be $x^4 + 2px^3 = qx^2 + rx + s$. Now $(x^2 + px + n)^2 = x^4 + 2px^3 + (p^2 + 2n)x^2 + 2pnx + n^2$: if therefore $(p^2 + 2n)x^2 + 2pnx + n^2$ be added to both sides of the proposed biquadratic, the first will become a complete square $(x^2 + px + n)^2$, and the latter part $(p^2 + 2n + q)x^2 + (2pn + r)x + n^2 + s$, is a complete square if $4(p^2 + 2n + q) . (n^2 + s) = 2pn + r^2$; that is, multiplying and arranging the terms according to the dimensions, of n, if $8n^3 + 4qn^2 + (8s - 4rp)n + 4qs + 4p^2s - r^2 = 0$. From this equation let a value of n be obtained, and substituted in the equation $(x^2 + px + n)^2 = (p^2 + 2n + q)x^2 + (2pn + r)x + n^2 + s$; then, extracting the square root on both sides

$$x^2 + px + n = \pm \left\{ \sqrt{(p^2+2n+q)}x + \sqrt{(n^2+s)} \right\} \begin{cases} \text{when } 2pn+r \\ \text{is positive;} \end{cases}$$

$$\text{or } x^2+px+n = \pm \left\{ \sqrt{(p^2+2n+q)}x - \sqrt{(n^2+s)} \right\} \begin{cases} \text{when } 2pn+r \\ \text{is negative.} \end{cases}$$

And from these two quadratics, the four roots of the given biquadratic may be determined*.

Note. Whenever, by taking away the second term of a biquadratic, after the manner described in cor. th. 3, that fourth term also vanishes, the roots may immediately be obtained by the solution of a quadratic only.

A biquadratic may also be solved independently of cubics, in the following cases :

1. When the difference between the coefficient of the third term, and the square of half that of the second term, is equal to the coefficient of the fourth term, divided by half that of the second. Then if p be the coefficient of the second term, the equation will be reduced to a quadratic by dividing it by $x^2 \pm \frac{1}{2}px$.

2. When the last term is negative, and equal to the square of the coefficient of the fourth term divided by 4 times that of the third term, *minus* the square of that of the second : then to complete the square, subtract the terms of the proposed biquadratic from $(x^2 \pm \frac{1}{2}px)^2$, and add the remainder to both its sides.

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* This rule for solving biquadratics, by conceiving each to be the difference of two squares, is frequently ascribed to Dr. *Waring*; but its original inventor was Mr. *Thomas Simpson*, formerly Professor of Mathematics in the Royal Military Academy.

3. When the coefficient of the fourth term divided by that of the second term, gives for a quotient the square root of the last term : then to complete the square, add the square of half the coefficient of the second term, to twice the square root of the last term, multiply the sum by x^2, from the product take the third term, and add the remainder to both sides of the biquadratic.

4. The fourth term will be made to go out by the usual operation for taking away the second term, when the difference between the cube of half the coefficient of the second term and half the product of the coefficients of the second and third term, is equal to the coefficient of the fourth term.

IV. *Euler's Rule for the Solution of Biquadratics.*

Let $x^4 - ax^2 - bx - c = 0$, be the given biquadratic equation wanting the second term. Take $f = \frac{1}{2}a$, $g = \frac{1}{16}a^2 + \frac{1}{4}c$, and $h = \frac{1}{64}b^2$, or $\sqrt{h} = \frac{1}{8}b$; with which values of f, g, h, form the cubic equation $z^3 - fz^2 + gz - h = 0$. Find the roots of this cubic equation, and let them be called p, q, r, then shall the four roots of the proposed biquadratic be these following: viz.

When $\frac{1}{8}b$ is positive :	When $\frac{1}{8}b$ is negative :
1. $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$.	$x = \sqrt{p} + \sqrt{q} - \sqrt{r}$.
2. $x = \sqrt{p} - \sqrt{q} - \sqrt{r}$.	$x = \sqrt{p} - \sqrt{q} + \sqrt{r}$.
3. $x = -\sqrt{p} + \sqrt{q} - \sqrt{r}$.	$x = -\sqrt{p} + \sqrt{q} + \sqrt{r}$.
4. $x = -\sqrt{p} + \sqrt{q} + \sqrt{r}$.	$x = -\sqrt{p} - \sqrt{q} - \sqrt{r}$.

Note. 1. In any biquadratic equation having all its terms, if $\frac{3}{8}$ of the square of the coefficient of the 2d term be greater than the product of the coefficients of the 1st and 3d terms, or $\frac{3}{8}$ of the square of the coefficient of the 4th term be greater than the product of the coefficients of the 3d and 5th terms, or $\frac{1}{3}$ of the square of the coefficient of the 3d term greater than the product of the coefficients of the 2d and 4th terms; then all the roots of that equation will be real and unequal : but if either of the said parts of those squares be less than either of those products, the equation will have imaginary roots.

2. In a biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$, of which two roots are impossible, and d an affirmative quantity, then the two possible roots will be both negative, or both affirmative, according as $a^3 - 4ab + 8c$, is an affirmative or a negative quantity, if the signs of the coefficients, a, b, c, d, are neither all affirmative, nor alternately — and + *.

* Various *general* rules for the solution of equations have been given by Demoivre, Bezout, Legrange, Atkinson, Horner, Holdred, &c.; but the most universal in their application are approximating rules, of which

EXAMPLES.

Ex. 1. Find the roots of the equation $x^2+\frac{7}{44}x=\frac{1695}{12716}$, by tables of sines and tangents.

Here $p=\frac{7}{44}$, $q=\frac{1695}{12716}$, and the equation agrees with the 1st form. Also $\tan A=\frac{88}{7}\sqrt{\frac{1695}{12716}}$, and $x=\tan\frac{1}{2}A=\sqrt{\frac{1695}{12716}}$.

In logarithms thus:

Log 1695 =	3·2291697
Arith. com. log 12716 =	5·8956495
sum + 10 =	19·1248192
half sum =	9·5624096
log 88 =	1·9444827
Arith. com. log 7 =	9·1549020
sum — 10 = log tan A =	10·6617943 = log tan 77°42′31″¾;
log tan ½A =	9·9061115 = log tan 38°51′15″⅞;
log $\sqrt{q}$, as above =	9·5624096
sum — 10 = log x = —	$\bar{1}$·4685211 = log ·2941176.

This value of x, viz. ·2941176, is nearly equal to $\frac{5}{17}$. To find whether that is the exact root, take the arithmetical complement of the last logarithm, viz. 0·5314379, and consider it as the logarithm of the denominator of a fraction whose numerator is unity: thus is the fraction found to be $\frac{1}{3\,4}$ exactly, and this is manifestly equal to $\frac{5}{17}$. As to the other root of the equation, it is equal to $-\frac{1695}{12716}\div\frac{5}{17}=-\frac{339}{748}$.

Ex. 2. Find the roots of the cubic equation $x^3-\frac{403}{441}x+\frac{46}{147}=0$, by a table of sines.

Here $p=\frac{403}{441}$, $q=\frac{46}{147}$, the second term is negative, and $4p^3>27q^2$: so that the example falls under the irreducible case. Hence, $\sin 3A=\frac{3\times 46}{147}\times\frac{441}{403}\times\frac{1}{2\sqrt{\frac{403}{3\cdot 441}}}=\frac{414}{463}\cdot\frac{1}{\sqrt{\frac{1612}{1323}}}$.

The three values of x therefore, are

$$x=\quad \sin A\sqrt{\frac{1612}{1323}}.$$

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a very simple and useful one is given in our first volume. See also *J. R. Young's Algebra.*

$$x = \sin(60^\circ - A)\sqrt{\frac{1612}{1323}}.$$

$$x = -\sin(60^\circ + A)\sqrt{\frac{1612}{1323}}.$$

The logarithmic computation is subjoined.

$$\begin{array}{rl}
\text{Log } 1612 = & 3{\cdot}2073650 \\
\text{Arith. com. log } 1323 = & 6{\cdot}8784402 \\
\hline
\text{sum} - 10 \ldots\ldots = & 0{\cdot}0858052 \\
\text{half sum} = & 0{\cdot}0429026 \text{ const. log.} \\
\hline
\text{Arith. com. const. log} = & 9{\cdot}9570974 \\
\log 414 \ldots = & 2{\cdot}6170003 \\
\text{Arith. com. log } 403\ . = & 7{\cdot}3946950 \\
\hline
\log \sin 3A \ldots = & 9{\cdot}9687927 = \log \sin 68^\circ\ 32'\ 18''\tfrac{1}{2}. \\
\hline
\text{Log } \sin A = & 9{\cdot}5891206 \\
\text{const. log} \quad & 0{\cdot}0429026 \\
\hline
1.\ \text{sum}-10 = \log x = - & \overline{1}{\cdot}6320232 = \log {\cdot}4285714 = \log \tfrac{3}{7}. \\
\hline
\text{Log } \sin(60^\circ - A) = & 9{\cdot}7810061 \\
\text{const. log} \ldots\ldots = & 0{\cdot}0429026 \\
\hline
2.\ \text{sum}-10 = \log x = - & \overline{1}{\cdot}8239087 = \log {\cdot}6666666 = \log \tfrac{2}{3}; \\
\hline
\text{Log } \sin(60^\circ + A) = & 9{\cdot}9966060 \\
\text{const. log} \ldots\ldots = & 0{\cdot}0429026 \\
\hline
3.\ \text{sum}-10 = \log -x = & 0{\cdot}0395086 = \log 1{\cdot}095238 = \log \tfrac{23}{21}.
\end{array}$$

So that the three roots are $\frac{3}{7}$, $\frac{2}{3}$, and $-\frac{23}{21}$; of which the first two are together equal to the third with its sign changed, as they ought to be.

Ex. 3. Find the roots of the biquadratic $x^4 - 25x^2 + 60x - 36 = 0$, by Euler's rule.

Here $a = 25$, $b = -60$, and $c = 36$; therefore

$$f = \frac{25}{2},\ g = \frac{625}{16} + 9 = \frac{769}{16},\ \text{and } h = \frac{225}{4}.$$

Consequently the cubic equation will be

$$z^3 - \frac{25}{2}z^2 + \frac{769}{16}z - \frac{225}{4} = 0.$$

The three roots of which are

$$z = \frac{9}{4} = p, \text{ and } z = 4 = q, \text{ and } z = \frac{25}{4} = r;$$

the square roots of these are $\sqrt{p} = \frac{3}{2}$, $\sqrt{q} = 2$ or $\frac{4}{2}$, $\sqrt{r} = \frac{5}{2}$. Hence, as the value of $\frac{1}{8}b$ is negative, the four roots are

$$\begin{array}{lrcl}
\text{1st } & x = & \tfrac{3}{2} + \tfrac{4}{2} - \tfrac{5}{2} = & 1, \\
\text{2d. } & x = & \tfrac{3}{2} - \tfrac{4}{2} + \tfrac{5}{2} = & 2, \\
\text{3d. } & x = & -\tfrac{3}{2} + \tfrac{4}{2} + \tfrac{5}{2} = & 3, \\
\text{4th. } & x = & -\tfrac{3}{2} - \tfrac{4}{2} - \tfrac{5}{2} = & -6.
\end{array}$$

Ex. 4. Produce a quadratic equation whose roots shall be $\frac{3}{4}$ and $\frac{4}{5}$. Ans. $x^2 - \frac{31}{20}x + \frac{3}{5} = 0$.

Ex. 5. Produce a cubic equation whose roots shall be 2, 5, and -3. Ans. $x^3-4x^2-11x+30=0$.

Ex. 6. Produce a biquadratic which shall have for the roots 1, 4, -5, and 6 respectively.
Ans. $x^4-6x^3-21x^2+146x-120=0$.

Ex. 7. Find x, when $x^2+347x=22110$.
Ans. $x=55$, $x=-402$.

Ex. 8. Find the roots of the quadratic $x^2-\frac{55}{12}x-\frac{325}{6}$.

Ans. $x=10$, $x=-\frac{65}{12}$.

Ex. 9. Solve the equation $x^2-\frac{264}{25}x=-\frac{695}{25}$.

Ans. $x=5$, $x=\frac{139}{25}$.

Ex. 10. Given $x^2-24113x=-481860$, to find x.
Ans. $x=20$, $x=24093$.

Ex. 11. Find the roots of the equation $x^3-3x-1=0$.
Ans. The roots are sine $70°$, $-\sin 50°$, and $-\sin 10°$, to a radius $=2$; or the roots are twice the sines of those arcs as given in the tables.

Ex. 12. Find the real root of $x^3-x-6=0$.
Ans. $\frac{2}{3}\sqrt{3}\times\sec 54°44'20''$.

Ex. 13. Find the real root of $25x^3+75x-46=0$.
Ans. $2\cot 74°27'48''$.

Ex. 14. Given $x^4-8x^3-12x^2+84x-63=0$, to find x by quadratics. Ans $x=2+\sqrt{7}\pm\sqrt{(11+\sqrt{7})}$.

Ex. 15. Given $x^4+36x^3-400x^2-3168x+7744=0$, to find x by quadratics. Ans. $x=11+\sqrt{209}$.

Ex. 16. Given $x^4+24x^3-114x^2-24x+1=0$, to find x.
Ans. $x=\pm\sqrt{197}-14$, $x=2\pm\sqrt{5}$.

Ex. 17. Find x, when $x^4-12x-5=0$.
Ans. $x=1\pm\sqrt{2}$, $x=-1\pm 2\sqrt{-1}$.

Ex, 18. Find x, when $x^4-12x^3+47x^2-72x+36=0$.
Ans. $x=1$, or 2, or 3, or 6.

Ex. 19. Given $x^5-5ax^4-80a^2x^3-68a^3x^2+7a^4x+a^5=0$, to find x.
Ans. $x=-a$, $x=6a\pm a\sqrt{37}$, $x=\pm a\sqrt{10}-3a$.

ON THE NATURE AND PROPERTIES OF CURVES, AND THE CONSTRUCTION OF EQUATIONS.

SECTION I.

Nature and Properties of Curves.

DEF. 1. A curve is a line whose several parts proceed in different directions, and are successively posited towards different points in space, which also may be cut by one right line in two or more points.

If all the points in the curve may be included in one plane, the curve is called a *plane* curve; but if they cannot all be comprised in one plane, then is the curve one of *double curvature.*

Since the word direction implies straight lines, and in strictness no part of a curve is a right line, some geometers prefer defining curves otherwise: thus, in a straight line, to be called the line of the abscissas, from a certain point let a line arbitrarily taken be called the abscissa, and denoted (commonly) by x: at the several points corresponding to the different values of x, let straight lines be continually drawn, making a certain angle with the line of the abscissas: these straight lines being regulated in length according to a certain law or equation, are called ordinates; and the line or figure in which their extremities are continually found is, in general, a curve line. This definition, however, is not free from objection; for a right line may be denoted by an equation between its abscissas and ordinates, such as $y=ax+b$.

Curves are distinguished into algebraical or geometrical, and transcendental or mechanical.

Def. 2. *Algebraical* or geometrical curves, are those in which the relations of the abscissas to the ordinates can be denoted by a common algebraical expression: such, for example, as the equations to the conic sections, given at page 536, &c. vol. i.

Def. 3. *Transcendental* or mechanical curves, are such as cannot be so defined or expressed by a pure algebraical equation; or when they are expressed by an equation, having one

of its terms a variable quantity, or a curve line. Thus, $y = \log x$, $y = \text{A} \,.\, \sin x$, $y = \text{A} \,.\, \cos x$, $y = \text{A}^x$, are equations to transcendental curves; and the latter in particular is an equation to an *exponential* curve.

Def. 4. Curves that turn round a fixed point or centre, gradually receding from it, are called *spiral* or *radial* curves.

Def. 5. *Family* or *tribe* of curves, is an assemblage of several curves of different kinds, all defined by the same equation of an indeterminate degree; but differently according to the diversity of their kind. For example, suppose an equation of an indeterminate degree, $a^{m-1}\,x = y^m$: if $m = 2$, then will $ax = y^2$; if $m = 3$, then will $a^2x = y^3$; if $m = 4$, then is $a^3x = y^4$; &c.: all which curves are said to be of the same family or tribe.

Def. 6. The *axis* of a figure is a right line passing through the centre of a curve, when it has one: if it bisects the ordinates, it is called a *diameter*.

Def. 7. An *asymptote* is a right line which continually approaches towards a curve, but never can touch it, unless the curve could be extended to an infinite distance.

Def. 8. An abscissa and an ordinate, whether right or oblique, are, when spoken of together, frequently termed *co-ordinates*.

ART. 1. The most convenient mode of classing algebraical curves, is according to the orders or dimensions of the equations which express the relation between the co-ordinates. For then the equation for the same curve, remaining always of the same order so long as each of the assumed systems of co-ordinates is supposed to retain constantly the same inclination of ordinate to abscissa, while referred to different points of the curve, however the axis and the origin of the abscissas, or even the inclination of the co-ordinates in different systems, may vary; the same curve will never be ranked under different orders, according to this method. If therefore we take, for a distinctive character, the number of dimensions which the co-ordinates, whether rectangular or oblique, form in the equation, we shall not disturb the order of the classes, by changing the axis and the origin of the abscissas, or by varying the inclination of the co-ordinates.

2. As algebraists call orders of different kinds of equations, those which constitute the greater or less number of dimensions, they distinguish by the same name the different kinds of resulting lines. Consequently the general equation of the first order being $0 = \alpha + \beta x + \gamma y$; we may refer to the first order all the lines which, by taking x and y for the co-

ordinates, whether rectangular or oblique, give rise to this equation. But this equation comprises the right line alone, which is the most simple of all lines ; and since, for this reason, the name of curve does not properly apply to the first order, we do not usually distinguish the different orders by the name of curve lines, but simply by the generic term of lines : hence the first order of lines does not comprehend any curves, but solely the right line.

As for the rest, it is indifferent whether the co-ordinates are perpendicular or not ; for if the ordinates make with the axis an angle φ whose sine is μ and consine ν, we can refer the equation to that of the rectangular co-ordinates, by making $y = \frac{u}{\mu}$, and $x = \frac{\nu u}{\mu} + t$; which will give for an equation between the perpendiculars t and u,

$$0 = \alpha + \beta t + (\frac{\beta\nu}{\mu} + \frac{\gamma}{\mu})u.$$

Thus it follows evidently, that the signification of the equation is not limited by supposing the ordinates to be rightly applied : and it will be the same with equations of superior orders, which will not be less general though the co-ordinates are perpendicular. Hence, since the determination of the inclination of the ordinates applied to the axis, takes nothing from the generality of a general equation of any order whatever, we put no restriction on its signification by supposing the co-ordinates rectangular ; and the equation will be of the same order whether the co-ordinates be rectangular or oblique.

3. All the lines of the second order will be comprised in the general equation

$$0 = \alpha + \beta x + \gamma y + \delta x^2 + \varepsilon xy + \zeta y^2 ;$$

that is to say, we may class among lines of the second order all the curve lines which this equation expresses. x and y denoting the rectangular co-ordinates. These curve lines are therefore the most simple of all, since there are no curves in the first order of lines ; it is for this reason that some writers call them curves of the first order. But the curves included in this equation are better known under the name of CONIC SECTIONS, because they all result from sections of the cone. The different kinds of these lines are the ellipse, the circle, or ellipse with equal axes, the parabola, and the hyperbola ; the properties of all which may be deduced with facility from the preceding general equation. Or this equation may be transformed into the subjoined one :

$$y^2 + \frac{\varepsilon x + \gamma}{\zeta} y + \frac{\delta x^2 + \beta x + \alpha}{\zeta} = 0 :$$

and this again may be reduced to the still more simple form $y^2 = fx^2 + gx + h$.

Here, when the first term fx^2 is *affirmative*, the curve expressed by the equation is a hyperbola; when fx^2 is *negative*, the curve is an ellipse: when that term is *absent*, the curve is a parabola. When x is taken upon a *diameter*, the equations reduce to those already given in sect. 4, ch. 1.

The mode of effecting these transformations is omitted for the sake of brevity. This section contains a *summary*, not an *investigation* of properties: the latter would require many volumes, instead of a section.

4. Under lines of the third order, or curves of the second, are classed all those which may be expressed by the equation
$$0 = \alpha + \beta x + \gamma y + \delta x^2 + \varepsilon xy + \zeta y^2 + \eta x^3 + \theta x^2 y + \iota xy^2 + \kappa y^3.$$
And in like manner we regard as lines of the fourth order, those curves which are furnished by the general equation
$$0 = \alpha + \beta x + \gamma y + \delta x^2 + \varepsilon xy + \zeta y^2 + \eta x^3 + \theta x^2 y + \iota xy^2 + \kappa y^3 + \lambda x^4 + \mu x^3 y + \nu x^2 y^2 + \xi xy^3 + o y^4;$$
taking always x and y for rectangular co-ordinates. In the most general equation of the third order, there are 10 constant quantities, and in that of the fourth order 15, which may be determined at pleasure; whence it results that the kinds of lines of the third order, and much more those of the fourth order, are considerably more numerous than those of the second.

5. It will not be easy to conceive, from what has gone before, what are the curve lines that appertain to the fifth, sixth, seventh, or any higher order; but as it is necessary to add to the general equation of the fourth order, the terms
$$x^5,\ x^4y,\ x^3y^2,\ x^2y^3,\ xy^4,\ y^5,$$
with their respective constant coefficients, to have the general equation comprising all the lines of the fifth order, this latter will be composed of 21 terms: and the general equation comprehending all the lines of the sixth order, will have 28 terms; and so on, conformably to the law of the triangular numbers. Thus, the most general equation for lines of the order n, will contain $\frac{(n+1)\,.\,(n+2)}{1\,.\,2}$ terms, and as many constant letters, which may be determined at pleasure.

6. Since the order of the proposed equation between the co-ordinates makes known that of the curve line; whenever we have given an algebraic equation between the co-ordinates x and y, or t and u, we know at once to what order it is necessary to refer the curve represented by that equation. If the equation be irrational, it must be freed from radicals, and if there be fractions, they must be made to disappear; this done, the greatest number of dimensions formed by the va-

riable quantities x and y, will indicate the order to which the line belongs. Thus, the curve which is denoted by this equation $y^2 - ax = 0$, will be of the second order of lines, or of the first order of curves; while the curve represented by the equation $y^2 = x\sqrt{(a^2 - x^2)}$, will be of the third order (that is, the fourth order of lines), because the equation is of the fourth order when freed from radicals; and the line which is indicated by the equation $y = \frac{a^3 - ax^2}{a^2 + x^2}$, will be of the third order, or of the second order of curves, because the equation, when the fraction is made to disappear, becomes $a^2y + x^2y = a^3 - ax^2$, where the term x^2y contains three dimensions.

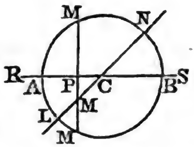
7. It is possible that one and the same equation may give different curves, according as the applicates or ordinates fall upon the axis perpendicularly or under a given obliquity. For instance, this equation, $y^2 = ax - x^2$, gives a circle, when the co-ordinates are supposed perpendicular; but when the co-ordinates are oblique, the curve represented by the same equation will be an ellipse. Yet all these different curves appertain to the same order, because the transformation of rectangular into oblique co-ordinates and the contrary, does not affect the order of the curve, or of its equation. Hence, though the magnitude of the angles which the ordinates form with the axis, neither augments nor diminishes the generality of the equation, which expresses the lines of each order; yet, a particular equation being given, the curve which it expresses can only be determined when the angle between the co-ordinates is determined also.

8. That a curve line may relate properly to the order indicated by the equation, it is requisite that this equation be not decomposable into rational factors; for if it could be composed of two or more such factors, it would then comprehend as many equations, each of which would generate a particular line, and the re-union of these lines would be all that the equation proposed could represent. Those equations, then, which may be decomposed into such factors, do not comprise one continued curve, but several at once, each of which may be expressed by a particular equation; and such combinations of separate curves are denoted by the term complex curves.

Thus, the equation $y^2 = ay + xy - ax$, which seems to appertain to a line of the second order, if it be reduced to zero by making $y^2 - ay - xy + ax = 0$, will be composed of the factors $(y - x)(y - a) = 0$; it therefore comprises the two equations $y - x = 0$, and $y - a = 0$, both of which belong to the right line: the first forms with the axis at the origin of the abscissas an angle equal to half a right angle;

and the second is parallel to the axis, and drawn at a distance $= a$. These two lines, considered together, are comprised in the proposed equation $y^2 = ay + xy - ax$. In like manner we may regard as complex this equation, $y^4 - xy^3 - a^2x^2 - ay^3 + ax^2y + a^2xy = 0$; for its factors being $(y-x)$ $(y-a)$ $(y^2 - ax) = 0$, instead of denoting one continued line of the fourth order, it comprises three distinct lines, viz. two right lines, and one curve denoted by the equa. $y^2 - ax = 0$.

9. We may therefore form at pleasure any complex lines whatever, which shall contain 2 or more right lines or curves. For, if the nature of each line is expressed by an equation referred to the same axis, and to the same origin of the abscissas, and after having reduced each equation to zero, we multiply them one by another, there will result a complex equation which at once comprises all the lines assumed. For example, if from the centre C, with a radius $CA = a$, a circle be described; and further, if a right line LN be drawn through the centre C; then we may, for any assumed axis, find an equation which will at once include the circle and the right line, as though these two lines formed only one.



Suppose there be taken for an axis the diameter AB, that forms with the right line LN an angle equal to half a right angle: having placed the origin of the abscissas in A, make the abscissa $AP = x$, and the applicate or ordinate $PM = y$; we shall have for the right line, $PM = CP = a - x$; and since the point M of the right line falls on the side of those ordinates which are reckoned negative, we have $y = -a + x$, or $y - x + a = 0$: but, for the circle, we have $PM^2 = AP \cdot PB$, and $BP = 2a - x$, which gives $y^2 = 2ax - x^2$, or $y^2 + x^2 - 2ax = 0$. Multiplying these two equations together we obtain the complex equation of the third order,

$y^3 - y^2x + yx^2 - x^3 + ay^2 - 2axy + 3ax^2 - 2a^2x = 0$, which represents, at once, the circle and the right line. Hence, we shall find that to the abscissa $AP = x$, correspond three ordinates, namely, two for the circle, and one for the right line. Let, for example, $x = \frac{1}{2}a$, the equation will become $y^3 + \frac{1}{2}ay^2 - \frac{3}{4}a^2y - \frac{3}{8}a^3 = 0$; whence we first find $y + \frac{1}{2}a = 0$, and by dividing by this root, we obtain $y^2 - \frac{3}{4}a^2 = 0$, the two roots of which being taken and ranked with the former, give the three following values of y:

I. $y = -\frac{1}{2}a$.
II. $y = +\frac{1}{2}a\sqrt{3}$.
III. $y = -\frac{1}{2}a\sqrt{3}$.

We see, therefore, that the whole is represented by one

equation, as if the circle together with the right line formed only one continued curve.

10. This difference between simple and complex curves being once established, it is manifest that the lines of the second order are either continued curves, or complex lines formed of two right lines; for if the general equation have rational factors, they must be of the first order, and consequently will denote right lines. Lines of the third order will be either simple, or complex, formed either of a right line and a line of the second order, or of three right lines. In like manner, line of the fourth order will be continued and simple, or complex, comprising a right line and a line of the third order, or two lines of the second order, or lastly, four right lines. Complex lines of the fifth and superior orders will be susceptible of an analogous combination, and of a similar enumeration. Hence it follows, that any order whatever of lines may comprise, at once, all the lines of inferior order, that is to say, that they may contain a complex line of any inferior orders with one or more right lines, or with lines of the second, third, &c. order; so that if we sum the numbers of each order, appertaining to the simple lines, there will result the number indicating the order of the complex line.

Def. 9. That is called an *hyperbolic* leg, or branch of a curve, which approaches constantly to some asymptote; and that a *parabolic* one which has no asymptote.

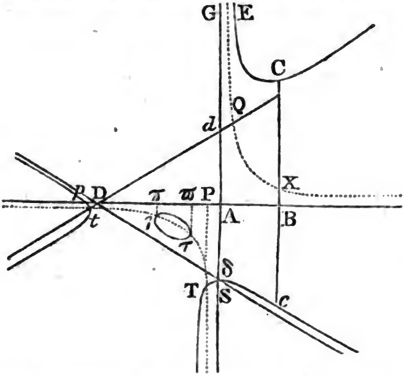
Art. 11. All the legs of curves of the second and higher kinds, as well as of the first, infinitely drawn out, will be of either the hyperbolic or the parabolic kind: and these legs are best known from the tangents. For if the point of contact be at an infinite distance, the tangent of a hyperbolic leg will coincide with the asymptote, and the tangent of a parabolic leg will recede *in infinitum*, will vanish and be nowhere found. Therefore the asymptote of any leg is found by seeking the tangent to that leg at a point infinitely distant: and the course, or way of an infinite leg, is found by seeking the position of any right line which is parallel to the tangent where the point of contact goes off *in infinitum*; for this right line is directed the same way with the infinite leg.

Sir Isaac Newton's reduction of all lines of the third order, to four cases of equations; with the enumeration of those lines.

CASE I.

12. All the lines of the first, third, fifth, and seventh order, or of any odd order, have at least two legs or sides proceeding on *ad infinitum*, and towards contrary parts. And all lines

of the *third* order have two such legs or branches running out contrary ways, and towards which no other of their infinite legs (except in the Cartesian parabola) tend. If the legs are of the *hyperbolic* kind, let GAS be their asymptote; and to it



let the parallel CBc be drawn, terminated (if possible) at both ends at the curve. Let this parallel be bisected in X, and then will the locus of that point X be the conical or common hyperbola XQ, one of whose asymptotes is AS. Let its other asymptote be AB. Then the equation by which the relation between the ordinate $BC = y$, and the abscissa $AB = x$, is determined, will always be of this form: viz.

$$xy^2 + ey = ax^3 + bx^2 + cx + d \quad . \quad . \quad . \quad \text{(I.)}$$

Here the coefficients e, a, b, c, d, denote given quantities, affected with their signs + and —, of which terms any one may be wanting, provided the figure through their defect does not become transformed into a conic section. The conical hyperbola XQ may coincide with its asymptotes, that is, the point X may come to be in the line AB; and then the term $+ ey$ will be wanting.

CASE II.

13. But if the right line CBc cannot be terminated both ways at the curve, but will come to it only in one point; then draw any line in a given position which shall cut the asymptote AS in A; as also any other right line, as BC, parallel to the asymptote, and meeting the curve in the point C; then the equation, by which the relation between the ordinate BC and the abscissa AB is determined, will always assume this form: viz. $xy = ax^3 + bx^2 + cx + d$ . . . . (II.)

CASE III.

14. If the opposite legs be of the parabolic kind, draw the right line CBC, terminated at both ends (if possible) at the curve, and running according to the course of the legs; which line bisect in B: then shall the locus of B be a right line. Let that right line be AB, terminated at any given point, as A: then the equation, by which the relation between the ordinate BC and the abscissa AB is determined, will always be of this form: $y^2 = ax^3 + bx^2 + cx + d$ . . . . (III.)

CASE IV.

15. If the right line CBC meet the curve only in one point, and therefore cannot be terminated at the curve at both ends: let the point where it comes to the curve be C, and let that right line at the point B, fall on any other right line given in position, as AB, and terminated at any given point, as A. Then will the equation expressing the relation between BC and AB, assume this form:

$$y = ax^3 + bx^2 + cx + d \quad . \ . \ . \ . \quad \text{(IV.)}$$

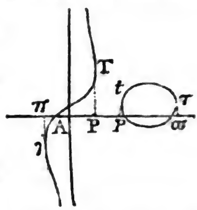
16. In the first case, or that of equation I, if the term ax^3 be affirmative, the figure will be a triple hyperbola with six hyperbolic legs, which will run on infinitely by the three asymptotes, of which none are parallel, two legs towards each asymptote, and towards contrary parts; and these asymptotes, if the term bx^2 be not wanting in the equation, will mutually intersect each other in 3 points, forming thereby the triangle $\text{D}d\delta$. But if the term bx^2 be wanting, they will all converge to the same point. This kind of hyperbola is called *redundant*, because it exceeds the conic hyperbola in the number of its hyperbolic legs.

In every redundant hyperbola, if neither the term ey be wanting, nor $b^2 - 4ac = ae\sqrt{a}$, the curve will have *no* diameter: but if either of those occur separately, it will have only *one* diameter; and *three*, if they both happen. Such diameter will always pass through the intersection of two of the asymptotes, and bisect all right lines which are terminated each way by those asymptotes, and which are parallel to the third asymptote.

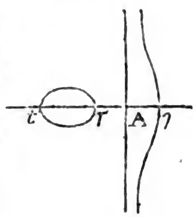
17. If the redundant hyperbola have no diameter, let the four roots or values of x in the equation $ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}e^2 = 0$, be sought; and suppose them to be AP, Aϖ, Aπ, and Ap (see the preceding figure). Let the ordinates PT, ϖT, $\pi\tau$, pt, be erected; they shall touch the curve in the points, T, τ, τ, t, and by that contact shall give the limits of the curve, by which its species will be discovered.

Thus, if all the roots AP, Aϖ, Aπ, Ap, be real, and have the same sign, and are unequal, the curve will consist of three hyperbolas and an oval: viz. an *inscribed hyperbola*, as EC; a *circumscribed hyperbola*, as Tδc; an *ambigeneal hyperbola*, (i. e. lying within one asymptote and beyond another) as *pt*; and an oval τ7. This is reckoned the *first* species. Other relations of the roots of the equation, give 8 more different species of redundant hyperbolas without diameters; 12 each with but *one* diameter; 2 each with *three* diameters; and 9 each with three asymptotes converging to a common point. Some of these have ovals, some points of decussation, and in some the ovals degenerate into nodes or knots.

18. When the term ax^3 in equa. I, is negative, the figure expressed by that equation will be a deficient or *defective hyperbola;* that is, it will have fewer legs than the complete conic hyperbola. Such is the marginal figure, representing Newton's 33d species; which is constituted of an *anguineal* or serpentine hyperbola (both legs approaching a common asymptote by means of a contrary flexure), and a conjugate oval. There are 6 species of defective hyperbolas, each having but one asymptote, and only two hyperbolic legs, running out contrary ways, *ad infinitum;* the asymptote being the first and principal ordinate. When the term ey is not absent, the figure will have no diameter; when it is absent, the figure will have one diameter. Of this latter class there are 7 different species, one of which, namely, Newton's 40th species, is exhibited in the margin.

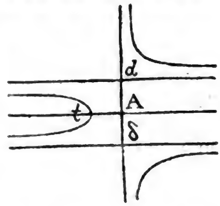


19. If, in equation I, the term ax^3 be wanting, but bx^2 not, the figure expressed by the equation remaining, will be a parabolic hyperbola, having two hyperbolic legs to one asymptote, and two parabolic legs converging one and the same way. When the term ey is not wanting, the figure will have no diameter; if that term be wanting, the figure will have one diameter. There are 7 species appertaining to the former case; and 4 to the latter.



20. When, in equa. I, the terms ax^3, bx^2, are wanting, or when that equation becomes $xy^2 + ey = cx + d$, it expresses a figure consisting of three hyperbolas opposite to one another, one lying between the parallel asymptotes, and the

other two without: each of these curves having three asymptotes, one of which is the first and principal ordinate, the other two parallel to the abscissa, and equally distant from it; as in the annexed figure of Newton's 60th species. Otherwise the said equation expresses two opposite circumscribed hyperbolas, and an anguineal hyperbola between the asymptotes. Under this class there are 4 species, called by Newton *Hyperbolismæ of an hyperbola*. By hyperbolismæ of a figure he means to signify when the ordinate comes out, by dividing the rectangle under the ordinate of a given conic section and a given right line, by the common abscissa.

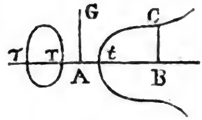


21. When the term cx^2 is negative, the figure expressed by the equation $xy^2 + ey = - cx^2 + d$, is either a serpentine hyperbola, having only one asymptote, being the principal ordinate; or else it is a conchoidal figure. Under this class there are three species, called *Hyperbolismæ of an ellipse.*

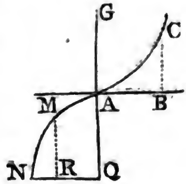
22. When the term cx^2 is absent, the equa. $xy^2 + ey = d$, expresses two hyperbolas, lying, not in the opposite angles of the asymptotes (as in the conic hyperbola), but in the adjacent angles. Here there are only 2 species, one consisting of an inscribed and an ambigeneal hyperbola, the other of two inscribed hyperbolas. These two species are called the *Hyperbolismæ of a parabola.*

23. In the second case of equations, or that of equation II, there is but one figure; which has four infinite legs. Of these, two are hyperbolic about one asymptote, tending towards contrary parts, and two converging parabolic legs, making with the former nearly the figure of a *trident*, the familiar name given to this species. This is the *Cartesian* parabola, by which equations of 6 dimensions are sometimes constructed: it is the 66th species of Newton's enumeration.

24. The third case of equations, or equa. III, expresses a figure having two parabolic legs running out contrary ways: of these there are 5 different species, called *diverging* or *bell-form* parabolas; of which 2 have ovals, 1 is nodate, 1 punctate, and 1 cuspidate. The figure shows Newton's 67th species; in which the oval must always be so small that no right line which cuts it twice can cut the parabolic curve *ct* more than once.



25. In the case to which equa. IV. refers, there is but one species. It expresses the *cubical* parabola with contrary legs. This curve may easily be described mechanically by means of a square and an equilateral hyperbola. Its most simple property is, that RM (parallel to AQ) always varies as $QN^3 - QR^3$.



26. Thus according to Newton there are 72 species of lines of the third order. But Mr. Stirling discovered four more species of redundant hyperbolas; and Mr. Stone two more species of deficient hyperbolas, expressed by the equation $xy^2 = bx^2 + cx + d$: i. e. in the case when $bx^2 + cx + d = 0$, has two unequal negative roots, and in that where the equation has two equal negative roots. So that there are *at least* 78 different species of lines of the third order. Indeed Euler, who classes all the varieties of lines of the third order under 16 general species, affirms that they comprehend more than 80 varieties; of which the preceding enumeration necessarily comprises nearly the whole.

27. Lines of the fourth order are divided by Euler into 146 classes; and these comprise more than 5000 varieties: they all flow from the different relations of the quantities in the 10 general equations subjoined.

$$\left.\begin{array}{ll}
1. & y^4 + fx^2y^2 + gxy^2 + hx^2y + iy^2 + kxy + ly \\
2. & y^4 + fxy^2 + gx^2y + hxy^2 + ixy + ky \\
3. & x^2y^2 + fy^3 + gx^2y + hy^2 + ky \\
4. & x^2y^2 + fy^3 + gy^2 + hxy + iy \\
5. & y^3 + fxy^2 + gx^2y + hy \\
6. & y^3 + fxy^2 + gxy + hy
\end{array}\right\} = ax^4 + bx^3 + cx^2 + dx + e.$$

$$\left.\begin{array}{ll}
7. & y^4 + ex^3y + fxy^3 + gxy^2 + hy^2 + ixy + ky \\
8. & x^3y + exy^2 + fx^2y + gy^2 + hxy + iy \\
9. & x^2y + ey^3 + fxy^2 + gxy + hy \\
10. & x^2y + ey^3 + fy^2 + gxy + hy
\end{array}\right\} = ax^3 + bx^2 + cx + d.$$

28. Lines of the fifth and higher orders of necessity become still more numerous; and present too many varieties to admit of any classification, at least in moderate compass. Instead, therefore, of dwelling upon these, we shall give a concise sketch of the most curious and important properties of curve lines in general, as they have been deduced from a contemplation of the nature and mutual relation of the roots of the equations representing those curves. Thus a curve being called of n dimensions, or a line of the nth order when its representative equation rises to n dimensions; then since for every different value of x there are n values of y, it will commonly happen that the ordinate will cut the curve in n or in $n-2$, $n-4$, &c., points, according as the equation has

n, or $n-2$, $n-4$, &c., possible roots. It is not however to be inferred, that a right line will cut a curve of n dimensions, in n, or $n-2$, $n-4$, &c., points, only ; for if this were the case, a line of the 2d order, a conic section for instance, could only be cut by a right line in two points :—but this is manifestly incorrect, for though a conic parabola will be cut in two points by a right line oblique to the axis, yet a right line parallel to the axis can only cut the curve in one point.

29. It is true in general, that lines of the n order cannot be cut by a right line in more than n points ; but it does not hence follow, that any right line whatever will cut in n points every line of that order ; it may happen that the number of intersections is $n-1$, $n-2$, $n-3$, &c., to $n-n$. The number of intersections that any right line whatever makes with a given curve line, cannot therefore determine the order to which a curve line appertains. For, as Euler remarks, if the number of intersections be $=n$, it does not follow that the curve belongs to the n order, but it may be referred to some superior order ; indeed it may happen that the curve is not algebraic, but transcendental. This case excepted, however, Euler contends that we may always affirm positively that a curve line which is cut by a right line in n points, cannot belong to an order of lines inferior to n. Thus, when a right line cuts a curve in 4 points, it is certain that the curve does not belong to either the second or third order of lines ; but whether it be referred to the fourth, or a superior order, or whether it be transcendental, is not to be decided by analysis.

30. Dr. Waring has carried this inquiry a step further than Euler, and has demonstrated that there are curves of any number of odd orders, that cut a right line in 2, 4, 6, &c., points only ; and of any number of even orders, that cut a right line in 3, 5, 7, &c. points only ; whence this author likewise infers, that the order of the curve cannot be announced from the number of points in which it cuts a right line. See his Proprietates Algebraicarum Curvarum.

31. Every geometrical curve being continued, either returns into itself, or goes on to an infinite distance. And if any plane curve has two infinite branches or legs, they join one another either at a finite, or at an infinite distance.

32. In any curve, if tangents be drawn to all points of the curve ; and if they always cut the abscissa at a finite distance from its origin ; that curve has an asymptote, otherwise not.

33. A line of any order may have as many asymptotes as it has dimensions, and no more.

34. An asymptote may intersect the curve is so many

points abating two, as the equation of the curve has dimensions. Thus, in a conic section, which is the second order of lines, the asymptote does not cut the curve at all ; in the third order it can only cut it in one point ; in the fourth order, in two points ; and so on.

35. If a curve have as many asymptotes as it has dimensions, and a right line be drawn to cut them all, the parts of that measured from the asymptotes to the curve, will together be equal to the parts measured in the same direction, from the curve to the asymptotes.

36. If a curve of n dimensions have n asymptotes, then the content of the n abscissas will be to the content of the n ordinates, in the same ratio in the curve and asymptotes ; the sum of their n subnormals, to ordinates perpendicular to their abscissas, will be equal to the curve and the asymptotes ; and they will have the same central and diametral curves.

37. If two curves of n and m dimensions have a common asymptote ; or the terms of the equations to the curves of the greatest dimensions have a common divisor ; then the curves cannot intersect each other in $n \times m$ points, possible or impossible. If the two curves have a common general centre, and intersect each other in $n \times m$ points, then the sum of the affirmative abscissas, &c., to those points, will be equal to the sum of the negative : and the sum of the n subnormals to a curve which has a general centre, will be proportional to the distance from that centre.

38. Lines of the third, fifth, seventh, &c. order, or any odd number, have, as before remarked, at least two infinite legs or branches, running contrary ways ; while in lines of the second, fourth, sixth, or any even number of dimensions, the figure may return into itself and be contained within certain limits.

39. If the right lines AP, PM, forming a given angle APM, cut a geometrical line of any order in as many points as it has dimensions, the product of the segments of the first terminated by P and the curve, will always be to the product of the segments of the latter, terminated by the same point and the curve, in an invariable ratio.

40. With respect to double, triple, quadruple, and other multiple points, or the points of intersection of 2, 3, 4, or more branches of a curve, their nature and number may be estimated by means of the following principles. 1. A curve of the n order is determinate when it is subjected to pass through the number $\frac{(n+1) \cdot (n+2)}{2} - 1$ points. 2. A curve of the n

order cannot intersect a curve of the m order in more than mn points.

Hence it follows that a curve of the second order, for example, can always pass through 5 given points (not in the same right line), and cannot meet a curve of the m order in more than mn points; and it is impossible that the curve of the m order should have 5 points whose degrees of multiplicity make together more than $2m$ points. Thus, a line of the fourth order cannot have four double points; because the line of the second order which would pass through these four double points, and through a fifth simple point of the curve of the fourth dimension, would meet 9 times; which is impossible, since there can only be an intersection 2×4 or 8 times.

For the same reason, a curve line of the 5th order cannot with one triple point, have more than three double points: and in a similar manner we may reason for curves of higher orders.

Again, from the known proposition, that we can always make a line of the third order pass through nine points, and that a curve of that order cannot meet a curve of the m order in more than $3m$ points, we may conclude that a curve of the m order cannot have nine points, the degrees of multiplicity of which make together a number greater than $3m$. Thus, a line of the fifth order cannot have more than 6 double points; a line of the 6th order, which cannot have more than one quadruple point, cannot have with that quadruple point more than 6 double points; nor with two triple points more than 5 double points; nor even with one triple point more than 7 double points. Analogous conclusions obtain with respect to a line of the fourth order, which we may cause to pass through 14 points, and which can only meet a curve of the m order in $4m$ points, and so on.

41. The properties of curves of a superior order agree, under certain modifications, with those of all inferior orders. For though some line or lines become evanescent, and others become infinite, some coincide, others become equal; some points coincide, and others are removed to an infinite distance; yet, under these circumstances, the general properties still hold good with regard to the remaining quantities; so that whatever is demonstrated generally of any order, holds true in the inferior orders: and, on the contrary, there is hardly any property of the inferior orders, but there is some similar to it, in the superior ones.

For as in the conic sections, if two parallel lines are drawn to terminate at the section, the right line that bisects these will bisect all other lines parallel to them; and is therefore called a *diameter* of the figure, and the bisected lines *ordi-*

nates, and the intersections of the diameter with the curve *vertices*; the common intersection of all the diameters the *centre*; and that diameter which is perpendicular to the ordinates, the *vertex*. So likewise in other curves, if two parallel lines be drawn, each to cut the curve in the number of points that indicate the order of the curve; the right line that cuts these parallels so, that the sum of the parts on one side of the line, estimated to the curve, is equal to the sum of the parts on the other side, it will cut in the same manner all other lines parallel to them that meet the curve in the same number of points; in this case also the divided lines are called *ordinates*, the line so dividing them a *diameter*, the intersection of the diameter and the curve *vertices*; the common intersection of two or more diameters the *centre*; the diameter perpendicular to the ordinates, if there be any such, the *axis*; and when all the diameters concur in one point, that is the *general centre*.

Again, the conic hyperbola, having a line of the second order, has two asymptotes; so likewise, that of the third order may have three; that of the fourth, four; and so on: and they can have no more. And as the parts of any right line between the hyperbola and its asymptotes are equal; so likewise in the third order of lines, if any line be drawn cutting the curve and its asymptotes in three points; the sum of two parts of it falling the same way from the asymptotes to the curve will be equal to the part falling the contrary way from the third asymptote to the curve; and so of higher curves.

Also in the conic sections which are not parabolic: as the square of the ordinate, or the rectangle of the parts of it on each side of the diameter, is to the rectangle of the parts of the diameter, terminating at the vertices, in a constant ratio, viz. that of the latus rectum, to the transverse diameter. So in non-parabolic curves of the next superior order, the solid under the three ordinates, is to the solid under the three abscissas, or the distances to the three vertices, in a certain given ratio. In which ratio if there be taken three lines proportional to the three diameters, each to each; then each of these three lines may be called a *latus rectum*, and each of the corresponding diameters a *transverse diameter*. And, in the common, or Apollonian parabola, which has but one vertex for one diameter, the rectangle of the ordinates is equal to the rectangle of the abscissa and latus rectum: so, in those curves of the second kind, or lines of the third kind, which have only two vertices to the same diameter, the solid under the three ordinates is equal to the solid under the two abscissas, and a given line, which may be reckoned the latus rectum.

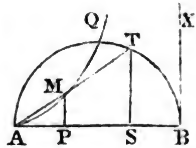
Lastly, since in the conic sections where two parallel lines terminating at the curve both ways, are cut by two other parallels likewise terminated by the curve; we have the rectangle of the parts of one of the first, to the rectangle of the parts of one of the second lines, as the rectangle of the parts of the second of the former, to the rectangle of the parts of the second of the latter pair, passing also through the common point of their division. So, when four such lines are drawn in a curve of the second kind, and each meeting it in three points; the solid under the parts of the first line, will be to that under the parts of the third, as the solid under the parts of the second, to that under the parts of the fourth. And the analogy between curves of different orders may be carried much further: but as enough is given for the objects of this work, we shall now present a few of the most useful problems.

PROBLEM I.

Knowing the characteristic property, or the manner of description of a curve, to find its equation.

This in most cases will be a matter of great simplicity; because the manner of description suggests the relation between the ordinates and their corresponding abscissas; and this relation, when expressed algebraically, is no other than the *equation to the curve*. Examples of this problem have already occurred at p. 536, &c. vol. i.: to which the following are now added to exercise the student.

Ex. 1. Find the equation to the cissoid of Diocles; whose manner of description is as below.



From any two points P, S, at equal distances from the extremities A, B, of the diameter of a semicircle, draw ST, PM, perpendicular to AB. From the point T where ST cuts the semicircle, draw a right line AT, it will cut PM in M, a point of the curve required.

Now, by theor. 87 Geom. $AS \cdot SB = ST^2$; and by the construction, $AS \cdot SB = AP \cdot PB$. Also the similar triangles APM, AST, give $AP : PM :: AS \cdot ST :: PB : ST = \frac{PM \cdot PB}{AP}$. Consequently $ST^2 = \frac{PM^2 \cdot PB^2}{AP^2} = AP \cdot PB$; and lastly, $\frac{PM^2 \cdot PB^2}{PB} = AP \cdot AP^2$,

or $PA^3 = PB \cdot PM^2$. Hence, if the diameter $AB = d$, $AP = x$, $PM = y$; the equation is $x^3 = y^2(d - x)$.

The complete cissoid will have another branch equal and similar to AMQ, but turned contrary ways; being drawn by means of points T′ falling in the other half of the circle. But the same equation will comprehend both branches of the curve; because the square of $-y$, as well as that of $+y$, is positive.

Cor. All cissoids are similar figures; because the abscissæ and ordinates of several cissoids will be in the same ratio, when either of them is in a given ratio to the diameter of its generating circle.

Ex. 2. Find the equation to the logarithmic curve, whose fundamental property is, that when the abscissas increase or decrease in arithmetical progression, the corresponding ordinates increase or decrease in geometrical progression.

Ans. $y = a^x$, a being the number whose logarithm is 1, in the system of logarithms represented by the curve.

Ex. 3. Find the equation to the curve called the *Witch*, whose construction is this: a semicircle whose diameter is AB being given; draw, from any point P in the diameter, a perpendicular ordinate, cutting the semicircle in D, and terminating in M, so that AP : PD :: AB : PM : then is M always a point in the curve.

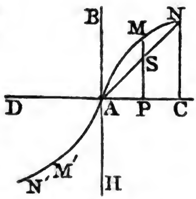
Ans. $y = d\sqrt{\frac{d-x}{x}}$.

PROBLEM II.

Given the equation to a curve, to describe it, and trace its chief properties.

The method of effecting this is obvious: for any abscissas being assumed, the corresponding values of the ordinates become known from the equation; and thus the curve may be traced, and its limits and properties developed.

Ex. 1. Let the equation $y^3 = a^2x$, or $y = \sqrt[3]{a^2x}$, to a line of the third order, be proposed.



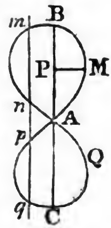
First, drawing the two indefinite lines BH, DC, to make an angle BAC equal to the assumed angle of the co-ordinates; let the values of x be taken upon AC, and those of y upon AB, or upon lines parallel to AB. Then, let it be inquired whether the curve passes through the point A, or not. In order to this, we must ascertain what y will be when $x=0$: and in that case $y=\sqrt[3]{(a^2\times 0)}$, that is, $y=0$. There-

fore the curve passes through A. Let it next be ascertained whether the curve cuts the axis AC in any other point; in order to which, find the value of x when $y=0$; this will be $\sqrt[3]{a^2x}=0$, or $x=0$. Consequently the curve does not cut the axis in any other point than A. Make $x=$ AP $=\frac{1}{2}a$, and the given equa. will become $y=\sqrt[3]{\frac{1}{2}a^3}=a\sqrt[3]{\frac{1}{2}}$. Therefore draw PM parallel to AB, and equal to $a\sqrt[3]{\frac{1}{2}}$, so will M be a point in the curve. Again, make $x=$ AC $=a$; then the equation will give $y=\sqrt[3]{a^3}=a$. Hence, drawing CN parallel to AB, and equal to AC or a, N will be another point in the curve. And by assuming other values of y, other ordinates, and consequently other points of the curve, may be obtained. Once more, making x infinite, or $x=\infty$, we shall have $y=\sqrt[3]{(a^2\times\infty)}$; that is, y is infinite when x is so; and therefore the curve passes on to infinity. And further, since when x is taken $=0$, it is also $y=0$, and when $x=\infty$, it is also $y=\infty$; the curve will have no asymptotes that are parallel to the co-ordinates.

Let the right line AN be drawn to cut PM (produced if necessary) in S. Then because CN $=$ AC, it will be PS $=$ AP $=\frac{1}{2}a$. But PM $=a\sqrt[3]{\frac{1}{2}}=\frac{1}{2}a\sqrt[3]{4}$, which is manifestly greater than $\frac{1}{2}a$; so that PM is greater than PS, and consequently the curve is concave to the axis AC.

Now, because in the given equation $y^3=a^2x$ the exponent of x is *odd*, when x is taken negatively or on the other side of A, its sign should be changed, and the reduced equation will then be $y=\sqrt[3]{-a^2x}$. Here it is evident that, when the values of x are taken in the negative way from A towards D, but equal to those already taken the positive way, there will result as many negative values of y, to fall below AD, and each equal to the corresponding values of y, taken above AC. Hence it follows that the branch AM′N′ will be similar and equal to the branch AMN; but contrarily posited.

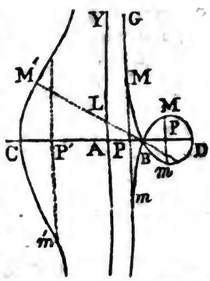
Ex. 2. Let the *lemniscate* be proposed, which is a line of the fourth order, denoted by the equation $a^2y^2=a^2x^2-x^4$.



In this equation we have $y=\pm\frac{x}{a}\sqrt{(a^2-x^2)}$; where, when $x=0$, $y=0$, therefore the curve passes through A, the point from which the values of x are measured. When $x=\pm a$, then $y=0$; therefore the curve passes through B and C, supposing AB and AC each $=\pm a$. If x were assumed greater than a, the value of y would become imaginary: therefore no part of the curve lies beyond B or C. When $x=\frac{1}{2}a$,

then $y=\frac{1}{2}\sqrt{(a^2-\frac{1}{4}a^2)}=\frac{1}{4}a\sqrt{3}$; which is the value of the semi-ordinate PM when $AP=\frac{1}{2}AB$. And thus, by assuming other values of x, other values of y may be ascertained, and the curve described. It has obviously two equal and similar parts, and a double point at A. A right line may cut this curve in either 2 points, or in 4: even the right line BAC is conceived to cut it in 4 points; becuuse the double point A is that in which two branches of the curve, viz. MAp and nAQ, are intersected.

Ex. 3. Let there be proposed the *Conchoid* of the ancients, which is a line of the fourth order defined by the equation $(a^2-x^2)\,.\,(x-b)^2=x^2y^2$, or $y=\pm\frac{x-b}{x}\sqrt{(a^2-x^2)}$.



Here, if $x=0$, then y becomes infinite; and therefore the ordinate at A (the origin of the abscissas) is an asymptote to the curve. If $AB=b$, and P be taken between A and B, then shall PM and pm be equal, and lie on different sides of the abscissa AP. If $x=b$, then the two values of y vanish, because $x-b=0$, and consequently the curve passes through B, having there a double point. If AP be taken greater than AB, then will there be two values of y, as before, having contrary signs; that value which was positive before being now negative, and *vice versa*. But if AD be taken $=a$, and P comes to D, then the two values of y vanish, because in that case $\sqrt{(a^2-x^2)}=0$. If AP be taken greater than AD or a, then a^2-x^2 becomes negative, and the value of y impossible: so that the curve does not go beyond D.

Now let x be considered as negative, or as lying on the side of A towards C. Then $y=\pm\frac{x+b}{x}\sqrt{(a^2-x^2)}$. Here if x vanish, both these values of y become infinite; and consequently the curve has two indefinite arcs on each side the asymptote or directrix AY. If x increase, y manifestly diminishes; and when $x=a$, then y vanishes: that is, if $AC=AD$, then one branch of the curve passes through C, while the other passes through D. Here also, if x be taken greater than a, y becomes imaginary; so that no part of the curve can be found beyond C.

If $a=b$, the curve will have a cusp in B, the node between B and D vanishing in that case. If a be less than b, then B will become a conjugate point.

In the figure, M′cm′ represents what is termed the *superior conchoid*, and GBmDMBm the *inferior conchoid*. The point B is called the *pole* of the conchoid : and the curve may be readily constructed by radial lines from this point, by means of the polar equation $z = \frac{b}{\cos\phi} \pm a$. It will merely be requisite to set off from any assumed point A, the distance $AB = b$; then to draw through B a right line mLM′ making any angle φ with CB, and from L, the point where this line cuts the directrix AY (drawn perpendicular to CB) set off upon it $LM' = Lm = a$; so shall M′ and m be points in the superior and inferior conchoids respectively.

Ex. 4. Let the principal properties of the curve whose equation is $yx^n = a^{n+1}$, be sought; when n is an odd number, and when n is an even number.

Ex. 5. Describe the line which is defined by the equation $xy + ay + cy = bc + bx$.

Ex. 6. Let the Cardioide, whose equation is $y^4 - 6ay^3 + (2x^2 + 12a^2)\,y^2 - (6ax^2 + 8a^3)\,y + (x^2 + 3a^2)\,x^2 = 0$, be proposed.

Ex. 7. Let the Trident, whose equation is $xy = ax^3 + bx^2 + cx + d$, be proposed.

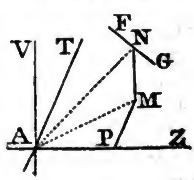
Ex. 8. Ascertain whether the *Cissoid* and the *Witch*, whose equations are found in the preceeding problem, have asymptotes.

PROBLEM III.

To determine the equation to any proposed curve surface.

Here the required equation must be deduced from the law or manner of construction of the proposed surface, the reference being to *three* co-ordinates, commonly rectangular ones, the variable quantities being x, y, and z. Of these, two, namely x and y, will be found in one plane, and the third z will always mark the distance from that plane.

Ex. 1. Let the proposed surface be that of a sphere, FNG.



The position of the fixed point A, which is the origin of the co-ordinates AP, PM, MN, being arbitrary; let it be supposed, for the greater convenience, that it is at the centre of the sphere. Let MA, NA, be drawn, of which the latter is manifestly equal to the radius of the sphere, and may be denoted by

r. Then, if AP $= x$, PM $= y$, MN $= z$; the right-angled triangle APM will give AM^2 $=$ AP^2 $+$ PM^2 $= x^2 + y^2$. In like manner, the right-angled triangle AMN, posited in a plane perpendicular to the former, will give AN^2 $=$AM^2 $+$ MN^2, that is, $r^2 = x^2 + y^2 + z^2$; or $z^2 = r^2 - x^2 - y^2$, the equation to the spherical surface, as required.

Scholium. Curve surfaces, as well as plane curves, are arranged in orders according to the dimensions of the equations, by which they are represented. And, in order to determine the properties of curve surfaces, processes must be employed, similar to those adopted when investigating the properties of plane curves. Thus, in like manner as in the theory of curve lines, the supposition that the ordinate y is equal to 0, gives the point or points where the curve cuts its axis; so, with regard to curve surfaces, the supposition of $z = 0$, will give the equation of the curve made by the intersection of the surface and its base, or the plane of the co-ordinates x, y. Hence, in the equation to the spherical surface, when $z = 0$, we have $x^2 + y^2 = r^2$, which is that of a circle whose radius is equal to that of the sphere. See p. 31.

Ex. 2. Let the curve surface proposed be that produced by a parabola turning about its axis.

Here the abscissas x being reckoned from the vertex or summit of the axis, and on a plane passing through that axis: the two other co-ordinates being, as before, y and z; and the parameter of the generating parabola being p: the equation of the parabolic surface will be found to be $z^2 + y^2 - px = 0$.

Now, in this equation, if z be supposed $= 0$, we shall have $y^2=px$, which (p. 538, vol. i.) is the equation to the generating parabola, as it ought to be. If we wished to know what would be the curve resulting from a section parallel to that which coincides with the axis, and at the distance a from it, we must put $z = a$; this would give $y^2 = px - a^2$, which is still an equation to a parabola, but in which the origin of the abscissas is distant from the vertex before assumed by the quantity $\frac{a^2}{p}$.

Ex. 3. Suppose the curve surface of a right cone were proposed.

Here we may most conveniently refer the equation of the surface to the plane of the circular base of the cone. In this case, the perpendicular distance of any point in the surface from the base, will be to the axis of the cone, as the distance of the foot of that perpendicular from the circumference (measured on a radius), to the radius of the base; that is, if

the values of x be estimated from the centre of the base, and r be the radius, z will vary as $r - \sqrt{(x^2 + y^2)}$. Consequently, the simplest equation of the conic surface, will be $z - r = - \sqrt{(x^2 + y^2)}$, or $r^2 - 2rz + z^2 = x^2 + y^2$.

Now, from this, the nature of curves formed by planes cutting the cone in different directions, may readily be inferred. Let it be supposed, first, that the cutting plane is inclined to the base of a right-angled cone in the angle of 45°, and passes through its centre: then will $z = x$, and this value of z substituted for it in the equation of the surface, will give $r^2 - 2rx = y^2$, which is the equation of the projection of the curve on the plane of the cone's base: and this (art. 3 of this chap.) is manifestly an equation to a *parabola*.

Or, taking the thing more generally, let it be supposed that the cutting plane is so situated, that the ratio of x to z shall be that of 1 to m: then will $mx = z$, and $m^2x^2 = z^2$. These substituted for z and z^2 in the equation of the surface, will give, for the equation of the projection of the section on the plane of the base, $r^2 - 2mx + (m^2 - 1)\, x^2 = y^2$. Now this equation, if m be greater than unity, or if the cutting plane pass between the vertex of the cone and the parabolic section, will be that of an *hyperbola:* and if, on the contrary, the cutting plane pass between the parabola and the base, i. e. if m be less than unity, the term $(m^2 - 1)\, x^2$ will be negative, when the equation will obviously designate an *ellipse*.

Schol. It might here be demonstrated, in a nearly similar manner, that every surface formed by the rotation of any conic section on one of its axes, being cut by any plane whatever, will always give a conic section. For the equation of such surface will not contain any power of x, y, or z, greater than the second; and therefore the substitution of any values of z in terms of x or of y, will never produce any powers of x or of y exceeding the square. The section therefore must be a line of the second order. See, on this subject, Hutton's Mensuration, part iii. sect. 4.

Ex. 4. Let the equation to the curve surface be $xyz = a^3$.

Then will the curve surface bear the same relation to the *solid* right angle, which the curve line whose equation is $xy = a^2$ bears to the *plane* right angle. That is, the curve surface will be posited between the three rectangular faces bounding such solid right angle, in the same manner as the equilateral hyperbola is posited between its rectangular asymptotes. And in like manner as there may be 4 equal equilateral hyperbolas comprehended beween the same rectangular asymptotes, when produced both ways from the angular point;

so there may be 6 equal hyperboloids posited within the 6 solid right angles which meet at the same summit, and all placed between the same three asymptotic planes.

SECTION II.

On the Construction of Equations.

PROBLEM I.

To construct simple equations, geometrically.

Here the sole art consists in resolving the fractions, to which the unknown quantity is equal, into proportional terms; and then constructing the respective proportions, by means of probs. 8, 9, 10, and 27 Geometry. A few simple examples will render the method obvious.

1. Let $x = \frac{ab}{c}$; then $c : a :: b : x$. Whence x may be found by constructing according to prob. 9 Geometry.

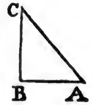
2. Let $x = \frac{abc}{de}$. First construct the proportion $d : a :: b : \frac{ab}{d}$, which 4th term call g; then $x = \frac{gc}{e}$; or $e : c :: g : x$.

3. Let $x = \frac{a^2-b^2}{c}$. Then, since $a^2-b^2=(a+b)\times(a-b)$; it will merely be necessary to construct the proportion $c : a+b :: a-b : x$.

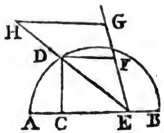
4. Let $x = \frac{a^2b-bc^2}{ad}$. Find, as in the first case, $g = \frac{ab}{d} = \frac{a^2b}{ad}$, and $h = \frac{bc}{d}$, so that $\frac{bc^2}{ad}$ may $= \frac{hc}{a}$. Then find by the first case $i = \frac{hc}{a}$. So shall $x = g-i$, the difference of those lines, found by construction.

5. Let $x = \frac{a^2b-bad}{af+bc}$. First find $\frac{af}{b}$, the fourth proportional to b, a and f, which make $= h$. Then $x = \frac{a(a-d)}{h+c}$; or, by construction it will be $h+c : a-d :: a : x$.

6. Let $x = \frac{a^2 + b^2}{c}$. Make the right-angled triangle ABC such that the leg $AB = a$, $BC = b$; then $AC = \sqrt{(AB^2 + BC^2)} = \sqrt{(a^2 + b^2)}$, by th. 34 Geom. Hence $x = \frac{AC^2}{c}$. Construct therefore the proportion $c : AC :: AC : x$, and the unknown quantity will be found as required.



7. Let $x = \frac{a^2 + cd}{h + c}$. First, find CD a mean proportional between $AC = c$, and $CB = d$, that is, find $CD = \sqrt{cd}$. Then make $CE = a$, and join DE, which will evidently be $= \sqrt{(a^2 + cd)}$. Next on any line EG set off $EF = h + c$, $EG = ED$; and draw GH parallel to FD, to meet DE (produced if need be) in H. So shall EH be $= x$, the third proportional to $h + c$, and $\sqrt{(a^2 + cd)}$, as required.

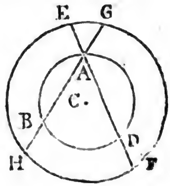


Note. Other methods suitable to different cases which may arise are left to the student's invention. And in all constructions the accuracy of the results will increase with the size of the diagrams; within convenient limits for operation.

PROBLEM II.

To find the roots of quadratic equations by construction.

In most of the methods commonly given for the construction of quadratics, it is required to set off the square root of the last term; an operation which can only be performed accurately when that term is a rational square. We shall here describe a method which, at the same time that it is very simple in practice, has the advantage of showing clearly the relations of the roots, and of dividing the third term into two factors, one of which at least may be a whole number.



In order to this construction, all quadratics may be classed under 4 forms: viz.

1. $x^2 + ax - bc = 0$.
2. $x^2 - ax - bc = 0$.
3. $x^2 + ax + bc = 0$.
4. $x^2 - ax + bc = 0$.

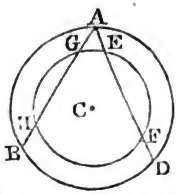
1. One general mode of construction will include the first two of these forms. Let $x^2 \mp ax - bc = 0$, and b be greater

than c. Describe any circle ABD having its diameter not less than the given quantities a and $b - c$, and within this circle inscribe two chords, AB $= a$, AD $= b - c$, both from any common assumed point A. Then, produce AD to F so that DF $= c$, and about the centre C of the former circle, with the radius CF, describe another circle, cutting the chords AD, AB, produced, in F, E, G, H: so shall AG be the *affirmative* and AH the *negative root* of the equation $x^2 + ax - bc = 0$; and contrariwise AG will be the *negative* and AH the *affirmative* root of the equation $x^2 - ax - bc = 0$.

For, AF or AD + DF $= b$, and DF or AE$=c$; and, making AG or BH $= x$, we shall have AH $= a + x$: and by the property of the circle EGFH (theor. 61 Geom.) the rectangle EA . AF = GA . AH, or $bc = (a + x)x$, or again by transposition $x^2 + ax - bc = 0$. Also if AH be $= -x$, we shall have AG or BH or AH — AB $= -x - a$: and conseq. GA . AH $= x^2 + ax$, as before. So that, whether AG be $= x$, or AH $= -x$, we shall always have $x^2 + ax - bc = 0$. And by an exactly similar process it may be proved that AG is the negative, and AH the positive root of $x^2 - ax - bc = 0$.

Cor. In quadratics of the form $x^2 + ax - bc = 0$, the positive root is always *less* than the negative root; and in those of the form $x^2 - ax - bc = 0$, the positive root is always *greater* than the negative one.

2. The third and fourth cases also are comprehended under one method of construction, with two concentric circles. Let $x^2 \mp ax + bc = 0$. Here describe any circle ABD, whose diameter is not less than either of the given quantities a and $b+c$; and within that circle inscribe two chords AB$=a$, AD $= b + c$, both from the same point A.



Then in AD assume DF $= c$, and about C the centre of the circle ABD, with the radius CF describe a circle, cutting the chords AD, AB, in the points F, E, G, H: so shall AG, AH, be the two *positive* roots of the equation $x^2 - ax + bc = 0$, and the two *negative* roots of the equation $x^2 + ax + bc = 0$. The demonstration of this also is similar to that of the first case.

Cor. 1. If the circle whose radius is CF just touches the chord AB, the quadratic will have two equal roots; which can only happen when $\frac{1}{4}a^2 = bc$.

Cor. 2. If that circle neither cut nor touch the chord AB, the roots of the equation will be imaginary; and this will

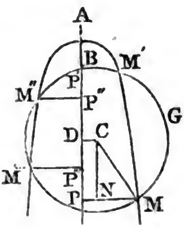
always happen, in these two forms, when bc is greater than $\frac{1}{4}a^2$.

PROBLEM III.

To find the roots of cubic and biquadratic equations, by construction.

1. In finding the roots of any equation, containing only one unknown quantity, by construction, the contrivance consists chiefly in bringing a new unknown quantity into that equation : so that various equations may be had, each containing the two unknown quantities ; and further, such that any two of them contain *together* all the known quantities of the proposed equation. Then from among these equations two of the most simple are selected, and their corresponding loci constructed ; the intersection of those loci will give the roots sought.

Thus it will be found that cubics may be constructed by two parabolas, or by a circle and a parabola, or by a circle and an equilateral hyperbola, or by a circle and an ellipse, &c. : and biquadratics by a circle and a parabola, or by a circle and an ellipse, or by a circle and an hyperbola, &c. Now, since a parabola of given parameter may be easily constructed by the rule in cor. 2 th. 4 Parabola, we select the circle and the parabola, for the construction of both biquadratic and cubic equations. The general method applicable to both, will be evident from the following description.



2. Let M″AM′M be a parabola whose axis is AP, M″M′GM a circle whose centre is C and radius CM, cutting the parabola in the points M, M′, M″, M‴ : from these points draw the ordinates to the axis MP, M′P′, M″P″, M‴P‴ ; and from C let fall CD perpendicularly to the axis ; also draw CN parallel to the axis, meeting PM in N. Let AD $= a$, DC $= b$, CM $= n$, the parameter of the parabola $= p$, AP $= x$, PM$=y$. Then (p. 538, vol. i.) $px=y^2$: also $\text{CM}^2 = \text{CN}^2 + \text{NM}^2$, or $n^2 = (x \mp a)^2 + (y \mp b)^2$; that is, $x^2 \pm 2ax + a^2 + y^2 \pm 2by + b^2 = n^2$. Substituting in this equation for x, its value $\frac{y^2}{p}$, and arranging the terms according to the dimensions of y, there will arise

$$y^4 \pm (2pa + p^2)y^2 \pm 2bp^2y + (a^2 + b^2 - n^2)p^2 = 0,$$

a biquadratic equation, whose roots will be expressed by the

ordinates PM, P′M′, P″M″, P‴M‴, at the points of intersection of the given parabola and circle.

3. To make this coincide with any proposed biquadratic whose second term is taken away by (cor. theor. 3); assume $y^4 - qy^2 + ry - s = 0$. Assume also $p = 1$; then comparing the terms of the two equations, it will be, $2a - 1 = q$, or $a = \frac{q+1}{2}$, $-2b = r$, or $b = \frac{-r}{2}$; $a^2 + b^2 - n^2 = -s$, or $n^2 = a^2 + b^2 + s$, and consequently $n = \sqrt{(a^2 + b^2 + s)}$. Therefore describe a parabola whose parameter is 1, and in the axis take AD $= \frac{q+1}{2}$: at right angles to it draw DC and $= -\frac{1}{2}r$; from the centre C, with the radius $\sqrt{(a^2 + b^2 + s)}$, describe the circle M‴M′GM, cutting the parabola in the points M, M′, M″, M‴; then the ordinates PM, P′M′, P″M″, P‴M‴, will be the roots required.

Note. This method, of making $p = 1$, has the obvious advantage of requiring only one parabola for any number of biquadratics, the necessary variation being made in the radius of the circle.

Cor. 1. When DC represents a negative quantity, the ordinates on the same side of the axis with C represent the negative roots of the equation; and the contrary.

Cor. 2. If the circle *touch* the parabola, two roots of the equation are equal; if it cut it only in two points, or touch it in one, two roots are impossible; and if the circle fall wholly within the parabola, all the roots are impossible.

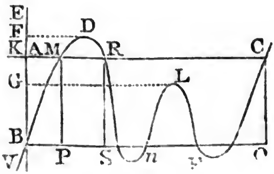
Cor. 3. If $a^2 + b^2 = n^2$, or the circle pass through the point A, the last term of the equation, i. e. $(a^2+b^2-n^2)\, p^2=0$; and therefore $y^4 \pm (2pa + p^2)\, y^2 \pm 2bq^2y = 0$, or $y^3 \pm (2pa + p^2)\, y \pm 2bq^2 = 0$. This cubic equation may be made to coincide with any proposed cubic, wanting its second term, and the ordinates PM, P″M″, P‴M‴, are its roots.

Thus, if the cubic be expressed generally by $y^3 \pm qy \pm s = 0$. By comparing the terms of this and the preceding equation, we shall have $\pm\, 2pa + p^2 = \pm\, q$, and $\pm\, 2bq^2 = \pm\, s$, or $\mp\, a = \frac{1}{2}p \pm \frac{q}{2q}$, and $b = \pm\, \frac{s}{2p^2}$. So that, to construct a cubic equation, with any *given* parabola, whose half parameter is AB (see the preceding figure): from the point B take, in the axis, (forward if the equation have $-q$, but backward if q be positive) the line BD $= \frac{q}{2q}$; then raise the perpendicular DC $= \frac{s}{2p^2}$, and from C describe a circle passing through the

vertex A of the parabola; the ordinates PM, &c. drawn from the points of intersection of the circle and parabola, will be the roots required.

PROBLEM IV.

To construct an equation of any order by means of a locus of the same degree as the equation proposed, and a right line.



As the general method is the same in all equations, let it be one of the 5th degree, as $x^5 - bx^4 + acx^3 - a^2dx^2 + a^3ex - a^4f = 0$. Let the last term a^4f be transposed; and, taking one of the linear divisors, f, of the last term, make it equal to z, for example, and divide the equation by a^4; then will $z = \frac{x^5 - bx^4 + acx^3 - a^2dx^2 + a^3ex}{a^4}$.

On the indefinite line BQ describe the curve of this equation, BMDRLFC, by the method taught in prob. 2, sect. 1, of this chapter, taking the values of x from the fixed point B. The ordinates PM, SR, &c. will be equal to z; and therefore, from the point B draw the right line $BA = f$, parallel to the ordinates PM, SR, and through the point A draw the indefinite right line KC both ways, and parallel to BQ. From the points in which it cuts the curve, let fall the perpendiculars MP, RS, CQ; they will determine the abscissas, BP, BS, BQ, which are the roots of the equation proposed. Those from A towards Q are positive, and those lying the contrary way are negative.

If the right line AC touch the curve in any point, the corresponding abscissa x will denote two equal roots; and if it do not meet the curve at all, all the roots will be imaginary.

If the sign of the last term, a^4f, had been positive, then we must have made $z = -f$, and therefore must have taken $BA = -f$, that is, below the point P, or on the negative side.

EXERCISES.

Ex. 1. Let it be proposed to divide a given arc of a circle into three equal parts.

Suppose the radius of the circle to be represented by r, the sine of the given arc by a, the unknown sine of its third part by x, and let the known arc be $3u$, and of course the

required arc be u. Then, by equa. VIII., IX., chap. iii. we shall have

$$\sin 3u = \sin(2u + u) = \frac{\sin 2u \,.\, \cos u + \cos 2u \,.\, \sin u}{r},$$

$$\sin 2u = \sin(\ u + u) = \frac{2 \sin u \,.\, \cos u}{r},$$

$$\cos 2u = \cos(\ u + u) = \frac{\cos^2 u - \sin^2 u}{r}.$$

Putting in the first of these equations, for $\sin 3u$ its given value a, and for $\sin 2u$, $\cos 2u$, their values given in the two other equations, there will arise

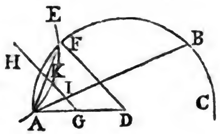
$$a = \frac{3 \sin u \,.\, \cos^2 u \,.\, \sin^3 u}{r}.$$

Then substituting for $\sin u$ its value x, and for $\cos^2 u$ its value $r^2 - x^2$, and arranging all the terms according to the powers of x, we shall have

$$x^3 - \tfrac{3}{4}r^2x + \tfrac{1}{4}ar^2 = 0,$$

a cubic equation of the form $x^3 - px + q = 0$, with the condition that $\frac{1}{27}p^3 > \frac{1}{4}q^2$; that is to say, it is a cubic equation falling under the irreducible case, and its three roots are represented by the sines of the three arcs u, $u + 120°$, and $u + 240°$.

Now, this cubic may evidently be constructed by the rule in prob. 3, cor. 3. But the trisection of an arc may also be effected by means of an equilateral hyperbola, in the following manner.



Let the arc to be trisected be AB. In the circle ABC draw the semidiameter AD, and to AD as a diameter, and to the vertex A, draw the equilateral hyperbola AE to which the right line AB (the chord of the arc to be trisected) shall be a tangent in the point A; then the arc AF, included within this hyperbola, is one third of the arc AB.

For, draw the chord of the arc AF, bisect AD at G, so that G will be the centre of the hyperbola, join DF, and draw GH parallel to it, cutting the chords AB, AF, in I and K. Then, the hyperbola being equilateral, or having its transverse and conjugate equal to one another, it follows from Def. 16 Conic Sections, that every diameter is equal to its parameter, and from cor. theor. 2 Hyperbola, that GK . KI = AK^2, or that GK : AK : : AK : KI; therefore the triangles GKA, AKI are similar, and the angle KAI = AGK, which is manifestly = ADF. Now the angle ADF at the centre of the circle being equal to KAI or FAB; and the former angle at the centre being mea-

sured by the arc AF, while the latter at the circumference is measured by half FB; it follows that AF $= \frac{1}{2}$FB, or $= \frac{1}{3}$AB, as it ought to be.

Mr. Lardner, in his *Analytical Geometry*, gives the following elegant solution of this problem.

"Let A be the given angle. By trigonometry

$$\cos^3 \tfrac{1}{3}A - \tfrac{3}{4}\cos \tfrac{1}{3}A - \tfrac{1}{4}\cos A = 0 ;$$

which, by supplying the radius r, and representing $\cos \frac{1}{3}A$ by x, becomes

$$4x^3 - 3r^2x - r^2 \cos A = 0 ;$$

which, multiplied by x, gives

$$4x^4 - 3r^2x^2 - r^2 \cos A \cdot x = 0.$$

"Let the equation of one of the curves be,

$$2x^2 = ry,$$

and the other by substitution will be,

$$2y^2 - 3ry - 2 \cos A \cdot x = 0.$$

The former is the equation of a parabola, the axis of which is the axis of y, the origin in the vertex, and the principal parameter equal to $\frac{1}{2}r$.

"The latter is also a parabola, the equation of its axis is $y = \frac{3}{4}r$; the co-ordinates of its vertex are $y = \frac{3}{4}r$, $x = -\frac{9r^2}{16 \cos A}$, and its principal parameter is cos A.

"These parabolas being described, their points of intersection give the roots of the equation. The intersection at the origin gives the root $x = 0$, which was introduced by the multiplication by x.

"The equation having more than one real root, it might appear that there were more values than one for the third of the given angle. But upon examining the process, it will be seen that the question really solved was not to find an angle equal to the third of a given angle, but *to find the cosine of an angle which is the third of an angle whose cosine is given.* Since, then, the arcs

$$\begin{array}{cc} & A \\ 2\pi - A, & 2\pi + A, \\ 4\pi - A, & 4\pi + A, \\ 6\pi - A, & 6\pi + A ; \end{array}$$

and in general all arcs which come under the general formula, $2m\pi \pm A$ have the same cosine, the question really solved is to find the cosine of the third of any of these arcs. And here again another apparent difficulty arises. If the number of arcs involved in the question be unlimited, shall there not be an unlimited number of values for the cosine of the third parts of these? To account for this it

should be considered that in general the arc $\frac{2m}{3}\pi \pm \frac{A}{3}$ must have the same cosine as some one of the three arcs,

$$\frac{A}{3},\ \tfrac{2}{3}\pi - \tfrac{1}{3}A,\ \tfrac{4}{3}\pi - \tfrac{1}{3}A\ ;$$

for the number $\frac{m}{3}$ must be either of these forms n, $n + \frac{1}{3}$, or $n + \frac{2}{3}$, where n is an integer. If it have the form n, that is, if 3 measures m, then

$$\frac{2m}{3}\pi \pm \tfrac{1}{3}A = 2n\pi \pm \tfrac{1}{3}A\ ;\ \text{therefore}$$

$$\cos\left(\frac{2m}{3}\pi \pm \tfrac{1}{3}A\right) = \cos\left(2n\pi \pm \tfrac{1}{3}A\right) = \cos \tfrac{1}{3}A.$$

"If it have the form $n+\frac{1}{3}$;

$$\frac{2m}{3}\pi \pm \tfrac{1}{3}A = 2n\,\pi + \tfrac{2}{3} \pm \pi\tfrac{1}{3}A\ ;\ \text{therefore}$$

$$\cos\left(\frac{2m\pi}{3} \pm \tfrac{1}{3}A\right) = \cos\left(2n\pi + \tfrac{2}{3}\pi \pm \tfrac{1}{3}A\right) = \cos \tfrac{1}{3}(2\,\pi \pm A).$$

"If it have the form $n+\frac{2}{3}$;

"2. $\frac{m}{3}\pi \pm \frac{1}{3}A = 2n\,\pi \pm \frac{4}{3}\pi \pm \frac{1}{3}A$; therefore

$$\cos\left(\frac{2m}{8}\pi \pm \tfrac{1}{3}A\right) = \cos\left(2n\,\pi + \tfrac{4}{3}\pi \pm \tfrac{1}{3}A\right) = \cos \tfrac{1}{3}\,(4\pi \pm A).$$

"And hence it follows that the $\cos\left(\frac{2m\pi}{3} \pm A\right)$, whatever be the value of m, must be equal to one or other of the quantities.

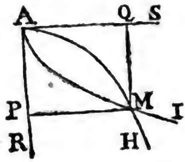
$$\cos.\ \tfrac{1}{3}A,$$
$$\cos.\ \tfrac{1}{3}\ (2\pi - A),$$
$$\cos.\ \tfrac{1}{3}\ (4\pi - A),$$

which correspond to the three roots of the cubic equation already found."

Ex. 2. Given the side of a cube, to find the side of another of double capacity.

Let the side of the given cube be a, and that of a double one y, then $2a^3 = y^3$, or, by putting $2a = b$, it will be $a^2 b = y^3$: there are therefore to be found two mean proportionals between the side of the cube and twice that side, and the first of those mean proportionals will be the side of the double cube. Now these may be readily found by means of two parabolas : thus :

Let the right lines AR, AS, be joined at right angles ; and a parabola AMH be described about the axis AR, with the parameter a ; and another parabola AMI about the axis AS, with the parameter b ; cutting the former in M. Then $AP = x$, $PM = y$,



are the two mean proportionals, of which y is the side of the double cube required.

For, in the parabola AMH the equation is $y^2=ax$, and in the parabola AMI it is $x^2=by$. Consequently $a : y : : y : x$, and $y : x : : x : b$. Whence $yx = ab$; or, by substitution, $y\sqrt{by} = ab$, or, by squaring, $y^3b = a^2b^2$; or lastly, $y^3=a^2b = 2a^3$, as it ought to be.

GENERAL SCHOLIUM.

On the Construction of Geometrical Problems.

Problems in Plane Geometry are solved either by means of the modern or algebraical analysis, or of the ancient or geometrical analysis. Of the former, some specimens are given in the Application of Algebra to Geometry, in the first volume of this Course. Of the latter, we here present a few examples, premising a brief account of this kind of analysis.

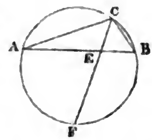
Geometrical analysis is the way by which we proceed from the thing demanded, granted for the moment, till we have connected it by a series of consequences with something anteriorly, known, or placed it among the number of principles known to be true.

Analysis may be distinguished into two kinds. In the one, which is named by Pappus contemplative, it is proposed to ascertain the truth or the falsehood of a proposition advanced; the other is referred to the solution of problems, or to the investigation of unknown truths. In the first we assume as true, or as previously existing, the subject of the proposition advanced, and proceed by the consequences of the hypothesis to something known; and if the result be thus found true, the proposition advanced is likewise true. The direct demonstration is afterwards formed, by taking up again, in an inverted order, the several parts of the analysis. If the consequence at which we arrive in the last place is found false, we thence conclude that the proposition analysed is also false. When a *problem* is under consideration, we first suppose it resolved, and then pursue the consequences thence derived till we come to something known. If the ultimate result thus obtained be comprised in what the geometers call data, the question proposed may be resolved: the demonstration (or rather the construction) is also constituted by taking the

parts of the analysis in an inverted order. The impossibility of the last result of the analysis will prove evidently, in this case as well as in the former, that of the thing required.

In illustration of those remarks take the following examples.

Ex. 1. It is required to draw, in a given segment of a circle, from the extremes of the base A and B, two lines AC, BC, meeting at a point C in the circumference, such that they shall have to each other a given ratio, viz. that of M to N.

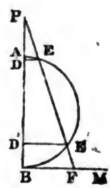


Analysis. Suppose that the thing is affected, that is to say, that AC : CB : : M : N, and let the base AB of the segment be cut in the same ratio in the point E. Then EC, being drawn, will bisect the angle ACB (by th. 83 Geom.); consequently, if the circle be completed, and CE be produced to meet it in F, the remaining circumference will also be bisected in F, or have FA = FB, because those arcs are the double measures of equal angles; therefore the point F, as well as E, being given, the point C is also given.

Construction. Let the given base of the segment AB be cut in the point E in the assigned ratio of M to N, and complete the circle; bisect the remaining circumference in F; join FE, and produce it till it meet the circumference in C: then drawing CA, CB, the thing is done.

Demonstration. Since the arc FA = the arc FB, the angle ACF = angle BCF, by theor. 49 Geom.; therefore AC : CB : : AE : EB, by th. 83. But AE : EB : : M : N, by construction; therefore AC : CB : : M : N. Q. E. D.

Ex. 2. From a given circle to cut off an arc, such that the sum of m times the sine, and n times the versed sine, may be equal to a given line.



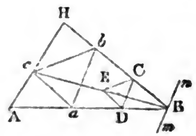
Analysis. Suppose it done, and that AEE'B is the given circle, BE'E the required arc, ED its sine, BD its versed sine; in DA (produced if necessary) take BP and nth part of the given sum; join PE, and produce it to meet BF ⊥ to AB or ∥ to ED, in the point F. Then, since $m \cdot \text{ED} + n \cdot \text{BD} = n \cdot \text{BP} = n \cdot \text{PD} + n \cdot \text{BD}$; consequently $m \cdot \text{ED} = n \cdot \text{PD}$; hence PD : ED : : $m : n$. But PD : ED : : (by sim. tri.) PB : BF; therefore PB : BF : : $m : n$. Now PB is given, therefore BF is given in magnitude, and, being at right angles to PB, is also given in position; therefore the point F is given and consequently PF given in position; and therefore the point E, its intersection

with the circumference of the circle AEE′B, or the arc BE is given. Hence the following

Construction. From B, the extremity of any diameter AB of the given circle, draw BM at right angles to AB; in AB (produced if necessary) take BP an nth part of the given sum; and on BM take BF so that BF : BP : : n : m. Join PF, meeting the circumference of the circle in E and E′, and BE or BE′ is the arc required.

Demonstration. From the points E and E′, draw ED and E′ D′ at right angles to AB. Then, since BF : BP : : n : m, and (by sim. tri.) BF : BP : : DE : DP; therefore DE : DP : : n : m. Hence m . DE $= n$. DP; add to each n . BD, then will m . DE $+ n$. BD $= n$. BD $+ a$. DP $= n$. PB, or the given sum.

Ex. 3. In the given triangle ABH, to inscribe another triangle abc, similar to a given one, having one of its sides parallel to a line mBn given by position, and the angular points a, b, c, situate in the sides AB, BH, AH, of the triangle ABH respectively.

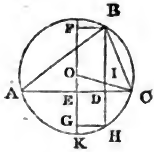


Analysis. Suppose the thing done, and that abc is inscribed as required. Through any point C in BH draw CD parallel to mBn or to ab, and cutting AB in D; draw CE parallel to bc, and DE to ac, intersecting each other in E. The triangles DEC, acb, are similar, and DC : ab : : CE : bc; also BDC, Bab, are similar, and DC : ab : : BC : Bb. Therefore BC : CE : : Bb : bc; and they are about equal angles, consequently B, E, c, are in a right line.

Construction. From any point C in BH, draw CD parallel to nm; on CD constitute a triangle CDE similar to the given one; and through its angles E draw BE, which produce till it cuts AH in c: through c draw ca parallel to ED and cb parallel to EC; join ab, then abc is the triangle required, having its side ab parallel to mn, and being similar to the given triangle.

Demonstration. For, because of the parallel lines ac, DE, and cb, EC, the quadrilaterals BDEC and Bacb, are similar; and therefore the proportional lines DC, ab, cutting off equal angles BDC, Bab; BCD, Bba; must make the angles EDC, ECD, respectively equal to the angles cab, cba; while ab is parallel to DC, which is parallel to mBn, by construction.

Ex. 4. Given, in a plane triangle, the vertical angle, the perpendicular, and the rectangle of the segments of the base made by that perpendicular; to construct the triangle.



Analysis. Suppose ABC the triangle required, BD the given perpendicular to the base AC, produce it to meet the periphery of the circumscribing circle ABCH, whose centre is O, in H; then, by th. 61 Geom. the rectangle BD . DH = AD . DC, the given rectangle: hence, since BD is given, DH and BH are given; therefore BI = HI is given: as also ID = OE: and the angle EOC is = ABC the given one, because EOC is measured by the arc KC, and ABC by half the arc AKC or by KC. Consequently EC and AC = 2EC are given. Whence this

Construction. Find DH such, that DB . DH = the given rectangle, or find $DH = \frac{AD \cdot DC}{BD}$; then on any right line GF take FE = the given perpendicular, and EG = DH; bisect FG in O, and make EOC = the given vertical angle; then will OC cut EC, drawn perpendicular to OE, in C. With centre O and radius OC, describe a circle, cutting CE produced in A: through F parallel to AC draw FB, to cut the circle in B; join AB, CB, and ABC is the triangle required.

Remark. In a similar manner we may proceed, when it is required to divide a given angle into two parts, the rectangle of whose tangents may be of a given magnitude. See prob. 40, Simpson's Select Exercises.

Note. For other exercises, the student may construct all the problems except the 24th, in the Application of Algebra to Geometry, at page 371, vol. i. And that he may be the better able to trace the relative advantages of the ancient and the modern analysis, it will be advisable that he solve those problems both geometrically and algebraically.

MECHANICS.

Definitions and preliminary Notions.

1. *Mechanics* is the science of equilibrium and of motion.

2. Every cause which moves, or tends to move a body, is called a *force*.

3. When the forces that are applied simultaneously to a body, destroy or annihilate each other's effects, then there is *equilibrium*.

4. *Statics* has for its object the equilibrium of forces applied to *solid* bodies.

5. By *Dynamics* we investigate the circumstances of the motion of solid bodies.

6. *Hydrostatics* is the science in which the equilibrium of fluids is considered.

7. *Hydrodynamics* is that in which the circumstances of their motion is investigated.

According to this division, *Pneumatics*, which relates to the properties of *elastic fluids*, is a branch of *Hydrostatics*.

For farther elucidation the following definitions, also, may advantageously find a place here, viz.

8. Body is the mass, or quantity of matter, in any material substance; and it is always proportional to its weight or gravity, whatever its figure may be.

Body is either Hard, Soft, or Elastic. A Hard Body is that whose parts do not yield to any stroke or percussion, but retains its figure unaltered. A Soft Body is that whose parts yield to any stroke or impression, without restoring themselves again; the figure of the body remaining altered. And an Elastic Body is that whose parts yield to any stroke, but which presently restore themselves again, and the body regains the same figure as before the stroke.

We know of no bodies that are absolutely, or perfectly, either hard, soft, or elastic; but all partaking these properties, more or less, in some intermediate degree.

9. Bodies are also either Solid or Fluid. A Solid Body is that whose parts are not easily moved among one another, and which retains any figure given to it. But a Fluid Body is that whose parts yield to the slightest impression, being easily moved among one another; and its surface, when left to itself, is always observed to settle in a smooth plane at the top.

10. Density is the proportional weight or quantity of matter in any body. So, in two spheres, or cubes, &c. of equal size or magnitude; if the one weigh only one pound, but the other two pounds; then the density of the latter is double the density of the former; if it weigh three pounds, its density is triple; and so on.

11. Motion is a continual and successive change of place.—If the body move equally, or pass over equal spaces in equal times, it is called Equable or Uniform Motion. But if it increase or decrease, it is Variable Motion; and it is called Accelerated Motion in the former case, and Retarded Motion in the latter.—Also, when the moving body is considered with

respect to some other body at rest, it is said to be Absolute Motion. But when compared with others in motion, it is called Relative Motion.

12. Velocity, or Celerity, is an affection of motion, by which a body passes over a certain space in a certain time. Thus, if a body in motion pass uniformly over 40 feet in 4 seconds of time, it is said to move with the velocity of 10 feet per second; and so on.

13. Momentum, or Quantity of Motion, is the power or force in moving bodies, by which they continually tend from their present places, or with which they strike any obstacle that opposes their motion.

14. Forces are distinguished into Motive, and Accelerative or Retarding. A Motive or Moving Force, is the power of an agent to produce motion; and it is equal or proportional to the momentum it will generate in any body, when acting, either by percussion, or for a certain time as a permanent force.

15. Accelerative, or Retardive Force, is commonly understood to be that which affects the velocity only: or it is that by which the velocity is accelerated or retarded; and it is equal or proportional to the motive force directly, and to the mass or body moved inversely. So, if a body of 2 pounds weight, be acted on by a motive force of 40; then the accelerating force is 20. But if the same force of 40 act on another body of 4 pounds weight; then the accelerating force in this latter case is only 10; and so is but half the former, and will produce only half the velocity.

16. Gravity or Weight, is that force by which a body endeavours to fall downwards. It is called Absolute Gravity, when the body is in empty space; and Relative Gravity, when immersed in a fluid.

17. Specific Gravity is the relation of the weights of different bodies of equal magnitude; and so is proportional to the density of the body.

NEWTONIAN AXIOMS.

18. Every body naturally endeavours to continue in its present state, whether it be at rest, or moving uniformly in a right line.

19. The change or Alteration of Motion, by any external force, is always proportional to that force, and in the direction of the right line in which it acts.

20. Action and Re-action, between any two bodies, are equal and contrary. That is, by Action and Re-action, equal

changes of motion are produced in bodies acting on each other; and these changes are directed towards opposite or contrary parts.

STATICS.

21. The relative magnitudes and directions of any two forces may be represented by two right lines, which shall bear to each other the relations of the forces, and which shall be inclined to each other in an angle equal to that made by the directions of the forces.

22. the name *resultant* is given to a force which is equivalent to two or more forces acting at once upon a point, or upon a body; these separate forces being named *constituents* or *composants*.

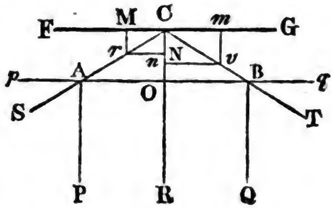
23. The operation by which the *resultant* of two or more forces applied to the same point, or line, or body, is determined, is called the *composition of forces*; the inverse problem is called the *decomposition*, or the *resolution of forces*.

24. The resultant of two or more forces which act upon the same line, in the same direction, is equal to their sum; and if some forces act in one direction, and others in a direction immediately opposite, the resultant will be equal to the excess of the sum of the forces which act in one direction above the sum of those which act in the opposite direction.

Composition and Resolution of Parallel Forces.

25. Prop. If to the extremities of an inflexible right line AB, are applied two forces, P and Q, whose directions are parallel and whose actions concur:—1st, The direction of the resultant, R, of those two forces is parallel to the right lines AP, BQ, and is equal to their sum. 2dly, That resultant divides the line AB into two parts reciprocally proportional to the two forces.

1. It is manifest that if two new forces, p and q, equal and in opposite directions are applied to the line AB, they will make no change in the state of the system: so that the resultant of the four forces p, q, P, Q, will be the same as that



of the resultant of the two original forces P, Q. Suppose, now, that S is the resultant of the two forces p, P, while T is that of the two forces q, Q. These resultants, lying in the same place, will, if prolonged, necessarily meet in some point C; to which, therefore, we may suppose the forces S and T applied.

Through this point let FG be drawn parallel to AB, and suppose each of the forces S and T resolved into two forces directed respectively in FG and CR. The forces, according to FG, being equal to p and q respectively, and applied in opposite directions, destroy each other's effects: the remaining forces, therefore, lying the same way on CR, must be added together for the resultant, which thus is equal to $P + Q$; being the first part of the proposition.

2. In order to establish the second part of the proposition, let MC, CN, be lines in proportion to each other as the forces p, P; and in C, Cn, respectively proportional as q, Q: and draw Nr, nv, parallel to AB.

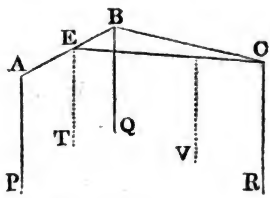
Then, by the sim. triangles CNr, COA; Cnv, COB;
$$\left\{\begin{array}{l} P : p :: CN : Nr :: CO : OA \\ q : Q :: nv : nC :: OB : OC. \end{array}\right.$$

Consequently, $P \cdot q : p \cdot Q :: CO \cdot OB : CO \cdot OA$,
or, since $p = q$, it is $P : Q :: OB : OA$. Q. E. D.

Corol. 1. If $P = Q$, $BO = OA$.

Corol. 2. When a single force R is applied to a point O, of an inflexible straight line AB, we may always resolve it, or conceive it resolved, into two others, which being applied to the two points A and B, in directions parallel to R, shall produce the same effect.

26. Prop. Any number of parallel forces, P, Q, R, S, &c. acting in the same sense, and their points of application being connected in an invariable manner; to determine their resultant.



Determining first, by the preceding prop. the resultant T of two of the forces P and Q, we shall have $T = P + Q$;

$$P + Q : Q :: AB : AE.$$

Thus, we may substitute for the forces P and Q, the single force T whose value and point of application are known. Draw EC from that point of application to the point C, at which another force, R, is applied. Compounding the forces T and R, their resultant V will be $= T + R = P + Q + R$; and its point of application, F, such that

$$P + Q + R : R :: EC : EF.$$

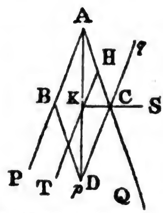
A similar method may, obviously, be pursued for any number of parallel forces.

27. If parallel forces act in opposite directions; some, for example, upwards, others downwards; find the resultants of the first and of the second class separately, the general resultant will be expressed by the difference of the two former.

28. The point through which the resultant of parallel forces passes, is called *the centre of parallel forces.* If the forces, without ceasing to be respectively parallel, and without changing either their magnitudes or their points of application, assume another general direction, the centre of those forces will still be the same, because the magnitudes and relations, on which its *position* depends, remain the same.

Concurring Forces.

29. PROP. The resultant of two forces P and Q acting in one plane, will be represented in direction and in magnitude, by the diagonal of the parallelogram constructed on the directions of those forces.



1. *In direction.* Take, on the directions AP, AQ, of the forces, P, Q, distances AB, AC, proportional to those forces, respectively. Suppose that the force Q is applied at the point C, and that at the same point two other forces p, q, equal to each other, act *in opposite directions*, each of those forces being, also, equal to Q.

The effect of the four forces P, Q, p, q, will evidently be the same as that of the primitive forces P, Q; since the other two annihilate each other's effects.

The forces Q, q, will have a resultant S, whose direction, CS, will bisect the angle QCq, made by the direction of the other two: since no reason can be assigned why it should lean to one rather than toward the other.

The forces P, p, acting in parallel directions, would have a resultant, T, whose direction TH (art. 25.) would be parallel to them, and pass through a point, H, such as that P : p : : HC : HA.

Now, the point K, where the directions CS, TH, of these two resultants intersect, will evidently be a point in the direction of the resultant of the *four* forces P, p, Q, q; and, consequently, of the original forces P, Q.

But the triangle CHK is isosceles: for, since HT, CP, are

parallel, the alternate angles DCK, HKC, are equal, and DCK, HCK, are equal, because SC bisects the angle QCq: hence HCK = HKC, and HK = HC.

But, from what has precedeed, P : Q : : HC : HA; and therefore P : p or Q : : HK : HA.

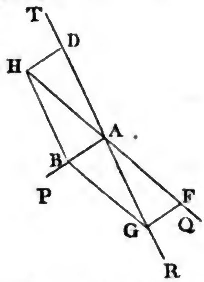
From B drawing BD parallel to AC, we shall have

P : Q : : AB : AC : : CD : AC,

whence CD : AC : : HK : HA;

a proportion which indicates that the three points, A, K, D, all fall on the diagonal of a parallelogram ABCD.

2. *In magnitude.* For, with regard to the forces P, Q, represented in magnitude and direction by AB and AF, let T be opposed to those two forces so as to keep the whole system in equilibrio: then it will, of necessity, be equal and opposite to their resultant, R, whose direction is AG. Now, if we suppose that the force Q is in equilibrio with the two forces P and T (which is consistent with our first hypothesis) the resultant of these latter will fall in the prolongation of QA, and will be represented by AH = AF. Also, if HD be drawn parallel to AB, and HB be joined, it will be equal and parallel to AG; and we shall have

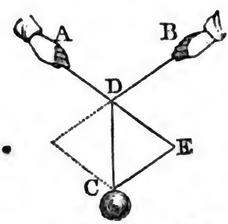


P : T : : AB : AD.

Consequently, since AB represents, or measures, the force P, AD will represent or measure the force T; and as that force is in equilibrio with the two forces P and Q, or with their resultant, R, this latter will be represented or measured by AG = AD; that is, by the diagonal of the parallelogram ABGF. Q. E. D.

30. *Corol.* 1. If three forces, as A, B, C, acting simultaneously in the same plane, keep one another in equilibrio, they will be respectively proportional to the three sides, DE, EC, CD, of a triangle which are drawn parallel to the directions of the forces AD, DB, CD.

For, producing AD, BD, and drawing CF, CE, parallel to them, then the force in CD is equivalent to the two AD, BD, by the supposition; but the force CD is also equivalent to the two ED and CE or FD; therefore, if CD represent the force C, then ED will represent its opposite force A, and CE, or FD,

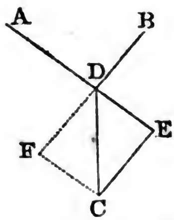


its opposite force B. Consequently the three forces, A, B, C, are proportional to DE, CE, CD, the three lines parallel to the directions in which they act.

31. *Corol.* 2. Because the three sides CD, CE, DE, are proportional to the sines of their opposite angles E, D, C; therefore the three forces, when in equilibrio, are proportional to the sines of the angles of the triangle made of their lines of direction; namely, each force proportional to the sine of the angle made by the direction of the other two.

32. *Corol.* 3. The three forces, acting against, and keeping one another in equilibrio, are also proportional to the sides of any other triangle made by drawing lines either perpendicular to the directions of the forces, or forming any given angle with those directions. For such a triangle is always similar to the former, which is made by drawing lines parallel to the directions; and therefore their sides are in the same proportion to one another.

33. *Corol.* 4. If any number of forces be kept in equilibrio by their actions against one another; they may be all reduced to two equal and opposite ones.—For, any two of the forces may be reduced to one force acting in the same plane; then this last force and another may likewise be reduced to another force acting in their plane: and so on, till at last they be all reduced to the action of only two opposite forces; which will be equal, as well as opposite, because the whole are in equilibrio by the supposition.



34. *Corol.* 5. If one of the forces, as C, be a weight, which is sustained by two strings drawing in the directions DA, DB: then the force or tension of the string AD, is to the weight C, or tension of the string DC, as DE to DC; and the force or tension of the other string BD, is to the weight C, or tension of CD, as CE to CD.

35. *Corol.* 6. Since in any triangle CDE we have, by the principles of trigonometry,

$$DC^2 = DE^2 + EC^2 \pm 2DE \cdot EC \cos. DEC,$$

it follows, that if F, f, be two forces that act simultaneously in directions, which make an angle A, then we may find the magnitude of the resultant, R, by the equation

$$R = \sqrt{(F^2 + f^2 \pm 2Ff \cos. A)}.$$

36. *Remark.*—The properties, in this proposition and its corollaries, hold true of all similar forces whatever, whether they be instantaneous or continual, or whether they act by

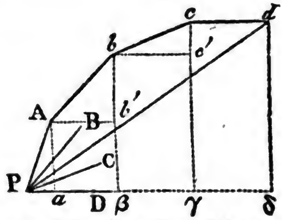
percussion, drawing, pushing, pressing, or weighing; and are of the utmost importance in mechanics and the doctrine of forces.

37. If three forces, whose directions concur in one point, are represented by the three contiguous edges of a parallelopiped, their resultant will be represented, both in magnitude and direction, by the diagonal drawn from the point of concourse, to the opposite angle of the parallelopiped.

The demonstration of this is left for the exercise of the student.

38. PROP. To find the resultant of several forces concurring in one point, and acting in one plane.

1st. *Graphically.*—Let, for example, four forces, A, B, C, D, act upon the point P, in magnitudes and directions represented by the lines PA, PB, PC, PD.



From the point A draw Ab parallel and equal to PB; from b draw bc parallel and equal to PC; from c draw cd parallel and equal to PD; and so on, till all the forces have thus been brought into the construction. Then join Pd, which will represent both the magnitude and the direction of the required resultant.

This is, in effect, the same thing as finding the resultant of two of the forces A and B; then blending that resultant with a third force C; their resultant with a fourth force D; and so on.

2d *By computation.* Drawing the lines Aa, Ab', &c. respectively parallel and perpendicular to the last force PD; we have

$$d\delta = \mathrm{A}a + bb' + cc' = \mathrm{A}\ \sin.\ \mathrm{APD} + \mathrm{B}\ \sin.\ \mathrm{BPD} + \mathrm{C}\ \sin.\ \mathrm{CPD}.$$

$$\mathrm{P}\delta = \mathrm{P}\alpha + \alpha\beta + \beta\gamma + \gamma\delta = \mathrm{A}\ \cos.\mathrm{APD} + \mathrm{B}\ \cos.\ \mathrm{BPD} + \mathrm{C}\cos.\mathrm{CPD} + \mathrm{D}$$

$$\tan d\mathrm{P}\delta = \frac{d\delta}{\mathrm{P}\delta} \ldots\ldots \mathrm{P}d = \sqrt{(\mathrm{P}\delta^2 + d\delta^2)} = \mathrm{P}\delta \sec.\ d\mathrm{P}\delta.$$

The numerical computation is best effected by means of a table of natural sines, &c.

39. *Remark.* Connected with this subject is the doctrine of *moments;* for an elucidation of which, however, the student should consult some of the books written expressly on mechanics, as those by *Marrat*, *Gregory*, or *Poisson.*

THE MECHANICAL POWERS, &c.

40. Weight and Power, when opposed to each other, signify the body to be moved, and the body that moves it; or the patient and agent. The power is the agent, which moves, or endeavours to move, the patient or weight.

41. Machine, or Engine, is any mechanical instrument contrived to move bodies. And it is composed of the mechanical powers.

42. Mechanical powers, are certain simple instruments, commonly employed for raising greater weights, or overcoming greater resistances, than could be effected by the natural strength without them. These are usually accounted six in number, viz. the Lever, the Wheel and Axle, the Pulley, the Inclined Plane, the Wedge, and the Screw.

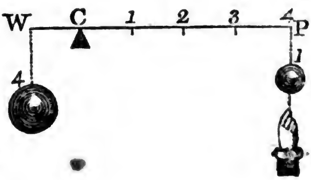
43. Centre of Motion, is the fixed point about which a body moves. And the Axis of Motion, is the fixed line about which it moves.

44. Centre of Gravity, is a certain point, on which a body being freely suspended, it will rest in any position.

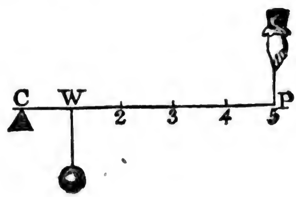
OF THE LEVER.

45. A Lever is any inflexible rod, bar, or beam, which serves to raise weights, while it is supported at a point by a fulcrum or prop, which is the centre of motion. The lever is supposed to be void of gravity or weight, to render the demonstrations easier and simpler. There are three kinds of levers.

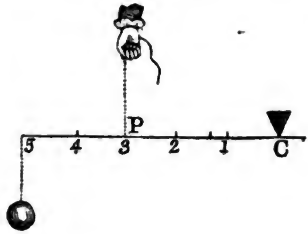
46. A Lever of the First kind has the prop c between the weight w and the power P. And of this kind are balances, scales, crows, hand-spikes, scissors, pincers, &c.



47. A Lever of the Second kind has the weight between the power and the prop. Such as oars, rudders, cutting knives that are fixed at one end, &c.



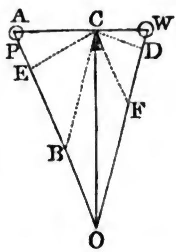
48. A Lever of the Third kind has the power between the weight and the prop. Such as tongs, the bones and muscles of animals, a man rearing a ladder, &c.



49. A Fourth kind is sometimes added, called the Bended Lever. As a hammer drawing a nail.

50. In all these instruments the power may be represented by a weight, which is its most natural measure, acting downward; but having its direction changed, when necessary, by means of a fixed pulley.

51. PROP. When the weight and power keep the lever in equilibrio, they are to each other reciprocally as the distances of their lines of direction from the prop. That is, P : W : : CD : CE; where CD and CE are perpendicular to WO and AO, the directions of the two weights, or the weight and power W and A.



For, draw CF parallel to AO, and CB parallel to WO: Also, join CO, which will be the direction of the pressure on the prop C; for there cannot be an equilibrium unless the directions of the three forces all meet in, or tend to, the same point, as O. Then, because these three forces keep each other in equilibrio, they are proportional to the sides of the triangle CBO or CFO, drawn in the direction of those forces; therefore - - - - - P : W : : CF : FO or CB.

But, because of the parallels, the two triangles CDF, CEB are equiangular, therefore - - - CD : CE : : CF : CB.

Hence, by equality, - - P : W : : CD : CE.

That is, each force is reciprocally proportional to the distance of its direction from the fulcrum.

Another proof might easily be made out from art. 25, on parallel forces; but it will be found that this demonstration

will serve for all the other kinds of levers, by drawing the lines as directed.

52. *Corol.* 1. When the angle A is = the angle W, then is CD : CE : : CW : CA : : P : W. Or when the two forces act perpendicularly on the lever, as two weights, &c.; then, in case of an equilibrium, D coincides with W, and E with P; consequently then the above proportion becomes also P : W : : CW : CA, or the distances of the two forces from the fulcrum, taken on the lever, are reciprocally proportional to those forces.

53. *Corol.* 2. If any force P be applied to a lever at A; its effect on the lever, to turn it about the centre of motion C, is as the length of the lever CA, and the sine of the angle of direction CAE. For the perp. CE is as CA × sin. ∠ A.

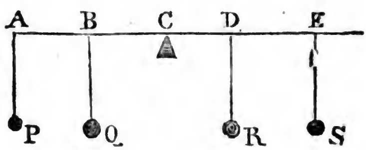
54. *Corol.* 3. Because the product of the extremes is equal to the product of the means, therefore the product of the power into the distance of its direction, is equal to the product of the weight into the distance of its direction.

That is, P × CE = W × CD.

55. *Corol.* 4. If the lever, with the weight and power fixed to it, be made to move about the centre C; the momentum of the power will be equal to the momentum of the weight; and their velocities will be in reciprocal proportion to each other. For the weight and power will describe circles whose radii are the distances CD, CE; and since the circumferences or spaces described are as the radii, and also as the velocities, therefore the velocities are as the radii CD, CE; and the momenta, which are as the masses and velocities, are as the masses and radii; that is, as P × CE and W × CD, which are equal by cor. 3.

56. *Corol.* 5. In a straight lever, kept in equilibrio by a weight and power acting perpendicularly; then, of these three, the power, weight, and pressure on the prop. any one is as the distance of the other two.

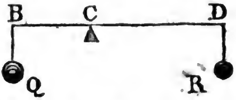
57. *Corol.* 6. If several weights P, Q, R, S, act on a straight lever, and keep it in equilibrio; then the sum of the products on one side of the prop. will be equal to the sum on the other side, made by multiplying each weight by its distance; namely,



(P × AC) + (Q × BC) = (R × DC) + (S × EC).

For, the effect of each weight to turn the level, is as the weight multiplied into its distance ; and in the case of an equilibrium, the sums of the effects, or of the products on both sides, are equal. The same would also follow from art. 26.

58. *Corol.* 7. Because, when two weights Q and R are in equilibrio, Q : R :: CD : CB ;

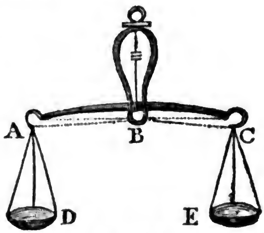


therefore, by composition, Q + R : Q :: BD : CD,
and, Q + R : R :: BD : CB.

That is, the sum of the weights is to either of them, as the sum of their distances is to the distance of the other.

SCHOLIUM.

59. On the foregoing principles depends the nature of scales and beams, for weighing all sorts of goods. For, if the weights be equal, then will the distances be equal also, which gives the construction of the common scales, which ought to have these properties :

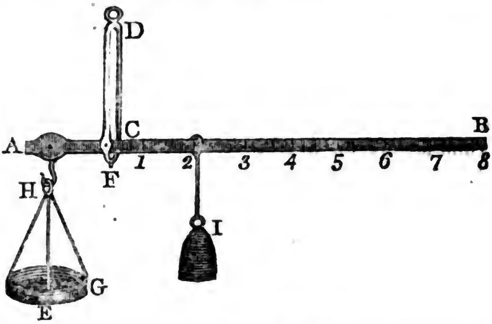


1*st*. That the points of suspension of the scales and the centre of motion of the beam, A, B, C, should be in a straight line : 2*d*, That the arms AB, BC, be of an equal length : 3*d*, That the centre of gravity be in the centre of motion B, or a little below it : 4*th*, That they be in equilibrio when empty : 5*th*, That there be as little friction as possible at the centre B. A defect in any of these properties makes the scales either imperfect or false. But it often happens that the one side of the beam is made shorter than the other, and the defect covered by making that scale the heavier, by which means the scales hang in equilibrio when empty : but when they are charged with any weights, so as to be still in equilibrio, those weights are not equal ; but the deceit will be detected by changing the weights to the contrary sides, for then the equilibrium will be immediately destroyed.

60. To find the true weight of any body by such a false balance :—First weigh the body in one scale, and afterwards weigh it in the other ; then the mean proportional between these two weights, will be the true weight required. For, if any body b weigh W pounds or ounces in the scale D, and only w pounds or ounces in the scale E : then we have these

two equations, namely, $AB \,.\, b = BC \,.\, W$,
and $BC \,.\, b = AB \,.\, w$;
the product of the two is $AB \,.\, BC \,.\, b^2 = AB \,.\, BC \,.\, Ww$;
hence then - - - $b^2 = Ww$,
and - - - $b = \sqrt{Ww}$,
the mean proportional, which is the true weight of the body b.

61. The Roman Statera, or Steelyard, is also a lever, but of unequal brachia or arms, so contrived, that one weight only may serve to weigh a great many, by sliding it backward and forward, to different distances, on the longer arm of the lever; and it is thus constructed:

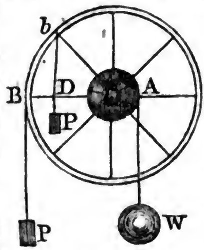


Let AB be the steelyard, and C its centre of motion, whence the divisions must commence if the two arms just balance each other: if not, slide the constant moveable weight I along from B towards C, till it just balance the other end without a weight, and there make a notch in the beam, marking it with a cipher 0. Then hang on at A a weight W equal to I, and slide I back towards B till they balance each other; there notch the beam, and mark it with 1. Then make the weight W double of I, and sliding I back to balance it, there mark it with 2. Do the same at 3, 4, 5, &c. by making W equal to 3, 4, 5, &c. times 1; and the beam is finished. Then, to find the weight of any body b by the steelyard: take off the weight W, and hang on the body b at A; then slide the weight I backward and forward till it just balance the body b, which suppose to be at the number 5; then is b equal to 5 times the weight of I. So, if I be one pound, then b is 5 pounds; but if I be 2 pounds, then b is 10 pounds; and so on.

OF THE WHEEL AND AXLE.

62. **Prop.** In the wheel and-axle; the weight and power will be in equilibrio, when the power P is to the weight W reciprocally as the radii of the circles where they act; that is, as the radius of the axle CA, where the weight hangs, to the radius of the wheel CB, where the power acts. That is, P : W : : CA : CB.

Here the cord, by which the power P acts, goes about the circumference of the wheel, while that of the weight W goes round its axle, or another smaller wheel, attached to the larger, and having the same axis or centre C. So that BA is a lever moveable about the point C, the power P acting always at the distance BC, and the weight W at the distance CA; therefore P : W : : CA : CB.

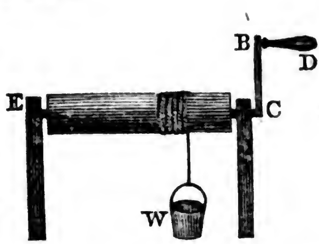


63. *Corol.* 1. If the wheel be put in motion; then, the spaces moved being as the circumferences, or as the radii, the velocity of W will be to the velocity of P, as CA to CB; that is, the weight is moved as much slower, as it is heavier than the power; so that what is gained in power, is lost in *time*. And this is the universal property of all machines and engines.

64. *Corol.* 2. If the power do not act at right angles to the radius C*b*, but obliquely; draw CD perpendicular to the direction of the power; then, by the nature of the lever, P : W : : CA : CD.

SCHOLIUM.

65. To this mechanical power belong all turning or wheel machines, of different radii. Thus, in the roller turning on the axis or spindle CE, by the handle CBD; the power applied at B is to the weight W on the roller as the radius of the roller is to the radius CB of the handle.



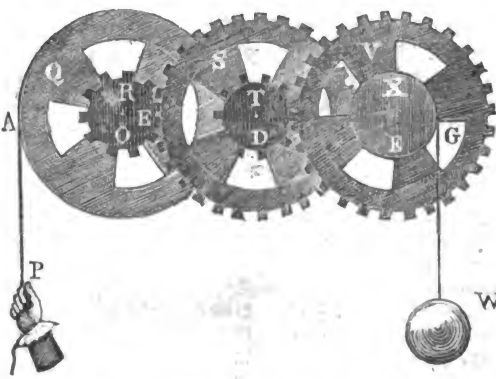
66. And the same for all cranes, capstans, windlasses, and

such like ; the power being to the weight, always as the radius or lever at which the weight acts, to that at which the power acts ; so that they are always in the reciprocal ratio of their velocities. And to the same principle may be referred the gimblet and augur for boring holes.

67. But all this, however, is on supposition that the ropes or cords, sustaining the weights, are of no sensible thickness. For, if the thickness be considerable, or if there be several folds of them, over one another, on the roller or barrel ; then we must measure to the middle of the outermost rope, for the radius of the roller ; or, to the radius of the roller, we must add half the thickness of the chord, when there is but one fold.

68. The wheel-and-axle has a great advantage over the simple lever, in point of convenience. For a weight can be raised but a little way by the lever ; whereas, by the continual turning of the wheel and roller, the weight may be raised to any height, or from any depth.

69. By increasing the number of wheels, too, the power may be multiplied to any extent, making always the less wheels to turn greater ones, as far as we please : and this is commonly called Tooth and Pinion Work, the teeth of one circumference working in the rounds or pinions of another, to turn the wheel. And then, in case of an equilibrium, the power is to the weight, as the continual product of the radii



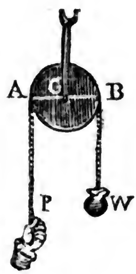
of all the axles, to that of all the wheels. So, if the power P turn the wheel Q, and this turn the small wheel or axle R, and this turn the wheel S, and this turn the axle T, and this turn the wheel V ; and this turn the axle X, which raises the

weight W ; then $P : W :: CB \cdot DE \cdot FG : AC \cdot BD \cdot EF$. And in the same proportion is the velocity of W slower than that of P. Thus, if each wheel be to its axle, as 10 to 1 ; then $P : W :: 1^3 : 10^3$ or as 1 to 1000. So that a power of one pound will balance a weight of 1000 pounds ; but then, when put in motion, the power will move 1000 times faster than the weight.

OF THE PULLEY.

70. A PULLEY is a small wheel, commonly made of wood or brass, which turns about an iron axis passing through the centre, and fixed in a block, by means of a cord passed round its circumference, which serves to draw up any weight. The pulley is either single, or combined together, to increase the power. It is also either fixed or moveable, according as it is fixed to one place, or moves up and down with the weight and power.

71. PROP. If a power sustain a weight by means of a fixed pulley : the power and weight are equal.

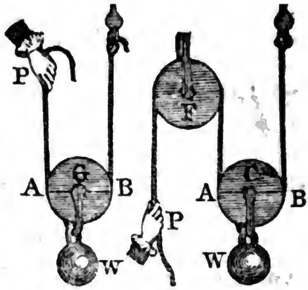


For through the centre C of the pulley draw the horizontal diameter AB : then will AB represent a lever of the first kind, its prop being the fixed centre C ; from which the points A and B, where the power and weight act, being equally distant, the power P is consequently equal to the weight W.

72. *Corol.* Hence, if the pulley be put in motion, the power P will descend as fast as the weight W ascends. So that the power is not increased by the use of the fixed pulley, even though the rope go over several of them. It is, however, of great service in the raising of weights, both by changing the direction of the force, for the convenience of acting, and by enabling a person to raise a weight to any height without moving from his place, and also by permitting a great many persons at once to exert their force on the rope at P, which they could not do to the weight itself; as is evident in raising the hammer or weight of a pile-driver, as well as on many other occasions.

73. PROP. If a power sustain a weight by means of one

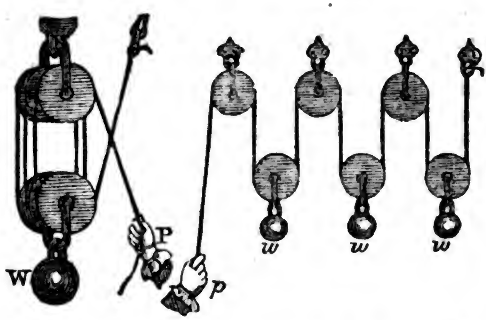
moveable pulley; the power is but half the weight, if the portions of the sustaining cord are parallel to each other.



For, here AB may be considered as a lever of the second kind, the power acting at A, the weight at C, and the prop or fixed point at B; and because P : W :: CB : AB, and CB = $\frac{1}{2}$AB, therefore P = $\frac{1}{2}$W, or W = 2P.

74. *Corol.* 1. Hence it is evident, that, when the pulley is put in motion, the velocity of the power will be double the velocity of the weight, as the point P moves twice as fast as the point C and weight W rises. It is also evident, that the fixed pulley F makes no difference in the power P, but is only used to change the direction of it, from upwards to downwards.

75. *Corol.* 2. Hence we may estimate the effect of a combination of any number of fixed and moveable pulleys; by which we shall find that every cord going over a moveable pulley always adds 2 to the power; since each moveable pulley's rope bears an equal share of the weight: while each rope that is fixed to a pulley, only increases the power by unity.



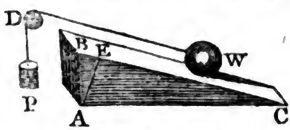
Here P = $\frac{1}{6}$W. Here $p = \frac{1}{2}w = \frac{w+w+w}{6}$.

Note.—If the portions of the sustaining cords between the pulleys are not parallel, the forces will be reduced upon the principle of art. 31.

OF THE INCLINED PLANE.

76. THE INCLINED PLANE, is a plane inclined to the horizon, or making an angle with it. It is often reckoned one of the simple mechanic powers; and the double inclined plane makes the wedge. It is employed to advantage in raising heavy bodies in certain situations, diminishing their weights by laying them on the inclined planes.

77. PROP. The power gained by the inclined plane, is in proportion as the length of the plane is to its height. That is, when a weight W is sustained on an inclined plane BC, by a power P acting in the direction DW, parallel to the plane; then the weight W, is in proportion to the power P, as the length of the plane is to its height; that is, W : P : : BC : AB.



For, draw AE perp. to the plane BC, or to DW. Then we are to consider that the body W is sustained by three forces, viz. 1st, its own weight or the force of gravity, acting perp. to AC, or parallel to BA; 2d, by the power P, acting in the direction WD, parallel to BC, or BE; and 3dly, by the re-action of the plane, perp. to its face, or parallel to the line EA. But when a body is kept in equilibrio by the action of three forces, it has been proved, (art. 30.) that the intensities of these forces are proportional to the sides of the triangle ABE, made by lines drawn in the directions of their actions; therefore those forces are to one another as the three lines - - - - AB, BE, AE; that is,
the weight of the body W is as the line AB,
the power P is as the line - - BE,
and the pressure on the plane as the line AE.
But the two triangles ABE, ABC, are equiangular, and have therefore their like sides proportional; that is,
the three lines - - - - - AB, BE, AE,
are to each other respectively as the three BC, AB, AC,
or also as the three - - - - AC, AE, CE,
which therefore are as the three forces, W, P, p,
which p denotes the pressure on the plane. That is, W : P : : BC : AB, or the weight is to the power, as the length of the plane is to its height.

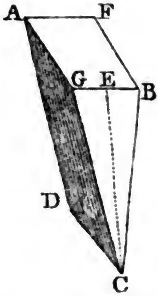
See more on the Inclined Plane in the Dynamics.

78. *Scholium.* The Inclined Plane comes into use in some situations in which the other mechanical powers cannot be conveniently applied, or in combination with them. As, in sliding heavy weights either up or down a plank or other plane laid sloping: or letting large casks down into a cellar, or drawing them out of it. Also, in removing earth from a lower situation to a higher by means of wheelbarrows, or otherwise, as in making fortifications, &c.; inclined planes, made of boards are employed. Rail-roads, or inclined planes, serve often to convey coals from the mouth of a mine.

Of all the various directions of drawing bodies up an inclined plane, or sustaining them on it, the most favourable is where it is parallel to the plane BC, and passing through the centre of the weight; a direction which is easily given to it, by fixing a pulley at D, so that a chord passing over it, and fixed to the weight, may act or draw parallel to the plane. In every other position, it would require a greater power to support the body on the plane, or to draw it up. For if one end of the line be fixed at W, and the other end inclined down towards B, below the direction WD, the body would be drawn down against the plane, and the power must be increased in proportion to the greater difficulty of the traction. And, on the other hand, if the line were carried above the direction of the plane, the power must be also increased; but here only in proportion as it endeavours to lift the body off the plane.

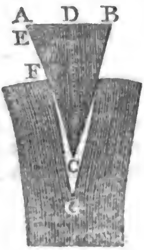
If the length BC of the plane be equal to any number of times its perp. height AB, as suppose 3 times; then a power P of 1 pound, hanging freely, will balance a weight W of 3 pounds, laid on the plane; and a power P of 2 pounds, will balance a weight W of 6 pounds; and so on, always 3 times as much. But then if they be set moving, the perp. descent of the power P, will be equal to 3 times as much as the perp. ascent of the weight W. For, though the weight W ascends up the direction of the oblique plane, BC, just as fast as the power P descends perpendicularly, yet the weight rises only the perp. height AB, while it ascends up the whole length of the plane BC, which is three times as much; that is, for every foot of the perp. rise of the weight, it ascends 3 feet up in the direction of the plane, and the power P descends just as much, or 3 feet.

OF THE WEDGE.



79. The Wedge is a piece of wood or metal, in form of half a rectangular prism. AF or BG is the breadth of its back; CE its height; GC, BC its sides: and its end GBC is composed of two equal inclined planes GCE, BCE.

80. Prop. When a wedge is in equilibrio; the power acting against the back, is to the force acting perpendicularly against either side, as the breadth of the back AB is to the length of the side AC or BC.



For, any three forces, which sustain one another in equilibrio, are as the corresponding sides of a triangle drawn perpendicular to the directions in which they act. But AB is perp. to the force acting on the back, to urge the wedge forward; and the sides AC, BC are perp. to the forces acting on them; therefore the three forces are as AB, AC, BC.

81. *Corol.* The force on the back	AB,
Its effect in direct. perp. to AC,	AC,
And its effect parallel to AB;	DC,
are as the three lines	which are per. to them.

And therefore the thinner a wedge is, the greater is its effect, in splitting any body, or in overcoming any resistance against the sides of the wedge.

SCHOLIUM.

82. But it must be observed, that the resistance, or the forces above-mentioned, respect one side of the wedge only. For if those against both sides be taken in, then, in the foregoing proportions, we must take only half the back AD, or else we must take double the line AC or DC. Various other theories of the wedge are given by different authors, but they need not here be detailed, on account of the irregularities introduced by friction.

In the wedge, the friction against the sides is very great, at least equal to the force to be overcome, because the wedge retains any position to which it is driven ; and therefore the resistance is doubled by the friction. But then the wedge has a great advantage over all the other powers, arising from the force of percussion or blow with which the back is struck, which is a force incomparably greater than any dead weight or pressure, such as is employed in other machines. And accordingly we find it produces effects vastly superior to those of any other power ; such as the splitting and raising the largest and hardest rocks, the raising and lifting the largest ship, by driving a wedge below it, which a man can do by the blow of a mallet : and thus it appears that the small blow of a hammer, on the back of a wedge is incomparably greater than any mere pressure, and will overcome it.

OF THE SCREW.

83. The Screw is one of the six mechanical powers, chiefly used in pressing or squeezing bodies close, though sometimes also in raising weights.

The screw is a spiral thread or groove cut round a cylinder, and every where making the same angle with the length of it. So that if the surface of the cylinder, with this spiral thread on it, where unfolded and stretched into a plane, the spiral thread would form a straight inclined plane, whose length would be to its height, as the circumference of the cylinder, is to the distance between two threads of the screw : as is evident by considering that, in making one round, the spiral rises along the cylinder the distance between the two threads.

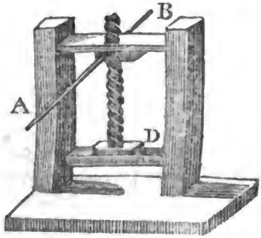
84. Prop. The energy of a power applied to turn a screw round, is to the force with which it presses upward or downward, setting aside the friction, as the distance between two threads, is to the circumference where the power is applied.

The screw being an inclined plane, or half wedge, whose height is the distance between two threads, and its base the circumference of the screw ; and the force in the horizontal direction, being to that in the vertical one, as the lines perpendicular to them, namely, as the height of the plane, or distance of the two threads, is to the base of the plane, or circumference of the screw ; therefore the power is to the pressure, as the distance of two threads is to that circumference. But, by means of a handle or lever, the gain in

power is increased in the proportion of the radius of the screw to the radius of the power, or length of the handle, or as their circumferences. Therefore, finally, the power is to the pressure, as the distance of the threads, is to the circumference described by the power.

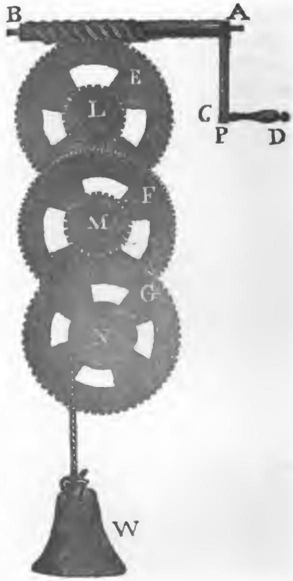
85. *Corol.* When the screw is put in motion; then the power is to the weight which would keep it in equilibrio, as the velocity of the latter is to that of the former; and hence their two momenta are equal, which are produced by multiplying each weight or power by its own velocity. So that this is a general property in all the mechanical powers, namely, that the momentum of a power is equal to that of the weight which would balance it in equilibrio; or that each of them is reciprocally proportional to its velocity.

SCHOLIUM.



86. Hence we can easily compute the force of any machine turned by a screw. Let the annexed figure represent a press driven by a screw, whose threads are each a quarter of an inch asunder: and let the screw be turned by a handle of 4 feet long, from A to B; then, if the natural force of a man, by which he can lift, pull, or draw, be 150 pounds; and it be required to determine with what force the screw will press on the board at D, when the man turns the handle at A and B, with his whole force. Then the diameter AB of the power being 4 feet, or 48 inches, its circumference is $48 \times 3{\cdot}1416$ or $150\frac{4}{5}$ nearly; and the distance of the threads being $\frac{1}{4}$ of an inch; therefore the power is to the pressure, as 1 to $603\frac{1}{5}$; but the power is equal to 150lb; theref. as $1 : 603\frac{1}{5} :: 150 : 90480$; and consequently the pressure at D is equal to a weight of 90480 pounds, independent of friction.

87. Again, if the endless screw AB be turned by a handle AC of 20 inches, the threads of the screw being distant half an inch each; and the screw turns a toothed wheel E, whose pinion L turns another wheel F, and the pinion M of this another wheel G, to the pinion or barrel of which is hung a weight W; it is required to determine what weight the man will be able to raise, working at the handle O; supposing



the diameters of the wheels to be 18 inches, and those of the pinions and barrel 2 inches; the teeth and pinions being all of a size.

Here $20 \times 3 \cdot 1416 \times 2 = 125 \cdot 664$, is the circumference of the power.

And $125 \cdot 664$ to $\frac{1}{2}$, or $251 \cdot 328$ to 1, is the force of the screw alone.

Also, 18 to 2, or 9 to 1, being the proportion of the wheels to the pinions; and as there are three of them, therefore 9^3 to 1^3, or 729 to 1, is the power gained by the wheels.

Consequently $251 \cdot 328 \times 729$ to 1, or $183218\frac{1}{4}$ to 1 nearly, is the ratio of the power to the weight, arising from the advantage both of the screw and the wheels.

But the power is 150lb; therefore $150 \times 183218\frac{1}{4}$, or 27482716 pounds, is the weight the man can sustain, which is equal to 12269 tons weight.

But the power has to overcome, not only the weight, but also the friction of the screw, which is very great, in some cases equal to the weight itself, since it is sometimes sufficient to sustain the weight, when the power is taken off.

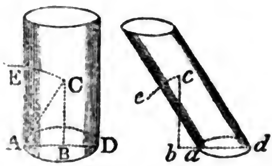
88. Upon the same principle the advantage of any other combination of the mechanical powers may be computed: allowance, however, being always to be made for stiffness of cords, friction, and other causes of resistance.

ON THE CENTRE OF GRAVITY.

89. The Centre of Gravity of a body, or of a system of bodies, is a certain point within it, or connected with it, on which the body being freely suspended, it will rest in any

position, and that centre will always tend to descend to the lowest place to which it can get, when it is not the point of suspension.

90. PROP. If a perpendicular to the horizon, from the centre of gravity of any body, fall within the base of the body, it will rest in that position; but if the perpendicular fall out of the base, the body will not rest in that position, but will fall down.



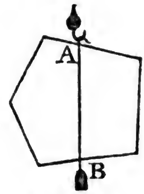
For, if CB be the perp. from the centre of gravity C, within the base: then the body cannot fall over towards A; because, in turning on the point A, the centre of gravity C would describe an arc which would rise from C to E; contrary to the nature of that centre, which only rests permanently when in the lowest place. For the same reason, the body will not fall towards D. And therefore it will stand in that position.

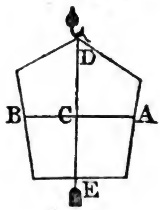
But if the perpendicular fall out of the base, as *cb*; then the body will fall over on that side: because, in turning on the point *a*, the centre *c* descends by describing the descending arc *ce*.

91. *Corol.* 1. If a perpendicular, drawn from the centre of gravity, fall just on the extremity of the base, the body may stand; but any the least force will cause it to fall that way. And the nearer the perpendicular is to any side, or the narrower the base is, the easier it will be made to fall, or be pushed over that way; because the centre of gravity has the less height to rise: which is the reason that a globe is made to roll on a smooth plane by any the least force. But the nearer the perpendicular is to the middle of the base, or the broader the base is, the firmer the body stands.

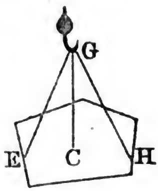
92. *Corol.* 2. Hence if the centre of gravity of a body be supported, the whole body is supported. And the place of the centre of gravity may, in many inquiries, be accounted the place of the body; for into that point the whole matter of the body may be supposed to be collected, and therefore all the force also with which it endeavours to descend.

93. *Corol.* 3. From the property which the centre of gravity has, of tending to descend to the lowest point, is derived an easy mechanical method of finding that centre.





Thus, if the body be hung up by any point A, and a plumb line AB be hung by the same point, it will pass through the centre of gravity; because that centre is not in the lowest point till it fall in the plumb line. Mark the line AB on it. Then hang the body up by any other point D, with a plumb line DE, which will also pass through the centre of gravity, for the same reason as before; and therefore that centre must be at C where the two plumb lines cross each other.



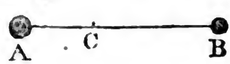
94. Or, if the body be suspended by two or more cords, GF, GH, &c. then a plumb line from the point G, will cut the body in its centre of gravity C.

95. Likewise, because a body rests when its centre of gravity is supported, but not else; we hence derive another easy method of finding that centre mechanically. For, if the body be laid on the edge of a prism, or over one side of a table, and moved backward and forward till it rest, or balance itself; then is the centre of gravity just over the line of the edge. And if the body be then shifted into another position, and balanced on the edge again, this line will also pass by the centre of gravity; and consequently the intersection of the two will indicate the place of the centre itself.

The place of the centre of gravity may be investigated, from its analogy to the centre of parallel forces; but the following method is adopted here, as in some respects easier of comprehension.

96. PROP. The common centre of gravity C of any two bodies A, B, divides the line joining their respective centres, into two parts, which are reciprocally as the bodies.

That is, AC : BC : : B : A.



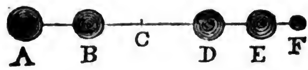
For, if the centre of gravity C be supported, the two bodies A and B will be supported, and will

rest in equilibrio. But, by the nature of the lever, when two bodies are in equilibrio about a fixed point C, they are reciprocally as their distances from that point; therefore A : B : : CB : CA.

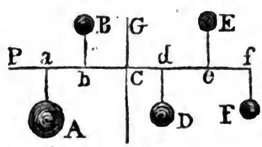
97. *Corol.* 1. Hence AB : AC : : A + B : B; or, the whole distance between the two bodies, is to the distance of either of them from the common centre, as the sum of the bodies is to the other body.

98. *Corol.* 2. Hence also, CA . A = CB . B; or the two products are equal, which are made by multiplying each body into its distance from the centre of gravity.

99. *Corol.* 3. As the centre C is pressed with a force equal to both the weights A and B, while the points A and B are each pressed with the respective weights A and B; therefore, if the two bodies be both united in their common centre C, and only the ends A and B of the line AB be supported, each will still bear, or be pressed by the same weight A and B as before. So that, if a weight of 100lb. be laid on a bar at C, supported by two men at A and B, distant from C, the one 4 feet, and the other 6 feet; then the nearer will bear the weight of 60lb. and the farther only 40lb. weight. This should be noted as a principle of extensive application.



100. *Corol.* 4. Since the effect of any body to turn a lever about the fixed point C, is as that body and as its distance from that point; therefore, if C be the common centre of gravity of all the bodies A, B, D, E, F, placed in the straight line AF; then is CA . A + CB . B = CD . D + CE . E + CF . F; or, the sum of the products on one side, equal to the sum of the products on the other, made by multiplying each body into its distance from that centre. And if several bodies be in equilibrio on any straight lever, then the prop is in the centre of gravity.



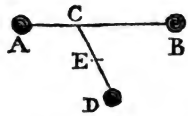
101. *Corol.* 5. And though the bodies be not situated in a straight line, but scattered about in any promiscuous manner, the same property as in the last corollary still holds true, if perpendiculars to any line whatever *af* be drawn through the several bodies, and their common centre of gravity, namely, that *ca* . A + *cb* . B = *cd* . D + *ce* . E + *cf* . F. For

the bodies have the same effect on the line af, to turn it about the point C, whether they are placed at the points a, b, d, e, f, or in any part of the perpendiculars Aa, Bb, Dd, Ee, Ff.

102. Prop. If there be three or more bodies, and if a line be drawn from any one body D to the centre of gravity of the rest C; then the common centre of gravity E of all the bodies, divides the line CD into two parts in E, which are reciprocally proportional as the body D to the sum of all the other bodies.

That is, $\text{CE} : \text{ED} :: \text{D} : \text{A} + \text{B}$, &c.

For, suppose the bodies A and B to be collected into the common centre of gravity C, and let their sum be called S. Then, by the last prop. $\text{CE} : \text{ED} :: \text{D} : \text{S}$ or $\text{A} + \text{B}$, &c.

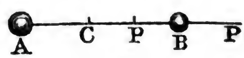


Corol. Hence we have a method of finding the common centre of gravity of any number of bodies; namely, by first finding the centre of any two of them, then the centre of that centre and a third, and so on for a fourth, or fifth, &c.

103. Prop. If there be taken any point P, in the line passing through the centres of two bodies; then the sum of the two products, of each body multiplied into its distance from that point, is equal to the product of the sum of the bodies multiplied into the distance of their common centre of gravity C from the same point P.

That is, $\text{PA} \cdot \text{A} \pm \text{PB} \cdot \text{B} = \text{PC} \cdot \overline{\text{A} + \text{B}}$.

For, by art. 98th, $\text{CA} \cdot \text{A} = \text{CB} \cdot \text{B}$, that is, $(\text{PA} - \text{PC}) \cdot \text{A} = (\text{PC} \pm \text{PB}) \cdot \text{B}$; therefore, by adding, $\text{PA} \cdot \text{A} \pm \text{PB} \cdot \text{B} = \text{PC} \cdot (\text{A} + \text{B})$.



104. *Corol.* 1. Hence, the two bodies A and B have the same force to turn the lever about the point P, as if they were both placed in C their common centre of gravity.

Or, if the line, with the bodies, move about the point P; the sum of the momenta of A and B, is equal to the momentum of the sum S or $\text{A} + \text{B}$ placed at the centre C.

105. *Corol.* 2. The same is also true of any number of bodies whatever, as will appear by cor. 4, art. 100. namely, $\text{PA} \cdot \text{A} + \text{PB} \cdot \text{B} + \text{PD} \cdot \text{D}$, &c. $= \text{PC} \cdot (\text{A} + \text{B} + \text{D}$, &c.) where P is in any point whatever of the line AC.

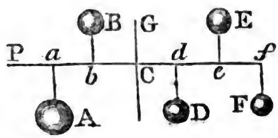
And, by cor. 5, art. 101, the same thing is true when the

bodies are not placed in that line, but any where in the perpendiculars passing through the points A, B, D, &c.; namely, $Pa \cdot A + Pb \cdot B + Pd \cdot D$, &c. $= PC \cdot (A + B + D$, &c.$)$.

106. *Corol.* 3. And if a plane pass through the point P perpendicular to the line CP; then the distance of the common centre of gravity from that plane, is

$$PC = \frac{Pa \cdot A + Pb \cdot B + Pd \cdot D, \ \&c.}{A + B + D, \ \&c.},$$ that is, equal to the sum of all the moments divided by the sum of all the bodies. Or, if A, B, D, &c. be the several particles of one mass or compound body; then the distance of the centre of gravity of the body, below any given point P, is equal to the forces of all the particles divided by the whole mass or body, that is, equal to all the $Pa \cdot A$, $Pb \cdot B$, $Pd \cdot D$, &c. divided by the body or sum of particles A, B, D, &c.

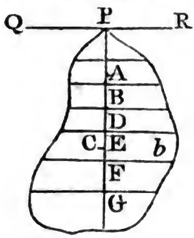
107. PROP. To find the centre of gravity of any body, or of any system of bodies.



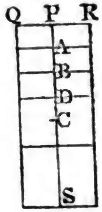
Through any point P draw a plane, and let Pa, Pb, Pd, &c. be the distance of the bodies A, B, D, &c. from the plane; then, by the last cor. the distance of the common centre of gravity from the plane, will be

$$PC = \frac{Pa \cdot A + Pb \cdot B + PD \cdot D, \ \&c.}{A + B + D, \ \&c.}$$

108. Or, if b be any body, and QPR any plane; draw PAB, &c. perpendicular to QR, and through A, B, &c. draw innumerable sections of the body b parallel to the plane QR. Let s denote any one of these sections, and $d =$ PA, or PB, &c. its distance from the plane QR. Then will the distance of the centre of gravity of the body from the plane be $PC = \dfrac{\text{sum of all the } ds}{b}$. And if the distance be thus found for two intersecting planes, they will give the point in which the centre is placed.



109. But the distance from one plane is sufficient for any regular body, because it is evident that, in such a figure, the centre of gravity is in the axis, or line passing through the centres of all the parallel sections.



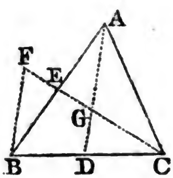
Thus, if the figure be a parallelogram, or a cylinder, or any prism whatever; then the axis or line, or plane PS, which bisects all the sections parallel to QR, will pass through the centre of gravity of all those sections, and consequently through that of the whole figure C. Then, all the sections s being equal, and the body $b = \text{PS} \cdot s$, the distance of the centre will be PC =

$$\frac{\text{PA} \cdot s + \text{PB} \cdot s + \&c.}{b} = \frac{\text{PA} + \text{PB} + \text{PD}\ \&c.}{\text{PS} \cdot s} \times s = \frac{\text{PA} + \text{PB} + \&c.}{\text{PS}}.$$

But PA + PB + &c. is the sum of an arithmetical progression, beginning at 0, and increasing to the greatest term PS, the number of the terms being also equal to PS; therefore the sum $\text{PA} + \text{PB} + \&c. = \frac{1}{2}\text{PS} \cdot \text{PS}$; and consequently $\text{PC} = \frac{\frac{1}{2}\text{PS} \cdot \text{PS}}{\text{PS}} = \frac{1}{2}\text{PS}$; that is, the centre of gravity is in the middle of the axis of any figure whose parallel sections are equal.

110. In other figures, whose parallel sections are not equal, but varying according to some general law, it will not be easy to find the sum of all the $\text{PA} \cdot s$, $\text{PB} \cdot s'$, $\text{PD} \cdot s''$, &c. except by the general method of Fluxions; which case therefore will be best reserved till we come to treat of that doctrine. It will be proper, however, to add here some examples of another method of finding the centre of gravity of a triangle, or any other right-lined plane figure.

111. PROP. To find the centre of gravity of a triangle.



From any two of the angles draw lines AD, CE, to bisect the opposite sides; so will their intersection G be the centre of gravity of the triangle.

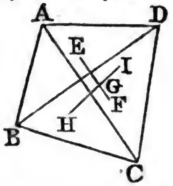
For, because AD bisects BC, it bisects also all its parallels, namely, all the parallel sections of the figure: therefore AD passes through the centres of gravity of all the parallel sections or component parts of the figure; and consequently the centre of gravity of the whole figure lies in the line AD. For the same reason, it also lies in the line CE. Consequently it is in their common point of intersection G.

112. *Corol.* The distance of the point G, is $\text{AG} = \frac{2}{3}\text{AD}$, and $\text{CG} = \frac{2}{3}\text{CE}$: or $\text{AG} = 2\text{GD}$, and $\text{CG} = 2\text{GE}$.

For, draw BF parallel to AD, and produce CE to meet it in F. Then the triangles AEG, BEF are similar, and also

equal, because AE = BE ; consequently AG = BF. But the triangles CDG, CBF are also equiangular, and CB being = 2CD, therefore BF = 2GD. But BF is also = AG ; consequently AG = 2GD or $\frac{2}{3}$AD. In like manner, CG = 2GE or $\frac{2}{3}$CE.

113. PROP. To find the centre of gravity of a trapezium.



Divide the trapezium ABCD into two triangles, by the diagonal BD, and find E, F, the centres of gravity of these two triangles : then shall the centre of gravity of the trapezium lie in the line EF connecting them. And therefore if EF be divided, in G, in the alternate ratio of the two triangles, namely, EG : GF :: triangle BCD : triangle ABD, then G will be the centre of gravity of the trapezium.

114. Or, having found the two points E, F, if the trapezium be divided into two other triangles BAC, DAC, by the other diagonal AC, and the centres of gravity H and I of these two triangles be likewise found ; then the centre of gravity of the trapezium will also lie in the line HI.

So that, lying in both the lines, EF, HI, it must necessarily lie in their intersection G.

115. And thus we are to proceed for a figure of any greater number of sides, finding the centres of their component triangles and trapeziums, and then finding the common centre of every two of these, till they be all reduced into one only.

PROBLEMS FOR EXERCISE.

1. Find, geometrically, the centre of gravity of a trapezoid.

2. Find, geometrically, the centre of gravity of a triangular pyramid.

3. Infer, thence, the centre of gravity of any pyramid.

4. Find, algebraically, the centre of gravity of the frustum of a pyramid.

5. Let a sphere whose diameter is 4 inches, and a cone whose altitude is 8 inches, and diameter of its base 3 inches, be fastened upon a thin wire which shall pass through the centre of the globe and the axis of the cone ; let the vertex of the cone be toward the sphere, and let its distance from the sphere's surface be 12 inches. Required the place of their common centre of gravity.

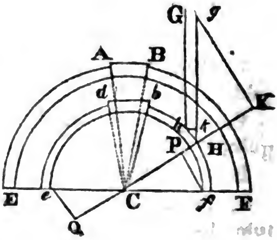
6. Demonstrate 1st, That the surface produced by a plane line or curve by revolving about an axis in the plane of that curve, is equal to the product of the generating line or curve into the path described by the centre of gravity.

And 2dly. That the solid produced by the revolution of a plane figure about an axis posited in the plane of that figure, is equal to the product of the generating surface into the circumstance described by the centre of gravity.

ON THE EQUILIBRIUM OF ARCHES.

116. A very interesting department of the science of Statics, is that which relates to the stability of arches, as introduced in the construction of bridges, powder-magazines, &c. Every such structure is a system of forces, and the examination of its firmness, therefore, requires the application of the general principles of equilibrium. We shall here present a few useful propositions in elucidation of the more received theories.

117. Prop. The force of a voussoir depending on the magnitude of the angle formed by its sides, the impelling force, and the resistance to be overcome, is on the first account directly as the radius of curvature of the arch at that point, on the second as the square of the sine of the angle included between the tangent of the curve at the given point and the vertical passing through that point, and on the third, as the sine of the same angle.



1. Let EABF, *eabf*, be two similar concentric curves, and AB, *ab*, two voussoirs similarly situated, whose sides perpendicular to the curve converge to the centre C. The forces of these voussoirs considered as portions of wedges, are inversely as the sines of the half vertical angles (schol. art. 82.) or, because each wedge occupies an equal portion of its respective arch, directly as the radii of curvature.

2ndly, Let H*h* be the invariable breadth of the voussoirs on the arch *eabf*, G*g*H*h* the incumbent weight, which, since GH is supposed given, is as the breadth *hk*, or as the sine of the angle *h*H*k*: by the resolution of the force *g*H into two *h*H, HK, the latter is the force impelling the voussoir to split the

arch, which, since gH is given, varies as the sine of Hgk, or hHk: wherefore, the force impelling the voussoir is as the square of the sine of hHk.

3dly, The wedge impelled in a direction perpendicular to the curve endeavours to split the arch, and therefore to move one segment about the fulcrum e, the other about the fulcrum f. Hence the force of the voussoir acting on the levers Hf, He, being as either of the perpendiculars fP, eQ, is as the sine of the angle fCP or hHk.

We have supposed the centre of curvature of the arches at the points A, a, h, H, to be at C: but this is merely to prevent the figure from being too complex, and makes no alteration in the nature of the demonstration.

Corol. Hence, if the height of the wall incumbent on any point H of the intrados is inversely as the cube of the sine of hHk into radius of curvature at that point, or directly as cube of the secant of the angle formed by hH and the horizon, and inversely as the radius of curvature, all the voussoirs will endeavour to split the arch with equal forces, and will be in perfect equilibrium with each other.

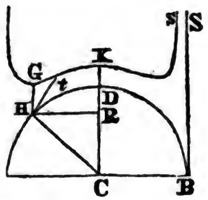
The general expression, therefore, for the thickness GH over any point of an arch, is

$$GH = \sec.^3 \text{elev}^n. \text{ at } H \times \frac{ar}{R}$$

where r = rad. of curvature at the vertex
a = thickness of material there
R = rad. of curvature at H.

The radii of curvature for the different curves are determinable by the method of fluxions, or by other means: they are here supposed known.

I. Suppose, for example, it were required to find the requisite thickness over any point of a circular arc, to ensure equilibration, the thickness a = DK, at the crown of the arch being given.



Here, rad. of curv. at H = rad. of curv. at D
that is $R = r$.
and sec. RHt = sec. arc. DH.

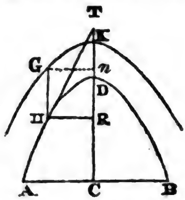
$$\text{Conseq. } GH = \sec.^3 DH \times \frac{ar}{R} = \sec.^3 DH \,.\, a.$$

Hence we have a convenient logarithmic expression for computation; viz.

$$\text{Log. DK} + 3 \text{ log. sec. DH} = \text{log. GH}.$$

In this example, the curve of equilibration, GKs, runs up to an infinite height over B, the springing of a semicircular arch. But over a portion of 30° or 35° on each side the vertex, as DH, the curve KG of the extrados accords very well with what would be required for a roadway.

Ex. 2. Determine the requisite thickness for equilibration over any point of a parabola.



Here if DR $= x$, RH $= y$, R $= \dfrac{(4x+p)^{\frac{3}{2}}}{2\sqrt{p}}$; which, at the vertex, where x vanishes becomes $r = \frac{1}{2}p$:

$$\text{RT} = 2x,\ \text{RH} = y = \sqrt{px},\ \text{TH} = \sqrt{\text{HR}^2 + \text{RT}^2} = \sqrt{px + 4x^2}$$

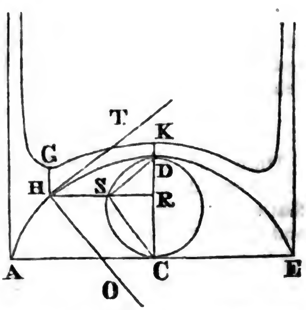
$$\text{sec. THR} = \frac{\text{TH}}{\text{HR}} = \sqrt{\frac{px + 4x^2}{px}} = \sqrt{\frac{p + 4x}{p}}$$

$$\therefore \text{GH} = \text{sec.}^3 \text{THR} \cdot \frac{r}{\text{R}} a$$

$$= \frac{(p + 4x)^{\frac{3}{2}}}{p^{\frac{3}{2}}} \cdot \tfrac{1}{2}p \cdot \frac{2\sqrt{p}}{(p + 4x)^{\frac{3}{2}}} a = a = \text{KD}.$$

So that the extrados is a parabola equal to the intrados, and every where vertically equidistant from it.

Ex. 3. To determine the requisite thickness over any point of a cycloidal arch.



Here, putting DK $= a$, DR$=x$, DC$=d$; we have, from the known properties of the cycloid, the tangent HT parallel to the corresponding chord SD, or angle THR $=$ ∠ DSR, SD $= \sqrt{dr}$, SR $= \sqrt{dx - x^2}$;

R (parallel to SC) $=$ HO $=$ 2SC $= 2\sqrt{d^2 - dx}$; r=2CD=$2d$:

$$\text{sec. THR} = \frac{\text{SD}}{\text{SR}} = \sqrt{\frac{d}{dx - x^2}} = \sqrt{\frac{dx}{d-x}}.$$

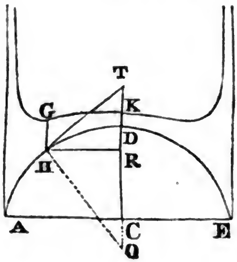
$$\text{GH} = \text{sec.}^3\,\text{THR} \,.\, \frac{r}{\text{R}} \,.\, a$$

$$= \frac{d^{\frac{3}{2}}}{(d-x)^{\frac{3}{2}}} \,.\, 2d \,.\, \frac{a}{2(d^2 - dx)^{\frac{1}{2}}},$$

$$= \frac{d^{\frac{3}{2}}}{(d-x)^{\frac{3}{2}}} \,.\, \frac{2da}{2d^{\frac{1}{2}}(d-x)^{\frac{1}{2}}},$$

$$= \frac{d^{\frac{4}{2}}}{(d-x)^{\frac{4}{2}}}a = \frac{ad^2}{(d-x)^2} = \frac{\text{DK} \,.\, \text{CD}^2}{\text{CR}^2}.$$

By computing the value of GH for several corresponding values of DR, and CR, and thence constructing the extrados by points, it will, as in the figure, appear analogous to that for the circle, but rather flatter till it approach the extremities of the arch, where the curve runs off to infinity, as in the case for the circle.



Exam. 4. To determine the requisite thickness over any point of an elliptical arch.

Here, taking x, y, and a, as before, take AC $= t$, DC $= c$, HQ $= \pi$, being perpendicular to the tangent HT. Then, by the property of the ellipse,

$$\text{DC}^2 : \text{AC}^2 :: \text{CR} : \text{QR},$$

$$\text{or, } c^3 : t^2 :: c - x : \frac{t^2}{c^2}(c - x) = \text{QR}.$$

Also, $\text{sec. THR} = \text{sec. HQR} = \frac{\text{HQ}}{\text{QR}} = \pi \div \frac{t^3}{c^2}(c-x) = \frac{\pi c^2}{t^2(c-x)}.$

Radius of curvature at H $= \text{R} = \frac{4\pi^3}{p^2}$, p being the parameter to CD $= \frac{2t^2}{c}$:

$$\therefore \text{R} = \frac{4\pi^3 c^2}{4t^4} = \frac{\pi^3 c^2}{t^4}\text{; and } r \text{ (rad. curv. at D)} = \frac{t^2}{c}.$$

Whence, lastly, $GH = \sec.^3 THR \cdot \frac{r}{R} \cdot a$,

$$= \frac{\pi^3 c^6}{t^6 (c-x)^3} \cdot \frac{t^2}{c} \cdot \frac{t^4}{\pi^3 c^2} a,$$

$$= \frac{c^3}{(c-x)^3} a = \frac{DK \cdot DC^3}{CR^3}:$$

as before, a convenient expression for logarithmic operation.

Here, again, computing values of GH for several assumed values of CR, the curve of the extrados may thence be constructed, and, like that for the cycloid, it will be found rather flatter than that for the circle, but still analogous to it.

EXAM. 5. For the Catenary. (See the fig. to Exam. 2.)

Here, put $DR = x$, $GR = y$, $DG = z$, $t =$ tension at the vertex D when the chain hangs from A and B. Then, by the nature of the curve $z^2 = 2tx + x^2$, subtang. $TR = \frac{zy}{t}$

Rad. curv. at $G = \frac{t^2 + z^2}{t} = R$, and therefore at D where z vanishes $r = t$.

$$HT = \sqrt{HR^2 + RT^2} = \sqrt{y^2 + \frac{z^2 y^2}{t^2}},$$

$$\sec. THR = \frac{TH}{HR} = \sqrt{y^2 + \frac{z^2 y^2}{t^2}} \div y = \sqrt{1 + \frac{z^2}{t^2}} = \sqrt{\frac{t^2 + z^2}{t^2}}.$$

$$\therefore GH = \sec.^3 THR \cdot \frac{r}{R} a = \frac{(t^2 + z^2)^{\frac{3}{2}}}{t^3} \cdot t \cdot \frac{t}{t^2 + z^2} \cdot a$$

$$= \frac{(t^2 + z^2)^{\frac{1}{2}} t^2 a}{t^3} = \frac{(t^2 + z^2)^{\frac{1}{2}} a}{t} \text{ [sub.}^g \text{ for } z^2 \text{ its value.]}$$

$$= \frac{(t^2 + 2tx + x^2)^{\frac{1}{2}} a}{t} = \frac{a(t+x)}{t} = a + \frac{ax}{t}.$$

Corol. If $a = t$, or the thickness at the crown equal to a line whose weight expresses the tension,

then $GH = a + x = KD + DR$.

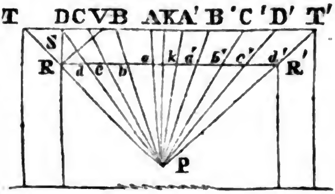
Corol. 2. If $a \gtrless t$, the exterior curve will proceed $\left\{\begin{matrix}\text{upwards}\\\text{downwards}\end{matrix}\right\}$ both ways from K.

Corol. 3. If DK, the thickness at the crown, be very small compared with t, then will the thickness over H be nearly

the same throughout : thus, suppose $a = \frac{1}{10000}t$, then GH $= a + \frac{tx}{10000t} = a + \frac{x}{10000} = a$ very nearly. Consequently, a heavy flexible cord or chain, left to adjust itself into a hanging catenary, and inverted, would support itself upon props perpendicular to the tangents at A and B.

Or $\text{K}n = a + x - (a + \frac{ax}{t}) = x - \frac{ax}{t} = (t - a)\frac{x}{t}$; which when a vanishes becomes $= x$, or $n\text{R} = \text{GH} = a$.

118. Prop. To point out the construction, and investigate the chief properties of the plat-band, or "flat arch," as it is sometimes called.



Let RR′ be the proposed width, and Kk the proposed thickness of a plat-band. Assume a point P in the inferior prolongation of Kk the middle of the structure ; and, supposing aa', ab, bc, cd, &c. the proposed thicknesses at bottom, of the truncated wedges of which the plat-band is to be constituted, let straight lines Pa'A′, PaA, PbB, PcC, &c. be drawn, they will respectively show the directions in which the mutually abutting faces of the several wedges are to be cut, so that the whole shall be an equilibrated structure.

Now, 1st, If $ak = a'k$, be taken to represent half the *weight* of the central wedge, then Pk perpendicular to it will represent the *horizontal thrust* throughout the plat-band, and consequently, the *thrust*, *shoot*, or *drift*, acting at R or R′.

2dly, Therefore, by assuming P nearer or farther from RR′, the thrust may be diminished or increased at pleasure.

3dly, No one of the wedges has a greater tendency to fall downwards than another ; for those tendencies are throughout as their weights, each being represented by the successive lines ab, bc, cd, &c. on both sides the key-stone. The former are as the differences of the tangents ka, kb, kc, &c. to the radius Pk ; and the latter are as the areas of the trapezoids abBA, bcCB, &c. which are as ab + AB to bc + BC, or as ab to bc ; the common height of all the trapezoids being equal to kK.

4thly, The *pressure* on each joint of the plat-band is

proportional to the surface of that joint. For, pressure on aA to pressure on bB, as Pa to Pb, that is, as aA to bB; and so throughout. The pressures being exerted perpendicularly to the respective surfaces, are evidently measured by lines in the directions of those surfaces (art. 32.) when we have assumed a horizontal line for the measure of gravity, and a vertical line to measure the horizontal thrust.

5thly, Hence also it follows that in this construction the pressure upon each square inch of joint, is a constant quantity throughout; being the same upon every square inch of the face in direction aA, as upon every square inch of face in direction bB, in direction cC, &c. to the extreme abutments RT, R′T′.

These properties will not be found co-existent in any other equilibrated structure.

119. *Scholium.* Yet this construction has a limitation which it is highly important to observe. To ensure stability, the distance of the centre of gravity of the semi-vault from the vertical PK, must exceed KV, the distance from the same vertical to the intersection of RV (a perpendicular to the abutment TR) with the top TT′ of the plat-band. Unless this condition be fulfilled, perpendiculars cannot be let fall from the centre of gravity upon both TR and Kk; or, in other words, the semi-vault cannot be sustained by means of the two surfaces TR, and Kk alone.

Let Rk = kR′ = h, Kk = k, and TS = t, being the tangent of the ulterior angle of slope to the radius RS = k. Then the distance of the centre of gravity of the semi-vault KkRT from the middle, Kk, of the key-stone will be

$$= \tfrac{1}{2}h + \frac{3ht + 2t^2}{12h + 6t}.$$

Farther, we have $t : k :: k :$ DV $= \dfrac{k^2}{t}$.

Therefore KV = KD — DV $= h - \dfrac{k^2}{t} = \dfrac{th - k^2}{t}$.

Hence, to ensure stability, we must have

$$1. \quad \tfrac{1}{2}h + \frac{3ht + 2t^2}{12h + 6t} > \frac{th - k^2}{t} :$$

Or, taking the limit of tottering equilibrium, we have

$$2. \quad t^3 - 3(h^2 - k^2)t + 6hk^2 = 0 :$$

from which when two of the three letters are known the third may be found.

Suppose, for example, that a plat-band were constructed upon an equilateral triangle, or such that angle RPR′ = 60°. Then TS $= t =$ tan. 30° to rad. Kk. Or, if K$k = k$, be taken $= 1$, then $t =$ tan. $30° = \frac{1}{3}\sqrt{3}$.

$$\text{Hence } \tfrac{1}{27}\, 3^{\frac{3}{2}} - (h^2 - 1)\sqrt{3} + 6h = 0.$$

From this equation h, in the case of the limit is found

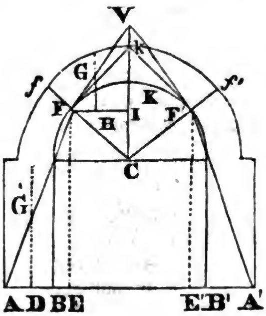
$$= \tfrac{1}{3}\sqrt{37} + \sqrt{3} = 3{\cdot}7596.$$

Consequently, in the proposed case, RR′ $= 2h$ must be less than 7·5192, or than $7\frac{1}{2}$ times the thickness, Kk, of the key-stone.

Of the Equilibrium of Vaults, regarding the Tenacity of Cements.

120. When the operation of cements is taken into the consideration, the conditions to ensure equilibrium are more easily investigated than when the gravitating tendency of the superincumbent matter is alone regarded. If the cohesive energy of the cement were insuperable, the arch might then be considered as one mass, which would be every where secure, whatever its form might be, provided the piers or abutments were sufficiently strong to resist the horizontal thrust. And, although this property cannot safely be imputed to any cement (strong as many cements are known to be), yet, in a structure, whose component parts are united with a very powerful cement, the matter above an arch will not yield, as when the whole is formed of simple wedges, or as when it would give way in vertical columns, but by the separation of the entire mass into three, or at most, into four pieces : that is, either into the two piers, and the whole mass between them, or into the two piers, and the including mass splitting into two at its crown. It may be advisable, therefore, to investigate the conditions of equilibrium for both these classes of dislocations.

121. PROP. Suppose that the arch Fff′F′ tend to fall vertically in one mass, by thrusting out the piers at the joints of fracture, Ff, F′f′ ; it is requir. to investigate the equations by which the equilibrium may be determined.



Let 2A denote the whole weight of the arch lying between Ff, and $F'f'$, G the centre of gravity of one half of that arch, the centre of gravity of the whole lying on CV; let P be the weight of one of the piers, reckoned as high as Ff, and G′ the place of its centre of gravity.

Now, FV, F′V′, being respectively perpendicular to Ff, $F'f'$, the weight 2A may be understood to act from V, in the directions VF, VF′, and pressing upon the two joints Ff, $F'f'$. The horizontal thrust which it exerts on F, will be $= a$ tan. FVI $=$ A cot. FCI $= \text{A} \cdot \frac{\text{CI}}{\text{FI}}$; and at the same time the vertical effort will $=$ A.

Now, the first of these forces tends to thrust out the solid AF horizontally, an effort which is resisted by friction; and since it is known that, *cæteris paribus*, the friction varies as the pressure, that is, here, as the weight, we shall have for the resisting force, $f \cdot \text{A} + f \cdot \text{P}$. Equating this with the above expression, $\text{A} \cdot \frac{\text{CI}}{\text{FI}}$, we obtain for the first equation of equilibrium

$$f \cdot \text{P} = \text{A}\left(\frac{\text{CI}}{\text{FI}} - f\right) \quad . \quad . \quad . \quad . \quad (\text{I.})$$

Moreover, the horizontal thrust that tends to overturn the pier AF about the angle A, must be regarded as acting at the arm of lever FE, and, therefore, as exerting altogether the energy, $\text{A} \cdot \frac{\text{CI}}{\text{FI}} \cdot \text{FE}$. This is counteracted by the vertical stress A, operating at the horizontal distance AE, and by the weight P, acting at the distance AD; DG′ being the vertical line passing through the centre of gravity, G′, of the pier. Hence we have

$$\text{A} \cdot \frac{\text{CI}}{\text{FI}} \cdot \text{FE} = \text{A} \cdot \text{AE} + \text{P} \cdot \text{AD} ;$$

and, after a little reduction, there results for the second equation of equilibrium:

$$\text{P} \cdot \frac{\text{AD}}{\text{FE}} = \text{A}\left(\frac{\text{CI}}{\text{FI}} - \frac{\text{AE}}{\text{FE}}\right) \quad . \quad . \quad . \quad . \quad (\text{II.})$$

122. Prop. Suppose that each of the two halves kF, kF′, of the arch, tend to turn about the vertex k, removing the points F, and F′: it is required to investigate the conditions of equilibrium in that case.

Referring the weight, A, of the semi-arch from its centre of gravity to the direction of the vertical joint kH; its energy is represented by $A \cdot \frac{FH}{EI}$; and the resulting horizontal thrust at A is, evidently, $A \cdot \frac{FH}{FI} \cdot \frac{FI}{kI} = A \cdot \frac{FH}{kI}$. The vertical stress is $= P + A$; and therefore the friction is represented by $f \cdot P + f \cdot A$. Equating this with the above value of the horizontal thrust, that the pier AF may not move horizontally, we have

$$f \cdot P = A\left(\frac{FH}{kI} - f\right) \quad . \quad . \quad . \quad . \quad (\text{I.})$$

Then, considering the arch and piers as a polygon capable of moving about the angles A, F, k, F′, A′, we must, in order to equilibrium, balance the joint action of P and the semi-arch A at the point F, with the horizontal thrust before-mentioned, acting at the arm of lever EF. Thus we shall have $P \cdot AD + A \cdot AE = A \cdot \frac{FH}{kI} \cdot EF$: from which, after due reduction, there results

$$P \cdot \frac{AD}{EF} = A\left(\frac{FH}{kI} - \frac{AE}{EF}\right) \quad . \quad . \quad . \quad . \quad (\text{II.})$$

123. *Corol.* Hence it will be easy to examine the stability of any arch whose parts are cemented as in the hypotheses of these two propositions. Assume different points such as F, in the arch, for which let the numerical values of the equations (I.) and (II.) be computed. To ensure stability, the first members of those equations, which represent the resistance to motion, must exceed the second members; the weakest points will be those in which the excess of the first above the second member is the least.

If the dimensions of the arch were given, and the thickness of the pier required, the same equations would serve for its determination*.

* The principles adopted in the two last propositions are due to De la Hire, and Coulomb, respectively. For a more comprehensive view of this interesting subject, the student may consult Hutton's Tracts, vol. i., the Appendix to Bossut's Mechanics, and Berard's Treatise on the Statics of Vaults and Domes. The pressure of earth, and the strength of materials, will be treated in a subsequent part of this volume.

DYNAMICS.

124. THAT department of mechanics which relates to the circumstances and effects of bodies in motion (art. 5.) is of great extent, and of very comprehensive application. A selection of its most interesting topics will here be presented; but numerous other problems which, while they fall within its scope, require the aid of the fluxional analysis, will be solved in the collections in a subsequent part of this volume.

GENERAL LAWS OF MOTION, &c.

125. PROP. THE quantity of matter, in all bodies, is in the compound ratio of their magnitudes and densities.

That is, b is as md; where b denotes the body or quantity of matter, m its magnitude, and d its density.

For, by art. 10, in bodies of equal magnitude, the mass or quantity of matter is as the density. But, the densities remaining, the mass is as the magnitude; that is, a double magnitude contains a double quantity of matter, a triple magnitude a triple quantity, and so on. Therefore the mass is in the compound ratio of the magnitude and density.

Corol. 1. In similar bodies, the masses are as the densities and cubes of the diameters, or of any like linear dimensions. —For the magnitudes of bodies are as the cubes of the diameters, &c.

Corol. 2. The masses are as the magnitudes and specific gravities.—For, by art. 10 and 17, the densities of bodies are as the specific gravities.

126. *Scholium.* Hence, if b denote any body, or the quantity of matter in it, m its magnitude, d its density, g its specific gravity, and a its diameter or other dimension; then, $\propto$ (pronounced or named *as*) being the mark for general proportion, from this proposition and its corollaries we have these general proportions:

$$b \propto md \propto mg \propto a^3d,$$

$$m \propto \frac{b}{d} \propto \frac{b}{g} \propto a^3,$$

$$d \propto \frac{b}{m} \propto g \propto \frac{mg}{a^3},$$

$$a^3 \propto \frac{b}{d} \propto m \propto \frac{mg}{d}.$$

127. Prop. The momentum, or quantity of motion, generated by a single impulse, or any momentary force, is as the generating force.

That is, m is as f; where m denotes the momentum, and f the force.

For every effect is proportional to its adequate cause. So that a double force will impress a double quantity of motion; a triple force, a triple motion; and so on. That is, the motion impressed, is as the motive force which produces it.

128. Prop. The momenta, or quantities of motion, in moving bodies, are in the compound ratio of the masses and velocities.

That is, m is as bv.

For, the motion of any body being made up of the motions of all its parts, if the velocities be equal, the momenta will be as the masses; for a double mass will strike with a double force; a triple mass with a triple force; and so on. Again, when the mass is the same, it will require a double force to move it with a double velocity, a triple force with a triple velocity, and so on; that is, the motive force is as the velocity; but the momentum impressed, is as the force which produces it, by art. 127; and therefore the momentum is as the velocity when the mass is the same. But the momentum was found to be as the mass when the velocity is the same. Consequently, when neither are the same, the momentum is in the compound ratio of both the mass and velocity.

Otherwise: $\text{M} : m :: \text{B} : b$, when V is constant:
and $m : \mu :: \text{V} : v$, when B is constant:
therefore, $\text{M} : \mu :: \text{BV} : bv$, when both vary.

129. Prop. In uniform motions, the spaces described are in the compound ratio of the velocities and the times of their description.

That is, s is as tv.

For, by the nature of uniform motion,
$\text{S} : s :: \text{T} : t$, when V is constant:
and $s : \sigma :: \text{V} : v$, when T is constant:
therefore $\text{S} : \sigma :: \text{TV} : tv$, when both vary.

Corol. 1. In uniform motions, the time is as the space directly, and velocity reciprocally ; or as the space divided by the velocity. And when the velocity is the same, the time is as the space. But when the space is the same, the time is reciprocally as the velocity.

Corol. 2. The velocity is as the space directly and the time reciprocally ; or as the space divided by the time. And when the time is the same, the velocity is as the space. But when the space is the same, the velocity is reciprocally as the time.

Scholium.

130. In uniform motions generated by momentary impulse, let b = any body or quantity of matter to be moved,
f = force of impulse acting on the body b,
v = the uniform velocity generated in b,
m = the momentum generated in b,
s = the space described by the body b,
t = the time of describing the space s with the veloc. v.

Then from the last three propositions and corollaries, we have these three general proportions, namely, $f \propto m$, $m \propto bv$, and $s \propto tv$; from which is derived the following table of the general relations of those six quantities, in uniform motions, and impulsive or percussive forces :

$$f \propto m \propto bv \propto \frac{bs}{t}.$$

$$m \propto f \propto bv \propto \frac{bs}{t}.$$

$$b \propto \frac{f}{v} \propto \frac{m}{v} \propto \frac{ft}{s} \propto \frac{mt}{s}.$$

$$s \propto tv \propto \frac{ft}{b} \propto \frac{tm}{b}.$$

$$v \propto \frac{s}{t} \propto \frac{f}{b} \propto \frac{m}{b}.$$

$$t \propto \frac{s}{v} \propto \frac{bs}{f} \propto \frac{bs}{m}.$$

By means of which, may be resolved all questions relating to uniform motions, and the effects of momentary or impulsive forces.

131. Prop. The momentum generated by a constant and uniform force, acting for any time, is in the compound ratio of the force and time of acting.

That is, m is as ft.

For, supposing the time divided into very small parts, by art. 127, the momentum in each particle of time is the same, and therefore the whole momentum will be as the whole time, or sum of all the small parts. But by the same prop. the momentum for each small time, is also as the motive force. Consequently the whole momentum generated, is in the compound ratio of the force and time of acting.

Corol. 1. The motion, or momentum, lost or destroyed in any time, is also in the compound ratio of the force and time. For whatever momentum any force generates in a given time; the same momentum will an equal force destroy in the same or equal time; acting in a contrary direction.

And the same is true of the increase or decrease of motion, by forces that conspire with, or oppose the motion of bodies.

Corol. 2. The velocity generated, or destroyed, in any time, is directly as the force and time, and reciprocally as the body or mass of matter.—For, by this and art. 128, the compound ratio of the body and velocity, is as that of the force and time; and therefore the velocity is as the force and time divided by the body. And if the body and force be given, or constant, the velocity will be as the time.

132. PROP. The spaces passed over by bodies, urged by any constant and uniform forces, acting during any times, are in the compound ratio of the forces and squares of the times directly, and the body or mass reciprocally.

Or, the spaces are as the squares of the times, when the force and body are given.

That is, s is as $\frac{ft^2}{b}$, or as t^2 when f and b are given. For, let v denote the velocity acquired at the end of any time t, by any given body b, when it has passed over the space s. Then, because the velocity is as the time, by the last corol. therefore $\frac{1}{2}v$ is the velocity at $\frac{1}{2}t$, or at the middle point of the time; and as the increase of velocity is uniform, the same space s will be described in the same time t, by the velocity $\frac{1}{2}v$ uniformly continued from beginning to end. But, in uniform motions, the space is in the compound ratio of the time and velocity; therefore s is as $\frac{1}{2}tv$, or indeed $s = \frac{1}{2}tv$. But, by the last corol. the velocity v is as $\frac{ft}{b}$, or as the force and time directly, and as the body reciprocally. Therefore s, or $\frac{1}{2}tv$, is as $\frac{ft^2}{b}$; that is, the space is as the force and square of the time directly, and as the body reciprocally.

Or s is as t^2, the square of the time only, when b and f are given.

Corol. 1. The space s is also as tv, or in the compound ratio of the time and velocity; b and f being given. For, $s = \frac{1}{2}tv$ is the space actually described. But tv is the space which might be described in the same time t, with the last velocity v, if it were uniformly continued for the same or an equal time. Therefore the space s, or $\frac{1}{2}tv$, which is actually described, is just half the space tv, which would be described with the last or greatest velocity, uniformly continued for an equal time t.

Corol. 2. The space s is also as v^2, the square of the velocity; because the velocity v is as the time t.

Scholium.

133. The last four propositions give theorems for resolving all questions relating to motions uniformly accelerated. Thus, put b = any body or quantity of matter,

f = the force constantly acting on it,
t = the time of its acting,
v = the velocity generated in the time t,
s = the space described in that time,
m = the momentum at the end of the time.

Then, from these fundamental relations, $m \propto bv$, $m \propto ft$, $s \propto tv$, and $v \propto \frac{ft}{b}$, we obtain the following table of the general relations of uniformly accelerated motions:

$$m \propto bv \propto ft \propto \frac{bs}{t} \propto \frac{fs}{v} \propto \frac{ft^2v}{s} \propto \sqrt{bfs} \propto \sqrt{bftv}.$$

$$b \propto \frac{m}{v} \propto \frac{ft}{v} \propto \frac{mt}{s} \propto \frac{ft^2}{s} \propto \frac{f^2t^3}{ms} \propto \frac{m^2}{fs} \propto \frac{m^2}{ftv} \propto \frac{fs}{v^2}.$$

$$f \propto \frac{m}{t} \propto \frac{bv}{t} \propto \frac{mv}{s} \propto \frac{ms}{t^2v} \propto \frac{m^2}{bs} \propto \frac{m^2}{btv} \propto \frac{bv^2}{s} \propto \frac{bs}{t^2}.$$

$$v \propto \frac{s}{t} \propto \frac{ft}{b} \propto \frac{m}{b} \propto \frac{ms}{ft^2} \propto \frac{fs}{m} \propto \frac{m^2}{bft} \propto \sqrt{\frac{fs}{b}} \propto \frac{f^2st}{m^2}.$$

$$s \propto tv \propto \frac{ft^2}{b} \propto \frac{mt}{b} \propto \frac{ft^2v}{m} \propto \frac{mv}{f} \propto \frac{m^2}{bf} \propto \frac{bv^2}{f} \propto \frac{m^2v}{f^2t}.$$

$$t \propto \frac{s}{v} \propto \frac{m}{f} \propto \frac{bv}{f} \propto \frac{ts}{m} \propto \sqrt{\frac{bs}{f}} \propto \sqrt{\frac{ms}{fv}} \propto \frac{m^2}{bfv}, \text{ \&c.}$$

134. From the above relations those quantities are to be left out which are given, or which are proportional to each other. Thus, if the body or quantity of matter be always

the same, then the space described is as the force and square of the time. And if the body be proportional to the force, as all bodies are in respect to their gravity; then the space described is as the square of the time, or square of the velocity; and in this case, if F be put $=\frac{f}{b}$, the accelerating force; then will

$$s \propto tv \propto Ft^2 \propto \frac{v^2}{F}.$$

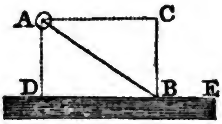
$$v \propto \frac{s}{t} \propto Ft \propto \sqrt{Fs}.$$

$$t \propto \frac{s}{v} \propto \frac{v}{F} \propto \sqrt{\frac{s}{F}}.$$

ON THE COLLISION OF BODIES.

135. PROP. If a body strike or act obliquely on a plain surface, the force or energy of the stroke, or action, is as the sine of the angle of incidence.

Or, the force on the surface is to the same if it had acted perpendicularly, as the sine of incidence is to radius.



Let AB express the direction and the absolute quantity of the oblique force on the plane DE; or let a given body A, moving with a certain velocity, impinge on the plane at B; then its force will be to the action on the plane, as radius to the sine of the angle ABD, or as AB to AD or BC, drawing AD and BC perpendicular, and AC parallel to DE.

For, by art. 29, the force AB is equivalent to the two forces AC, CB; of which the former AC does not act on the plane, because it is parallel to it. The plane is therefore only acted on by the direct force CB, which is to AB, as the sine of the angle BAC, or ABD, to radius.

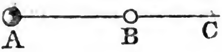
Corol. 1. If a body act on another, in any direction, and be any kind of force, the action of that force on the second body, is made only in a direction perpendicular to the surface on which it acts. For the force in AB acts on DE only by the force CB, and in that direction.

Corol. 2. If the plane DE be not absolutely fixed, it will move, after the stroke, in the direction perpendicular to its

surface. For it is in that direction that the force is exerted.

136. PROP. If one body A, strike another body B, which is either at rest or moving towards the body A, or moving from it, but with a less velocity than that of A; then the momenta, or quantities of motion, of the two bodies, estimated in any one direction, will be the very same after the stroke that they were before it.

For, because action and re-action are always equal, and in contrary directions, art. 20, whatever momentum the one body gains one way by the stroke, the other must just lose as much in the same direction; and therefore the quantity of motion in that direction, resulting from the motions of both the bodies, remains still the same as it was before the stroke.



137. Thus, if A with a momentum of 10, strike B at rest, and communicate to it a momentum of 4, in the direction AB. Then A will have only a momentum of 6 in that direction; which, together with the momentum of B, viz. 4, make up still the same momentum between them as before, namely 10.

138. If B were in motion before the stroke, with a momentum of 5, in the same direction, and receive from A an additional momentum of 2. Then the motion of A after the stroke will be 8, and that of B, 7; which between them make 15, the same as 10 and 5, the motions before the stroke.

139. Lastly, if the bodies move in opposite directions, and meet one another, namely, A with a motion of 10, and B, of 5; and A communicate to B a motion of 6 in the direction AB of its motion. Then, before the stroke, the whole motion from both, in the direction of AB, is $10-5$ or 5. But, after the stroke, the motion of A is 4 in the direction AB, and the motion of B is $6-5$ or 1 in the same direction AB; therefore the sum $4+1$, or 5, is still the same motion from both, as it was before.

140. PROP. The motion of bodies included in a given space, is the same with regard to each other, whether that space be at rest, or move uniformly in a right line.

For, if any force be equally impressed both on the body and the line on which it moves, this will cause no change in the motion of the body along the right line. For the same reason, the motions of all the other bodies, in their several directions, will still remain the same. Consequently their motions among themselves will continue the same, whether

the including space be at rest, or be moved uniformly forward. And therefore their mutual actions on one another, must also remain the same in both cases.

141. Prop. If a hard and fixed plane be struck by either a soft or a hard unelastic body, the body will adhere to it. But if the plane be struck by a perfectly elastic body, it will rebound from it again with the same velocity with which it struck the plane.

For, since the parts which are struck, of the elastic body, suddenly yield and give way by the force of the blow, and as suddenly restore themselves again with a force equal to the force which impressed them, by the definition of elastic bodies; the intensity of the action of that restoring force on the plane, will be equal to the force or momentum with which the body struck the plane. And, as action and re-action are equal and contrary, the plane will act with the same force on the body, and so cause it to rebound or move back again with the same velocity as it had before the stroke.

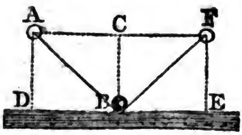
But hard or soft bodies, being devoid of elasticity, by the definition, having no restoring force to throw them off again, they must necessarily adhere to the plane struck.

142. *Corol.* 1. The effect of the blow of the elastic body, on the plane, is double to that of the unelastic one, the velocity and mass being equal in each.

For the force of the blow from the unelastic body, is as its mass and velocity, which is only destroyed by the resistance of the plane. But in the elastic body, that force is not only destroyed and sustained by the plane; but another also equal to it is sustained by the plane, in consequence of the restoring force, and by virtue of which the body is thrown back again with an equal velocity. And therefore the intensity of the blow is doubled.

143. *Corol.* 2. Hence unelastic bodies lose, by their collision, only half the motion lost by elastic bodies; their mass and velocities being equal.—For the latter communicate double the motion of the former.

144. Prop. If an elastic body A impinge on a firm plane DE at the point B, it will rebound from it in an angle equal to that in which it struck it; or the angle of incidence will be equal to the angle of reflection; namely, the angle ABD equal to the angle FBE.



Let AB express the force of the body A in the direction AB; which let be resolved into the two AC, CB, parallel and perpendicular to the plane.—Take BE and CF equal to AC, and

draw BF. Now action and re-action being equal, the plane will resist the direct force CB by another BC equal to it, and in a contrary direction; whereas the other AC, being parallel to the plane, is not acted on or diminished by it, but still continues as before. The body is therefore reflected from the plane by two forces BC, BE, perpendicular and parallel to the plane, and therefore moves in the diagonal BF by composition. But, because AC is equal to BE or CF, and that BC is common, the two triangles BCA, BCF are mutually similar and equal; and consequently the angles at A and F are equal, as also their equal alternate angles ABD, FBE, which are the angles of incidence and reflection.

145. PROP. To determine the motion of non-elastic bodies, when they strike each other directly, or in the same line of direction.

Let the non-elastic body B, moving with the velocity V in the direction Bb, and the body b with the velocity v, strike each other. Then, because the momentum of any moving body is as the mass into the velocity, $BV = M$ is the momentum of the body B, and $bv = m$ the momentum of the body b, which let be the less powerful of the two motions. Then, by art. 136, the bodies will both move together as one mass in the direction BC after the stroke, whether before the stroke the body b moved towards C or towards B. Now, according as that motion of b was from or towards B, that is, whether the motions were in the same or contrary ways, the momentum after the stroke, in direction BC, will be the sum or difference of the momentums before the stroke; namely, the momentum in direction BC will be

$BV + bv$, if the bodies moved the same way, or
$BV - bv$, if they moved contrary ways, and
BV only, if the body b were at rest.

Then divide each momentum by the common mass of matter $B + b$, and the quotient will be the common velocity after the stroke in the direction BC; namely, the common velocity will be, in the first case,

$$\frac{BV+bv}{B+b}, \text{ in the 2d } \frac{BV-bv}{B+b}, \text{ and in the 3d } \frac{BV}{B+b}.$$

Corol. $V - \frac{BV+bv}{B+b} = \frac{V-v}{B+b} \times b$, the veloc. lost by B.

146. For example, if the bodies, or weights, B and b, be as 5 to 3, and their velocities V and v, as 6 to 4, or as 3 to 2, before the stroke; then 15 and 6 will be as their momen-

tums, and 8 the sum of their weights ; consequently, after the stroke, the common velocity will be as

$$\frac{15+6}{8} = \frac{21}{8} \text{ or } 2\tfrac{5}{8} \text{ in the first case,}$$

$$\frac{15-6}{8} = \frac{9}{8} \text{ or } 1\tfrac{1}{8} \text{ in the second, and}$$

$$\frac{15}{8} \;.\;.\;.\;. \text{ or } 1\tfrac{7}{8} \text{ in the third.}$$

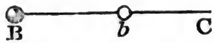
147. Prop. If two perfectly elastic bodies impinge on one another, their relative velocity will be the same both before and after the impulse ; that is, they will recede from each other with the same velocity with which they approached and met.

For the compressing force is as the intensity of the stroke ; which, in given bodies, is as the relative velocity with which they meet or strike. But perfectly elastic bodies restore themselves to their former figure, by the same force by which they were compressed ; that is, the restoring force is equal to the compressing force, or to the force with which the bodies approach each other before the impulse. But the bodies are impelled from each other by this restoring force ; and therefore this force, acting on the same bodies, will produce a relative velocity equal to that which they had before : or it will make the bodies recede from each other with the same velocity with which they before approached, or so as to be equally distant from one another at equal times before and after the impact.

148. *Remark.* It is not meant by this proposition, that each body will have the same velocity after the impulse as it had before ; for that will be varied according to the relation of the masses of the two bodies ; but that the velocity of the one will be, after the stroke, so much increased, and the other decreased, as to have the same difference as before, in one and the same direction. So, if the elastic body B move with a velocity V, and overtake the elastic body b moving the same way with the velocity v ; then their relative velocity, or that with which they strike, is $\mathrm{V}-v$, and it is with this same velocity that they separate from each other after the stroke. But if they meet each other, or the body b move contrary to the body B ; then they meet and strike with the velocity $\mathrm{V}+v$, and it is with the same velocity that they separate and recede from each other after the stroke. But whether they move forward or backward after the impulse, and with what particular velocities, are circumstances that depend on the various masses and velocities of the bodies

before the stroke, and which make the subject of the next proposition.—It may further be remarked, that the sums of the two velocities, of each body, before and after the stroke, are equal to each other. Thus, V, v being the velocities before the impact, if x and y be the corresponding ones after it; since $V - v = y - x$, therefore $V + x = v + y$.

149. Prop. To determine the motions of elastic bodies after striking each other directly.



Let the elastic body B move in the direction BC, with the velocity V; and let the velocity of the other body b be v in the same line; which latter velocity v will be positive if b move the same way as B, but negative if b move in the opposite direction to B. Then their relative velocity in the direction BC is $V - v$; also the momenta before the stroke are BV and bv, the sum of which is $BV + bv$ in the direction BC.

Again, put x for the velocity of B, and y for that of b, in the same direction BC, after the stroke; then their relative velocity is $y - x$, and the sum of their momenta $Bx + by$ in the same direction.

But the momenta before and after the collision estimated in the same direction, are equal, by art. 136, as also the relative velocities, by the last prop. Whence arise these two equations;

$$\text{viz. } BV + bv = Bx + by,$$
$$\text{and } V - v = y - x;$$

the resolution of which equations gives

$$x = \frac{(B-b)V+2bv}{B+b}, \text{ the velocity of } B,$$

$$y = \frac{-(B-b)v+2BV}{B+b}, \text{ the velocity of } b.$$

$$\text{Or, } x = V - \frac{2b}{B+b}(V-v), \text{ and } y = v + \frac{2B}{B+b}(V-v).$$

So that the velocity lost by B is $\dfrac{2b}{B+b}(V-v)$,

and the velocity gained by b is $\dfrac{2B}{B+b}(V-v)$;

which two velocities are in the ratio of b to B, or reciprocally as the two bodies themselves.

Corol. 1. The velocity lost by B drawn into B, and the velocity gained by b drawn into b, give each of them

$\frac{2\text{B}b}{\text{B}+b}(\text{V} - v)$, for the momentum gained by the one and lost by the other, by the stroke; which increment and decrement being equal, they cancel one another, and leave the same momentum $\text{BV}+bv$ after the impact, as it was before it.

Corol. 2. Hence also, $\text{BV}^2 + bv^2 = \text{B}x^2 + by^2$, or the sum of the vires vivarum is always preserved the same, both before and after the impact. For, since

$$\text{BV} + bv = \text{B}x + by,$$

$$\text{or BV} - \text{B}x = by - bv,$$

$$\text{and V} + x = y + v, \text{ these two equas. multiplied,}$$

$$\text{give BV}^2 - \text{B}x^2 = by^2 - bv^2,$$

$$\text{or BV}^2 + bv^2 = \text{B}x^2 + by^2,$$

the equation of the so called living forces.

Corol. 3. But if v be negative, or the body b moved in the contrary direction before collision, or towards B; then, changing the sign of v, the same theorems become

$$x = \frac{(\text{B}-b)\text{V}-2bv}{\text{B}+b}, \text{ the velocity of B,}$$

$$y = \frac{(\text{B}-b)v+2\text{BV}}{\text{B}+b}, \text{ the veloc. of } b, \text{ in the direction BC.}$$

And if b were at rest before the impact, making its velocity $v = 0$, the same theorems give

$$x = \frac{\text{B}-b}{\text{B}+b}\text{V}, \text{ and } y = \frac{2\text{B}}{\text{B}+b}\text{V}, \text{ the velocities in this case.}$$

And, in this case, if the two bodies B and b be equal to each other; then $\text{B}-b=0$, and $\frac{2\text{B}}{\text{B}+b} = \frac{2\text{B}}{2\text{B}} = 1$; which give $x = 0$, and $y = \text{V}$; that is, the body B will stand still, and the other body b will move on with the whole velocity of the former; a thing which we sometimes see happen in playing at billiards; and which would happen much oftener if the balls were perfectly elastic.

Scholium.

150. If the bodies be elastic only in a partial degree, the sum of the momenta will still be the same, both before and after collision, but the velocities after, will be less than in the case of perfect elasticity, in the ratio of the imperfection. Hence, with the same notation as before, the two equations will now be $\text{BV} + bv = \text{B}x + by$,

$$\text{and V} - v = \frac{m}{n}(y - x),$$

where m to n denotes the ratio of perfect to imperfect elasticity. And the resolution of these two equations, gives the following values of x and y, viz.

$$x = \text{V} - \frac{m + n}{m} \cdot \frac{b}{\text{B}+b}(\text{V} - v),$$

$$y = v + \frac{m+n}{m} \cdot \frac{\text{B}}{\text{B}+b}(\text{V} - v),$$

for the velocities of the two bodies after impact in the case of imperfect elasticity: and these would become the same as the former if n were $= m$.

Hence, if the two bodies B and b be equal, then

$$x = \text{V} - \frac{m+n}{2m}(\text{V} - v), \text{ and } y = v + \frac{m+n}{2m}(\text{V} - v),$$

where the velocity lost by B is just equal to that gained by b. And if in this case b was at rest before the impact, or $v = 0$, then the resulting motions would be

$$x = \frac{m-n}{2m}\text{V}, \text{ and } y = \frac{m+n}{2m}\text{V},$$

which are in the ratio of $m - n$ to $m + n$.

Also, if $m = n$, or the bodies perfectly elastic, then $x = 0$, and $y = \text{V}$; or B would be at rest, and b go on with the first motion of B.

Further, in this case also, the velocity of B before the impact, is to that of b after it, as V to $\frac{m+n}{2m}$ V, or as $2m$ to $m + n$. But, if the bodies be now supposed to vibrate in circles, as pendulums, in which case the chords (C and c) of the arcs described are known to be proportional to the velocities; then it will be $2m : m + n :: \text{C} : c$; hence $m : n :: \text{C} : 2c - \text{C}$. So that, by measuring these chords, of the arcs thus experimentally described, the ratio of m to n, or the degree of elasticity in the bodies, may be determined.

151. PROP. The greatest velocity which can be generated by the propagation of motion through a row of contiguous perfectly elastic bodies, will be when those bodies are in geometrical progression.

First, take three bodies, A, X, and C: then (art. 149) the velocity communicated from A to X $= \frac{2\text{A}a}{\text{A}+\text{X}}$, a being the velocity of A: and when the body X impinges upon C at rest with this velocity, the vel. communicated to C will

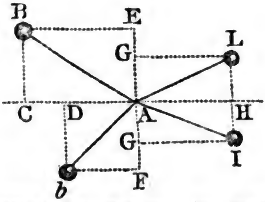
$$= \frac{2\text{X}}{\text{X}+\text{C}} \cdot \frac{2\text{A}a}{\text{A}+\text{X}} = \frac{4\text{A}a\text{X}}{(\text{A}+\text{X})(\text{X}+\text{C})}$$

$$= \frac{4\text{A}a}{(\text{A} \div \text{x}+1)(\text{x}+\text{c})} = \frac{4\text{A}a}{\text{A}+\text{x}+(\text{AC}\div\text{x})+\text{c}}.$$

This fraction is evidently a max. when its denominator is a min. that is, since A and C are given, when $x^2 = \text{AC}$, or when x is a mean proportional between A and C.

For the same reason the velocity communicated from the second body through C the third, to a fourth, D, will be greatest when C is a mean proportional between the second and fourth. Like reasoning will evidently hold for a series of perfectly elastic bodies. Further, if the number of bodies in the geometrical progression be increased without limit, the quantity of motion communicated to the last, from a given quantity of motion in the first, however small, may also be increased without limit.

152. PROP. If bodies strike one another obliquely, it is proposed to determine their motions after the stroke.



Let the two bodies B, b, move in the oblique directions BA, bA, and strike each other at A, with velocities which are in proportion to the lines BA, bA; to find their motions after the impact. Let CAH represent the plane in which the bodies touch in the point of concourse; to which draw the perpendiculars BC, bD, and complete the rectangles CE, DF. Then the motion in BA is resolved into the two BC, CA; and the motion in bA is resolved into the two bD, DA; of which the antecedents BC, bD, are the velocities with which they directly meet, and the consequents CA, DA, are parallel; therefore by these the bodies do not impinge on each other, and consequently the motions, according to these directions, will not be changed by the impulse; so that the velocities with which the bodies meet, are as BC and bD, or their equals EA and FA. The motions therefore of the bodies B, b, directly striking each other with the velocities EA, FA, will be determined by art. 145 or 149. according as the bodies are elastic or non-elastic; which being done, let AG be the velocity, so determined, of one of them, as A; and since there remains also in the body a force of moving in the direction parallel to BE, with a velocity as BE, make AH equal to BE, and complete the rectangle CH: then the two motions in AH and AG, or HI, are compounded into the diagonal AI, which therefore will be the path and velocity of the body B after the stroke. And after the same manner is the motion of the other body b determined after the impact.

If the elasticity of the bodies be imperfect in any given degree, then the quantity of the corresponding lines must be diminished in the same proportion. For the full consideration of this branch of the inquiry the student is referred to the Treatises of Mechanics by *Gregory* and *Bridge*.

Problems for Exercise on Collision.

EXAM. 1. A cannon ball weighing 12lbs. moving with a velocity of 1200 feet per second, *meets* another of 18lbs. moving with a velocity of 1000 feet per second. Required the velocity of each after impact, supposing both to be non-elastic.

EXAM. 2. B and b are as 3 to 2, and the velocity of B is to that of b as 5 to 4. They are perfectly hard, and move before impact in the same direction; what are the velocities lost by B and gained by b?

EXAM. 3. B and b are perfectly elastic, and move in opposite directions. B is triple of b, but b's velocity is double that of B. How do those bodies move after impact?

EXAM. 4. A body whose elasticity is to perfect elasticity as 15 to 16, falls from the height of 100 feet upon a perfectly hard horizontal plane. It then rebounds and falls again, and so on, always in a vertical direction. It is required to find the whole space described by the body before its motion ceases, as well as the entire time of its motion.

EXAM. 5. Investigate what must be the force of elasticity, so that the sums of the products formed by multiplying each body into *any* assumed power, n, of its velocity, may not be altered by the impact of the two bodies.

THE LAWS OF GRAVITY; THE DESCENT OF HEAVY BODIES; AND THE MOTION OF PROJECTILES IN FREE SPACE.

153. PROP. ALL the properties of motion delivered in art. 132, its corollaries and scholium, for constant forces, are true in the motions of bodies freely descending by their own gravity; namely, that the velocities are as the times, and the

spaces as the squares of the times, or as the squares of the velocities.

For, since the force of gravity is uniform, and constantly the same, at all places near the earth's surface, or at nearly the same distance from the centre of the earth; and since this is the force by which bodies descend to the surface; they therefore descend by a force which acts constantly and equally; consequently all the motions freely produced by gravity, are as above specified, by that proposition, &c.

SCHOLIUM.

154. Now it has been found, by numberless experiments, that gravity is a force of such a nature, that all bodies, whether light or heavy, fall vertically through equal spaces in the same time, abstracting from the resistance of the air; as lead or gold and a feather, which in an exhausted receiver fall from the top to the bottom in the same time. It is also found that the velocities acquired by descending, are in the exact proportion of the times of descent: and further, that the spaces descended are proportional to the squares of the times, and therefore to the squares of the velocities. Hence then it follows, that the weights or gravities, of bodies near the surface of the earth, are proportional to the quantities of matter contained in them; and that the spaces, times, and velocities, generated by gravity, have the relations contained in the three general proportions before laid down. Further, as it is found, by accurate experiments, that a body in the latitude of London, falls nearly $16\frac{1}{12}$ feet in the first second of time, and consequently that at the end of that time it has acquired a velocity double, or of $32\frac{1}{6}$ feet by corol. 1, art. 132; therefore, if $\frac{1}{2}g$ denote $16\frac{1}{12}$ feet, the space fallen through in one second of time, or g the velocity generated in that time; then, because the velocities are directly proportional to the times, and the spaces to the squares of the times; therefore it will be,

$$\text{as } 1'' : t'' :: \ gt : \ gt = v \text{ the velocity,}$$
$$\text{and } 1^2 : t^2 :: \tfrac{1}{2}g : \tfrac{1}{2}gt^2 = s \text{ the space.}$$

So that, for the descents of gravity, we have these general equations, namely,

$$s = \tfrac{1}{2}gt^2 = \frac{v^2}{2g} = \tfrac{1}{2}tv.$$

$$v = gt = \frac{2s}{t} = \sqrt{2gs}.$$

$$t = \frac{v}{g} = \frac{2s}{v} = \sqrt{\frac{2s}{g}}.$$

$$g = \frac{v}{t} = \frac{2s}{t^2} = \frac{v^2}{2s}.$$

Hence, because the times are as the velocities, and the spaces as the squares of either, therefore,

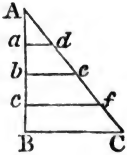
if the times be as the numbs.	1,	2,	3,	4,	5,	&c.
the velocities will also be as	1,	2,	3,	4,	5,	&c.
and the spaces as their squares	1,	4,	9,	16,	25,	&c.
and the space for each time as	1,	3,	5,	7,	9,	&c.

namely, as the series of the odd numbers, which are the differences of the squares denoting the whole spaces. So that if the first series of natural numbers be seconds of time,

namely, the times in seconds,	1″,	2″,	3″,	4″,	&c.
the velocities in feet will be	$32\frac{1}{6}$,	$64\frac{1}{3}$,	$96\frac{1}{2}$,	$128\frac{2}{3}$,	&c.
the spaces in the whole times	$16\frac{1}{12}$,	$64\frac{1}{3}$,	$144\frac{3}{4}$,	$257\frac{1}{3}$,	&c.
and the space for each second	$16\frac{1}{12}$,	$48\frac{1}{4}$,	$80\frac{5}{12}$,	$112\frac{7}{12}$,	&c.

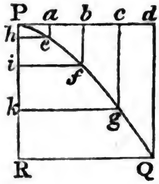
of which spaces the common difference is $32\frac{1}{6}$ feet, the natural and obvious measure of g, the force of gravity.

155. These relations, of the times, velocities, and spaces, may be represented by certain lines and geometrical figures. Thus, if the line AB denote the time of any body's descent, and BC, at right angles to it, the velocity gained at the end of that time; by joining AC, and dividing the time AB into any number of parts at the points a, b, c;



then shall ad, be, cf, parallel to BC, be the velocities at the points of time, a, b, c, or at the ends of the times, Aa, Ab, Ac; because these latter lines, by similar triangles, are proportional to the former ad, be, cf, and the times are proportional to the velocities. Also, the area of the triangle ABC will represent the space descended by the force of gravity in the time AB, in which it generates the velocity BC; because that area is equal to $\frac{1}{2}$AB × BC, and the space descended is $s = \frac{1}{2}tv$, or half the product of the time and the last velocity. And, for the same reason, the less triangles Aad, Abe, Acf, will represent the several spaces described in the corresponding times Aa, Ab, Ac, and velocities ad, be, cf; those triangles or spaces being also as the squares of their like sides Aa, Ab, Ac, which represent the times, or of ad, be, cf, which represent the velocities.

156. But as areas are rather unnatural representations of the spaces passed over by a body in motion, which are lines, the relations may better be represented by the abscisses



and ordinates of a parabola. Thus, if PQ be a parabola, PR its axis, and RQ its ordinate ; and P*a*, P*b*, P*c*, &c. parallel to RQ, represent the times from the beginning, or the velocities, then *ae*, *bf*, *cg*, &c. parallel to the axis PR, will represent the spaces described by a falling body in those times ; for, in a parabola, the abscisses P*h*, P*i*, P*k*, &c. or *ae*, *bf*, *cg*, &c. which are the spaces described, are as the squares of the ordinates *he*, *if*, *kg*, &c. or P*a*, P*b*, P*c*, &c. which represent the times or velocities.

157. And because the laws for the destruction of motion, are the same as those for the generation of it, by equal forces, but acting in a contrary direction ; therefore,

1*st*, A body thrown directly upward, with any velocity, will lose equal velocities in equal times.

2*d*, If a body be projected upward, with the velocity it acquired in any time by descending freely, it will lose all its velocity in an equal time, and will ascend just to the same height from which it fell, and will describe equal spaces in equal times, in rising and falling, but in an inverse order ; and it will have equal velocities at any one and the same point of the line described, both in ascending and descending.

3*d*, If bodies be projected upward, with any velocities, the height ascended to, will be as the squares of those velocities, or as the squares of the times of ascending, till they lose all their velocities.

158. In solving problems, where a body, instead of being permitted to fall freely, is projected vertically upwards or downwards with a given velocity, it will assist the comprehension of what takes place, to ascertain what results from the original projection, and what from the force of gravity. Thus, if a body be projected with a velocity v, it will, in the time t, described the space tv (art. 129) apart from the operation of gravity or any other force. Blending this with the preceding expression for the space described by a falling body, we have

$$s = tv \mp \tfrac{1}{2}gt^2,$$

in which the *lower* sign must be employed when the projection is vertically *downwards*, the *upper* when the projection is vertically *upwards*.

EXERCISES ON RISING AND FALLING BODIES.

1. Find the space descended vertically by a body in 7 seconds of time, and the velocity acquired ?

Ans. $788\frac{1}{12}$, space ; $225\frac{1}{6}$, velocity.

2. Required the time of generating a velocity of 100 feet per second, and the whole space descended.

Ans. $3''\frac{21}{193}$, time ; $155\frac{85}{193}$ *f.* space.

3. Find the time of descending 400 feet, and the velocity at the end of that time.

Ans. $4''\frac{76}{77}$, time ; $160\frac{32}{77}$, velocity.

4. If a body fall freely for 5″, how far will it descend during the last second of its motion ?

5. If an arrow be propelled vertically upwards from a bow with a velocity of $96\frac{1}{2}$ feet per second, how high will it rise, and how long will it be before it returns again to the ground ?

6. If a ball be projected vertically *downwards* with a velocity of 100 feet per second, how far will it have descended in three seconds ?

7. If a ball be projected *upwards* with a velocity of 100 feet per second, how far will it have arisen in three seconds ?

8. If a ball be projected vertically upwards with a velocity of 44 feet per second, will it be above or below the point of projection in four seconds, the force of gravity tending all the time to draw it downwards ?

9. A drop of rain falls through $176\frac{1}{12}$ feet in the last second ; how high is the cloud from which it descended ?

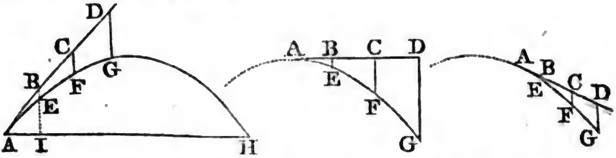
10. A body falling freely was observed to pass through half its descent in the last second ; how far did it fall, and how long was it in falling ?

11. Two weights, one of 5lbs. the other of 3lbs. hang freely over a pulley : after motion is allowed to commence how far will the larger weight descend, or the smaller arise, in four seconds ?

N. B. The theorem for operation is $s = \frac{W-w}{W+w} \cdot \frac{1}{2}gt^2$.

12. Two equal weights are balanced over a pulley. A pound weight being added to one of them, and motion in consequence taking place, the preponderating weight descended through $16\frac{1}{12}$ feet in four seconds. Required the measure of the two equal weights ?

158. **Prop.** If a body be projected in free space, either parallel to the horizon, or in an oblique direction, by the force of gunpowder, or any other impulse; it will, by this motion, in conjunction with the action of gravity, describe the curve line of a parabola.



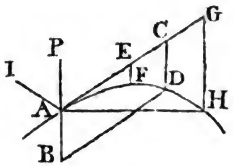
Let the body be projected from the point A, in the direction AD, with any uniform velocity: then, in any equal portions of time, it would, by art. 129, describe the equal spaces AB, BC, CD, &c. in the line AD, if it were not drawn continually down below that line by the action of gravity. Draw BE, CF, DG, &c. in the direction of gravity, or perpendicular to the horizon, and equal to the spaces through which the body would descend by its gravity in the same time in which it would uniformly pass over the corresponding spaces AB, AC, AD, &c. by the projectile motion. Then, since by these two motions the body is carried over the space AB, in the same time as over the space BE, and the space AC in the same time as the space CF, and the space AD in the same time as the space DG, &c.; therefore, by the composition of motions, at the end of those times, the body will be found respectively in the points E, F, G, &c.; and consequently the real path of the projectile will be the curve line AEFG, &c. But the spaces AB, AC, AD, &c. described by uniform motion, are as the times of description; and the spaces BE, CF, DG, &c. described in the same times by the accelerating force of gravity, are as the squares of the times; consequently the perpendicular descents are as the squares of the spaces in AD, that is BE, CF, DG, &c. are respectively proportional to AB^2, AC^2, AD^2, &c.; which is the property of the parabola by theor. 8, Con. Sect. Therefore the path of the projectile is the parabolic line AEFG, &c. to which AD is a tangent at the point A.

159. *Corol.* 1. The horizontal velocity of a projectile, is always the same constant quantity, in every point of the curve: because the horizontal motion is in a constant ratio to the motion in AD, which is the uniform projectile motion. And the projectile velocity is in proportion to the constant horizontal velocity, as radius to the cosine of the angle DAH, or angle of elevation or depression of the piece above or below the horizontal line AH.

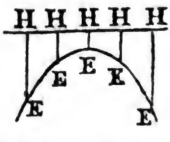
160. *Corol.* 2. The velocity of the projectile in the direction of the curve, or of its tangent at any point A, is as the secant of its angle BAI of direction above the horizon. For the motion in the horizontal direction AI is constant, and AI is to AB, as radius to the secant of the angle A; therefore the motion at A, in AB, is everywhere as the secant of the angle A.

161. *Corol.* 3. The velocity in the direction DG of gravity, or perpendicular to the horizon, at any point G of the curve, is to the first uniform projectile velocity at A, or point of contact of a tangent, as 2GD is to AD. For, the times in AD and DG being equal, and the velocity acquired by freely descending through DG, being such as would carry the body uniformly over twice DG in an equal time, and the spaces described with uniform motions being as the velocities, therefore the space AD is to the space 2DG, as the projectile velocity at A, to the perpendicular velocity at G.

162. PROP. The velocity in the direction of the curve, at any point of it, as A, is equal to that which is generated by gravity in freely descending through a space which is equal to one-fourth of the parameter of the diameter of the parabola at that point.



Let PA or AB be the height due to the velocity of the projectile at any point A in the direction of the curve or tangent AC, or the velocity acquired by falling through that height; and complete the parallelogram ACDB. Then is CD=AB or AP, the height due to the velocity in the curve at A; and CD is also the height due to the perpendicular velocity at D, which must be equal to the former; but by the last corol. the velocity at A is to the perpendicular velocity at D, as AC to 2CD; and as these velocities are equal, therefore AC or BD is equal to 2CD, or 2AB; and hence AB or AP is equal to $\frac{1}{2}$BD, or $\frac{1}{4}$ of the parameter of the diameter AB, by corol. to theor. 13 of the parabola.

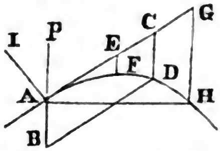


163. *Corol.* 1. Hence, and from cor. 2 theor. 13 of the parabola, it appears that if from the directrix of the parabola which is the path of the projectile, several lines HE be drawn perpendicular to the directrix, or parallel to the axis; then the velocity of the projectile in the direction of the curve, at any point E, is always equal to the velocity acquired by a body falling freely through the perpendicular line HE.

164. *Corol.* 2. If a body, after falling through the height PA (last fig. but one), which is equal to AB, and when it arrives at A, have its course changed, by reflection from an elastic plane AI, or otherwise, into any direction AC, without altering the velocity ; and if AC be taken = 2AP or 2AB, and the parallelogram be completed ; then the body will describe the parabola passing through the point D.

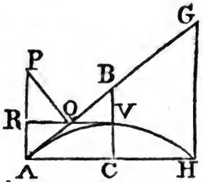
165. *Corol.* 3. Because AC=2AB or 2CD or 2AP, therefore AC^2 = 2AP × 2CD or AP . 4CD ; and, because all the perpendiculars EF, CD, GH, are as AE^2, AC^2, AG^2 ; therefore also AP . 4EF = AE^2, and AP . 4GH = AG^2, &c. ; and because the rectangle of the extremes is equal to the rectangle of the means of four proportionals, therefore always

it is AP : AE : : AE : 4EF,
and AP : AC : : AC : 4CD,
and AP : AG : : AG : 4GH,
and so on.



166. PROP. Having given the direction, and the impetus, or altitude due to the first velocity of a projectile ; to determine the greatest height to which it will rise, and the random or horizontal range.

Let AP be the height due to the projectile velocity at A, AG the direction, and AH the horizon. On AG let fall the perpendicular PQ, and on AP the perpendicular QR ; so shall AR be equal to the greatest altitude CV, and 4QR equal to the horizontal range AH. Or, having drawn PQ, perp. to AG, take AG = 4AQ, and draw GH perp. to AH ; then AH is the range.



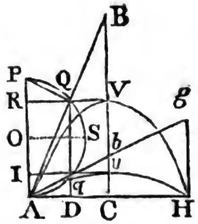
For, by the last corollary, AP : AG : : AG : 4GH ;
and, by similar triangles, AP : AG : : AQ : GH,
or - - - AP : AG : : 4AQ : 4GH ;
therefore AG = 4AQ ; and, by similar triangles, AH = 4QR.

Also, if V be the vertex of the parabola, then AB or $\frac{1}{2}$AG = 2AQ, or AQ = QB ; consequently AR = BV, which is = CV by the property of the parabola.

167. *Corol.* 1 Because the angle Q is a right angle, which is the angle in a semicircle, therefore if, on AP as a diameter, a semicircle be described, it will pass through the point Q.

168. *Corol.* 2. If the horizontal range and the projectile velocity be given, the direction of the piece so as to hit the object H, will be thus easily found : Take AD = $\frac{1}{4}$AH, draw

DQ perpendicular to AH, meeting the semicircle, described on the diameter AP, in Q and q; then AQ or Aq will be the direction of the piece. And hence it appears, that there are two directions AB, Ab, which, with the same projectile velocity, give the very same horizontal range AH. And these two directions make equal angles qAD, QAP, with AH and AP, because the arc PQ = the arc Aq.



169. *Corol.* 3. Or, if the range AH, and direction AB, be given; to find the altitude and velocity or impetus. Take AD = $\frac{1}{4}$AH, and erect the perpendicular DQ, meeting AB in Q; so shall DQ be equal to the greatest altitude CV. Also, erect AP perpendicular to AH, and QP to AQ; so shall AP be the height due to the velocity.

170. *Corol.* 4. When the body is projected with the same velocity, but in different directions; the horizontal ranges AH will be as the sines of double the angles of elevation.— Or, which is the same, as the rectangle of the sine and cosine of elevation. For AD or RQ, which is $\frac{1}{4}$AH, is the sine of the arc AQ, which measures double the angle QAD of elevation.

And when the direction is the same, but the velocities different; the horizontal ranges are as the square of the velocities, or as the height AP, which is as the square of the velocity; for the sine AD or RQ or $\frac{1}{4}$AH is as the radius or as the diameter AP.

Therefore, when both are different, the ranges are in the compound ratio of the squares of the velocities, and the sines of double the angles of elevation.

171. *Corol.* 5. The greatest range is when the angle of elevation is 45°, or half a right angle; for the double of 45 is 90, which has the greatest sine. Or the radius OS, which is $\frac{1}{4}$ of the range, is the greatest sine.

And hence the greatest range, or that at an elevation of 45°, is just double the altitude AP which is due to the velocity, or equal to 4VC. Consequently, in that case, C is the focus of the parabola, and AH its parameter. Also the ranges are equal, at angles equally above and below 45°.

172. *Corol.* 6. When the elevation is 15°, the double of which, or 30°, has its sine equal to half the radius; consequently then its range will be equal to AP, or half the greatest range at the elevation of 45°; that is, the range at 15°, is equal to the impetus or height due to the projectile velocity.

173. *Corol.* 7. The greatest altitude CV, being equal to AR, is as the versed sine of double the angle of elevation, and also as AP or the square of the velocity. Or as the square of the sine of elevation, and the square of the velocity; for the square of the sine is as the versed sine of the double angle.

174. *Corol.* 8. The time of flight of the projectile, which is equal to the time of a body falling freely through GH or 4CV, four times the altitude, is therefore as the square root of the altitude, or as the projectile velocity and sine of the elevation.

SCHOLIUM.

175. From the last proposition and its corollaries, may be deduced the following set of theorems, for finding all the circumstances of projectiles on horizontal planes, having any two of them given. Thus, let s, c, t, denote the sine, cosine, and tangent of elevation; S, v the sine and versed sine of the double elevation; R the horizontal range; T the time of flight; V the projectile velocity; H the greatest height of the projectile; $g = 32\frac{1}{6}$ feet, and a the impetus, or the altitude due to the velocity V. Then,

$$\mathrm{R} = 2a\mathrm{S} = 4asc = \frac{\mathrm{SV}^2}{g} = \frac{sc\mathrm{V}^2}{\frac{1}{2}g} = \frac{\frac{1}{2}gc\mathrm{T}^2}{s} = \frac{\frac{1}{2}g\mathrm{T}^2}{t} = \frac{4\mathrm{H}}{t}.$$

$$\mathrm{V} = \sqrt{2ag} = \sqrt{\frac{g\mathrm{R}}{\mathrm{S}}} = \sqrt{\frac{g\mathrm{R}}{2sc}} = \frac{\frac{1}{2}g\mathrm{T}}{s} = \frac{2}{s}2\sqrt{\tfrac{1}{2}g\mathrm{H}}.$$

$$\mathrm{T} = \frac{{}^*s\mathrm{V}}{\frac{1}{2}g} = 2s\sqrt{\frac{a}{\frac{1}{2}g}} = \sqrt{\frac{t\mathrm{R}}{\frac{1}{2}g}} = \frac{\mathrm{SR}}{\frac{1}{2}gc} = 2\sqrt{\frac{\mathrm{H}}{\frac{1}{2}g}}.$$

$$\mathrm{H} = as^2 = \tfrac{1}{2}av = \tfrac{1}{4}t\mathrm{R} = \frac{\mathrm{SR}}{4c} = \frac{s^2\mathrm{V}^2}{2g} = \frac{v\mathrm{V}^2}{4g} = \tfrac{1}{8}g\mathrm{T}^2.$$

And from any of these, the angle of direction may be found. Also, in these theorems, g may, in many cases, be taken $= 32$, without the small fraction $\frac{1}{6}$, which will be near enough for common use.

176. PROP. To determine the range on an oblique plane; having given the impetus or velocity, and the angle of direction.

Let AE be the oblique plane, at a given angle, either above or below the horizontal plane AH; AG the direction of the piece, and AP the altititude due to the projectile velocity at A.

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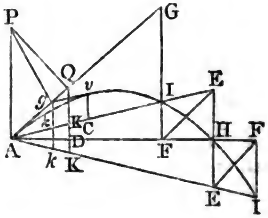
* This time, with 30° elevation, is just equal to the time of perpendicular ascent, with the same velocity v.

By the last proposition, find the horizontal range AH to the given velocity and direction; draw HE perpendicular to AH, meeting the oblique plane in E; draw EF parallel to AG, and FI parallel to HE; so shall the projectile pass through I, and the range on the oblique plane will be AI. As is evident by theor. 15 of the Parabola, where it is proved, that if AH, AI be any two lines terminated at the curve, and IF, HE parallel to the axis; then is EF parallel to the tangent AG.



177. *Otherwise*, without the Horizontal Range.

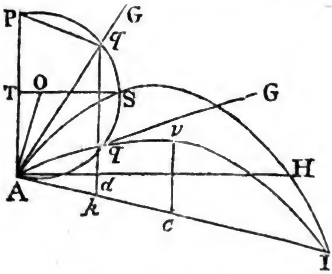
Draw PQ perp. to AG, and QD perp. to the horizontal plane AF, meeting the inclined plane in K; take AE = 4AK, draw EF parallel to AG, and FI parallel to AP or DQ; so shall AI be the range on the oblique plane. For AH = 4AD, therefore EH is parallel to FI, and so on, as above.

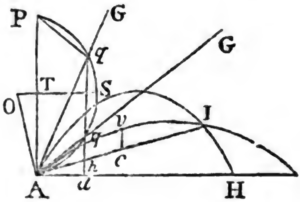
Otherwise.

178. Draw Pq making the angle APq = the angle GAI; then take AG = 4Aq, and draw GI perp. to AH. Or, draw qk perp. to AH, and take AI = 4Ak. Also kq will we equal to cv the greatest height above the plane.

For, by cor. 2, art. 164, AP : AG :: AG : 4GI;
and by sim. triangles, AP : AG :: Aq : GI,
or - - - AP : AG :: 4Aq : 4GI;
therefore AG = 4Aq; and by sim. triangles, AI = 4Ak.

Also, qk, or $\frac{1}{4}$GI, is = to cv by theor. 13 of the Parabola,





179. *Corol.* 1. If AO be drawn perp. to the plane AI, and AP be bisected by the perpendicular STO; then with the centre O describing a circle through A and P, the same will also

pass through q, because the angle GAI, formed by the tangent AI and AG, is equal to the angle AIq, which will therefore stand on the same arc Aq.

180. *Corol.* 2. If there be given the range AI and the velocity, or the impetus, the direction will hence be easily found thus: Take Ak = $\frac{1}{4}$AI, draw kq perp. to AH, meeting the circle described with the radius AO in two points q and q; then Aq or Pq will be the direction of the piece. And hence it appears that there are two directions, which, with the same impetus, give the very same range AI. And these two directions make equal angles with AI and AP, because the arc Pq is equal the arc Aq. They also make equal angles with a line drawn from A through S, because the arc Sq is equal the arc Sq.

181. *Corol.* 3. Or, if there be given the range AI, and the direction Aq; to find the velocity or impetus. Take Ak = $\frac{1}{4}$AI, and erect kq perp. to AH, meeting the line of direction in q; then draw qP making the $\angle$ AqP = $\angle$ Akq; so shall AP be the impetus, or the altitude due to the projectile velocity.

182. *Corol.* 4. The range on an oblique plane, with a given elevation, is directly proportional to the rectangle of the cosine of the direction of the piece above the horizon, and the sine of the direction above the oblique plane, and reciprocally to the square of the cosine of the angle of the plane above or below the horizon.

For, put s = sin. $\angle q$AI or APq,
c = cos. $\angle q$AH or sin, PAq,
C = cos. $\angle$ IAH or sin. Akd or Akq or AqP.

Then in the triangle APq, C : s : : AP : Aq;
and in the triangle Akq, C : c : : Aq : Ak;
theref. by composition, C^2 : cs : : AP : AK = $\frac{1}{4}$AI.

So that the oblique range AI $= \frac{cs}{C^2} \times 4$AP.

183. The range is the greatest when Ak is the greatest; that is, when kq touches the circle in the middle point S; and then the line of direction passes through S, and bisects the angle formed by the oblique plane and the vertex. Also, the ranges are equal at equal angles above and below this direction for the maximum.

184. *Corol.* 5. The greatest height cv or kq of the projectile, above the plane, is equal to $\frac{s^2}{C^2} \times$ AP. And therefore it is as the impetus and square of the sine of direction above

the plane directly, and square of the cosine of the plane's inclination reciprocally.

For . c (sin. AqP) : s (sin. APq) : : AP : Aq,
and c (sin. Akq) : s (sin. kAq) : : Aq : kq,
theref. by comp. $c^2 : s^2 : : AP : kq$.

185. *Corol.* 6. The time of flight in the curve AVI is $= \frac{2s}{c}\sqrt{\frac{AP}{\frac{1}{2}g}}$, where $\frac{1}{2}g = 16\frac{1}{12}$ feet. And therefore it is as the velocity and sine of direction above the plane directly, and cosine of the plane's inclination reciprocally. For the time of describing the curve, is equal to the time of falling freely through GI or $4kq$ or $\frac{4s^2}{c^2} \times AP$. Therefore, the time being as the square root of the distance,

$\sqrt{\frac{1}{2}g} : \frac{2s}{c}\sqrt{AP} : : 1'' : \frac{2s}{c}\sqrt{\frac{AP}{\frac{1}{2}g}}$, the time of flight.

SCHOLIUM.

186. From the foregoing corollaries may be collected the following set of theorems, relating to projects made on any given inclined planes, either above or below the horizontal plane. In which the letters denote as before, namely,

c = cos. of direction above the horizon,
c = cos. of inclination of the plane,
s = sin. of direction above the plane,
R = the range on the oblique plane,
T the time of flight,
V the projectile velocity,
H the greatest height above the plane,
a the impetus, or alt. due to the velocity V,
$g = 32\frac{1}{6}$ feet. Then,

$$R = \frac{cs}{c^2}\times 4a = \frac{2cs}{c^2 g}V^2 = \frac{gc}{2s}T^2 = \frac{4c}{s}H.$$

$$H = \frac{s^2}{c^2}a = \frac{s^2V^2}{2gc^2} = \frac{sR}{4c} = \frac{g}{8}T^2.$$

$$V = \sqrt{2ag} = c\sqrt{\frac{gR}{2cs}} = \frac{gc}{2s}T = \frac{2c}{s}\sqrt{\tfrac{1}{2}gH}.$$

$$T = \frac{2s}{c}\sqrt{\frac{a}{\frac{1}{2}g}} = \frac{sV}{\frac{1}{2}gc} = \sqrt{\frac{sR}{\frac{1}{2}gc}} = 2\sqrt{\frac{H}{\frac{1}{2}g}}.$$

And from any of these, the angle of direction may be found.

187. Geometrical constructions of the principal cases in projectiles in a non-resisting medium, flow readily from the properties of the parabola; and in many cases those constructions suggest simple modes of computation. The following problems will serve by way of exercise.

1. Given the impetus and elevation; to find, by construction, the range, on a horizontal plane, the greatest height, and thence the time of flight.

2. Given the impetus, and the range, on a horizontal plane; to find, by construction, the elevation, and the greatest height.

3. Given the elevation, and the range on a horizontal plane; to find, by construction, the impetus, the greatest height, and thence by computation, the time.

4. Given the impetus, the point and direction of projection, to find the place where the ball will fall upon any plane given in position.

5. Given the impetus and the point of projection, to find the elevation necessary to hit any given point; and to show the limits of possibility. Both construction and mode of computation are required.

PRACTICAL GUNNERY.

188. We have now given the whole theory of projectiles, with theorems for all the cases, regularly arranged for use, both for oblique and horizontal planes. But, before they can be applied in resolving the several cases in the practice of gunnery, it is necessary that some more data be laid down, as derived from good experiments made with balls or shells discharged from cannon or mortars, by gunpowder, under different circumstances. For, without such experiments and data, those theorems can be of very little utility in real practice, on account of the imperfections and irregularities in the firing of gunpowder, and the expulsion of balls from guns, but more especially on account of the enormous resistance of the air to all projectiles made with any velocities that are considerable. As to the cases in which projectiles are made with small velocities, or such as do not exceed 200, or 300, or 400 feet per second of time, they may be resolved tolerably near the truth, especially for the larger shells, by the parabolic theory, laid down above. But, in cases of great projectile velocities, that theory is quite in-

adequate, without the aid of several data drawn from many and good experiments. For so great is the effect of the resistance of the air to projectiles of considerable velocity, that some of those which in the air range only between 2 and 3 miles at the most, would in vacuo range about ten times as far, or between 20 and 30 miles.

The effects of this resistance are also various, according to the velocity, the diameter, and the weight of the projectile. So that the experiments made with one size of ball or shell, will not serve for another size, though the velocity should be the same; neither will the experiments made with one velocity, serve for other velocities, though the ball be the same. And therefore it is plain that, to form proper rules for practical gunnery, we ought to have good experiments made with each size of mortar, and with every variety of charge, from the least to the greatest. And not only so, but these ought also to be repeated at many different angles of elevation, namely, for every single degree between 30° and 60° elevation, and at intervals of 5° above 60° and below 30°, from the vertical direction to point blank. By such a course of experiments it will be found, that the greatest range, instead of being constantly that at an elevation of 45°, as in the parabolic theory, will be at all intermediate degrees between 45 and 30, being more or less, both according to the velocity and the weight of the projectile; the smaller velocities and larger shells ranging farthest when projected almost at an elevation of 45°; while the greatest velocities, especially with the smaller shells, range farthest with an elevation of about 30°, or little more.

189. There have, at different times, been made certain small parts of such a course of experiments as is hinted at above. Such as the experiments or practice carried on in the year 1773, on Woolwich Common; in which all the sizes of mortars were used, and a variety of small charges of powder. But they were all at the elevation of 45°; consequently these are defective in the higher charges, and in all the other angles of elevation.

Other experiments were also carried on in the same place in the years 1784, and 1786, with various angles of elevation indeed, but with only one size of mortar, and only one charge of powder, and that but a small one too; so that all those nearly agree with the parabolic theory. Other experiments have also been carried on with the ballistic pendulum, at different times; from which have been obtained some of the laws for the quantity of powder, the weight and velocity of the ball, the length of the gun, &c. Namely, that the velocity of the ball varies as the square root of the charge direct-

ly, and as the square root of the weight of ball reciprocally; and that, some rounds being fired with a medium length of one-pounder gun, at 15° and 45° elevation, and with 2, 4, 8, and 15 ounces of powder, gave nearly the velocities, ranges, and times of flight, as they are here set down in the following table. But good experiments are wanted with large balls and shells.

Powder.	Elevation of gun.	Velocity of ball	Range.	Time of flight.
oz.		feet.	feet.	
2	15°	860	4100	9″
4	15	1230	5100	12
8	15	1640	6000	14½
12	15	1680	6700	15½
2	45	860	5100	21

190. But as we are not yet provided with a sufficient number and variety of experiments, on which to establish true rules for practical gunnery, independent of the parabolic theory, we must at present content ourselves with the data of some one certain experimental range and time of flight, at a given angle of elevation; and then, by help of these, and the rules in the parabolic theory, determine the like circumstances for other elevations that are not greatly different from the former, assisted by the following practical rules.—

191. SOME PRACTICAL RULES IN GUNNERY.

I. *To find the Velocity of any Shot or Shell.*

Rule. Divide double the weight of the charge of powder by the weight of the shot, both in lbs. Extract the square root of the quotient. Multiply that root by 1600, and the product will be the velocity in feet, or the number of feet the shot passes over per second, nearly.

Or say—As the root of the weight of the shot, is to the root of double the weight of the powder, so is 1600 feet, to the velocity*.

* In more recent experiments carried on at Woolwich, by the Editor of the present edition, in conjunction with the select committee of artillery officers, it has been found that a charge of *a third* of the weight of the ball, gives, at a medium, a velocity of 1600 feet; gunpowder being much improved in its manufacture since the time when

II. *Given the Range at One Elevation; to find the Range at Another Elevation.*

Rule. As the sine of double the first elevation, is to its range; so is the sine of double another elevation, to its range.

III. *Given the Range for one Charge; to find the Range for Another Charge, or the Charge for Another Range.*

Rule. The ranges have the same proportion as the charges; that is, as one range is to its charge, so is any other range to its charge: the elevation of the piece being the same in both cases.

192. Example 1. If a ball of 1lb. acquire a velocity of 1600 feet per second, when fired with 8 ounces of powder; it is required to find with what velocity each of the several kinds of shells will be discharged by the full charges of powder, viz.

Nature of the shells in inches .	13	10	8	$5\frac{1}{2}$	$4\frac{2}{5}$
Their weight in lbs. . .	196	90	48	16	8
Charge of powder in lbs. :	9	4	2	1	$\frac{1}{2}$
Ans. The velocities are .	485	477	462	566	566

Exam. 2. If a shell be found to range 1000 yards when discharged at an elevation of 45°; how far will it range when the elevation is 30° 16′, the charge of powder being the same? Ans. 2612 feet, or 871 yards.

Exam. 3. The range of a shell, at 45° elevation, being found to be 3750 feet; at what elevation must the piece be set, to strike an object at the distance of 2810 feet, with the same charge of powder? Ans. at 24° 16′, or at 65° 44′.

Exam. 4. With what impetus, velocity, and charge of powder, must a 13-inch shell be fired, at an elevation of 32° 12′, to strike an object at the distance of 3250 feet?
Ans. impetus 1802, veloc. 340, change 4lb. $7\frac{1}{2}$oz.

Exam. 5. A shell being found to range 3500 feet, when

Sir Tho. Blomfield and Dr. Hutton made their experiments. Putting B for the weight of the ball, and c for that of the charge, $v = 1600\sqrt{\frac{3c}{B}}$, is now found a good approximative theorem for the initial velocity.

discharged at an elevation of 25° 12′; how far then will it range at an elevation of 36° 15′ with the same charge of powder? Ans. 4332 feet.

Exam. 6. If, with a charge of 9lb. of powder, a shell range 4000 feet; what charge will suffice to throw it 3000 feet, the elevation being 45° in both cases?

Ans. $6\frac{3}{4}$lb. of powder.

Exam. 7. What will be the time of flight for any given range, at the elevation of 45°, or for the greatest range?

Ans. the time in secs. is $\frac{1}{4}$ the sq. root of the range in feet.

Exam. 8. In what time will a shell range 3250 feet, at an elevation of 32°? Ans. $11\frac{1}{4}$ sec. nearly.

Exam. 9. How far will a shot range on a plane which ascends 8° 15′, and another which descends 8° 15′; the impetus being 3000 feet, and the elevation of the piece 32° 30′?

Ans. 4244 feet on the ascent,
and 6745 feet on the descent.

Exam. 10. How much powder will throw a 13-inch shell 4244 feet on an inclined plane, which ascends 8° 15′, the elevation of the mortar being 32° 30′?

Ans. 7·3765lb. or 7lb. 6oz.

Exam. 11. At what elevation must a 13-inch mortar be pointed, to range 6745 feet, on a plane which descends 8° 15′; the charge $7\frac{3}{8}$lb. of powder? Ans. 32° $41'\frac{1}{2}$.

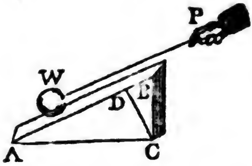
Exam. 12. In what time will a 13-inch shell strike a plane which rises 8° 30′, when elevated 45°, and discharged with an impetus of 2304 feet? Ans. $14\frac{4}{5}$ seconds.

THE DESCENT OF BODIES ON INCLINED PLANES AND CURVE SURFACES.—THE MOTION OF PENDULUMS.

193. Prop. If a weight w be sustained on an inclined plane AB, by a power P, acting in a direction WP, parallel to the plane. Then

The weight of the body, w,	The length AB,
The sustaining power P, and	The height BC, and
The pressure on the plane, p,	The base AC,
are respectively as	of the plane.

For, draw CD perpendicular to the plane. Now here are three forces, keeping one another in equilibrio; namely, the weight, or force of gravity, acting perpendicular to AC, or parallel to BC; the power acting parallel to DB; and the pressure perpendicular to AB, or parallel to DC: but when three forces keep one another in equilibrio, they are proportional to the sides of the triangle CBD, made by lines in the direction of those forces, by art. 30; therefore those forces are to one another as BC, BD, CD. But the two triangles ABC, CBD, are equiangular, and have their like sides proportional; therefore the three BC, BD, CD, are to one another respectively as the three AB, BC, AC; which therefore are as the three forces W, P, p.

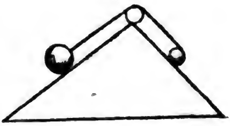


Corol. 1. Hence the weight W, power P, and pressure p, are respectively as radius, sine and cosine, of the plane's elevation BAC above the horizon.

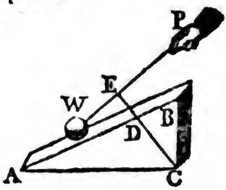
For, since the sides of triangles are as the sines of their opposite angles, therefore the three AB, BC, AC,
are respectively as - - sin. C, sin. A, sin. B,
or as - - - - radius, sine, cosine,
of the angle A of elevation.

Corol. 2. The power or relative weight that urges a body W down the inclined plane, is $= \frac{BC}{AB} \times W$; or the force with which it descends, or endeavours to descend, is as the sine of the angle A of inclination.

Corol. 3. Hence, if there be two planes of the same height, and two bodies be laid on them which are proportional to the lengths of the planes; they will have an equal tendency to descend down the planes. And consequently they will mutually sustain each other if they be connected by a string parallel to the planes.

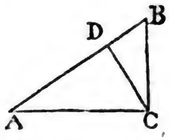


Corol. 4. In like manner, when the power P acts in any other direction whatever, WP; by drawing CDE perpendicular to the direction WP, the three forces in equilibrio, namely, the weight W, the power P, and the pressure on the plane, will still be respectively



as AC, CD, AD, drawn perpendicular to the direction of those forces.

194. PROP. The velocity acquired by a body descending freely down an inclined plane AB, is to the velocity acquired by a body falling perpendicularly, in the same time; as the height of the plane BC, is to its length AB.



For the force of gravity, both perpendicularly and on the plane, is constant; and these two, by corol. 2, art. 193, are to each other as AB to BC. But, by art. 131, the velocities generated by any constant forces, in the same time, are as those forces. Therefore the velocity down BA is to the velocity down BC, in the same time, as the force on BA to the force on BC: that is, as BC to BA.

Corol. 1. Hence, as the motion down an inclined plane is produced by a constant force, it will be a motion uniformly accelerated; and therefore the laws before laid down for accelerated motions in general, hold good for motions on inclined planes; such, for instance, as the following: That the velocities are as the times of descending from rest; that the spaces descended are as the squares of the velocities, or squares of the times; and that if a body be thrown up an inclined plane, with the velocity it acquired in descending, it will lose all its motion, and ascend to the same height, in the same time, and will repass any point of the plane with the same velocity as it passed it in descending.

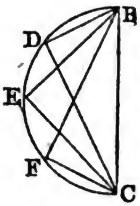
Corol. 2. Hence also, the space descended along an inclined plane, is to the space descended perpendicularly, in the same time, as the height of the plane CB, to its length AB, or as the sine of inclination to radius. For the spaces described by any forces, in the same time, are as the forces, or as the velocities.

Corol. 3. Consequently the velocities and spaces descended by bodies down different inclined planes, are as the sines of elevation of the planes.

Corol. 4. If CD be drawn perpendicular to AB; then, while a body falls freely through the perpendicular space BC, another body will, in the same time, descend down the part of the plane BD. For, by similar triangles, - - - BC : BD : : BA : BC, that is, as the space descended, by corol. 2.

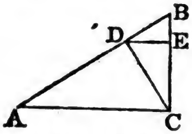
Or, in any right-angled triangle BDC, having its hypothenuse BC perpendicular to the horizon, a body will descend down any of its three sides, BD, BC, DC, in the same time. And therefore, if on the diameter BC a circle be described, the

time of descending down any chords BD, BE, BF, DC, EC, FC, &c. will be all equal, and each equal to the time of falling freely through the perpendicular diameter BC. Also the velocities acquired in descending down the chords BD, BE, BF, BC, are to one another as the lengths of those chords.



195. Prop. The time of descending down the inclined plane BA, is to the time of falling through the height of the plane BC, as the length BA is to the height BC.

Draw CD perpendicular to AB. Then the times of describing BD and BC are equal, by the last corol. Call that time t, and the time of describing BA call T.



Now, because the spaces described by constant forces, are as the squares of the times; therefore $t^2 : T^2 :: BD : BA$.

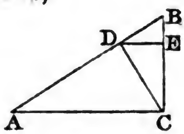
But the three BD, BC, BA, are in continual proportion; therefore $BD : BA :: BC^2 : BA^2$;
hence, by equality, $t^2 : T^2 :: BC^2 : BA^2$,
or - - $t : T :: BC : BA$.

Corol. Hence the times of descending different planes, of the same height, are to one another as the lengths of the planes.

196. Prop. A body acquires the same velocity in descending down any inclined plane BA, as by falling perpendicular through the height of the plane BC.

For, the velocities generated by any constant forces, are in the compound ratio of the forces and times of acting. But if we put

F to denote the whole force of gravity in BC,
f the force on the plane AB,
t the time of describing BC, and
T the time of descending down AB;
then by art. 193, $F : f :: BA : BC$;
and by art, 195, $t : T :: BC : BA$;
theref. by comp. $Ft : fT :: 1 : 1$.



That is, the compound ratio of the forces and times, or the ratio of the velocities, is a ratio of equality.

Corol. 1. Hence the velocities acquired, by bodies des-

cending down any planes, from the same height, to the same horizontal line, are equal.

Corol. 2. If the velocities be equal, at any two equal altitudes, D, E; they will be equal at all other equal altitudes A, C.

Corol. 3. Hence also, the velocities acquired by descending down any planes are as the square roots of the heights.

SCHOLIUM.

197. We may here introduce some useful formulæ, relative to motions along inclined planes, analogous to those already given for bodies falling freely (art. 154, 158.)

I. Let g, as before $= 32\frac{1}{6}$ feet, s the space along an inclined plane whose inclination is i, t the time, v the velocity; then

$$1.\ s = \tfrac{1}{2}gt^2 \sin.\ i = \frac{v^2}{2g \sin.\ i} = \tfrac{1}{2}tv$$

$$2.\ v = gt \sin.\ i = \sqrt{(2gs \sin.\ i)} = \frac{2s}{t}$$

$$3.\ t = \sqrt{\frac{2s}{g \sin.\ i}} = \frac{2s}{v},$$

II. Suppose V to be the velocity with which a body is projected up or down the plane; then, we have

$$4.\ v = V \mp gt \sin.\ i$$

$$5.\ s = Vt \mp \tfrac{1}{2}gt \sin.\ i = \frac{V^2 \mp v^2}{2g \sin.\ i}.$$

Making $v = o$, in equa. 4, and the latter member of equa. 5; the first will give the *time* at which the body will cease to rise, the latter the *space*.

III. If R be a constant resistance to motion on a horizontal plane, then

$$6.\ v = V - Rt$$

$$7.\ s = Vt - \tfrac{1}{2}Rt^2 = \frac{V^2 - v^2}{2R},$$

where, making $v = o$, we find when the motion ceases.

198. The first eight of the following problems will serve to exemplify these theorems.

1. How far will a body descend from quiescence in 4 seconds, along an inclined plane whose length is 400 and height 300 feet?

2. What velocity will such a body have acquired when it has reached the bottom A of the plane? (fig. to art. 194.)

3. Suppose AD = DB, in what time will the body pass over each of those portions?

4. How long would a body be in falling down 100 feet of a plane whose length AB is 150 feet, and height BC 60?

5. If AB = 90, and BC = 25 feet, what velocity would a body acquire in falling through 70 feet?

6. A body is projected up an inclined plane, whose length is 10 times its height, with a velocity of 30 feet per second; in what time will its velocity be destroyed, and it cease to ascend?

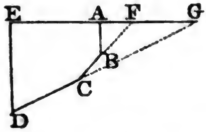
7. Suppose that at the moment a body is projected up AB with the velocity acquired by falling down it, another body begins to fall down it, where will they meet, the length of AB being given?

8. Given AB = 90, BC = 60 feet. And suppose two bodies to be let fall the same moment, one vertically, the other down the plane BA; what distance BD will the latter have moved, when the former has descended to C?

9. Ascertain, geometrically, the position of the right line of quickest descent, from a given point to a given plane.

10. Find, geometrically, the slope of a roof, down which rain may descend quickest.

199. Prop. If a body descend down any number of contiguous planes, AB, BC, CD; it will at last acquire the same velocity, as a body falling perpendicularly through the same height ED, supposing the velocity not altered by changing from one plane to another.



Produce the planes DC, CB, to meet the horizontal line EA produced in F and G. Then, by cor. 1, last art. the velocity at B is the same, whether the body descend thorugh AB or FB. And therefore the velocity at C will be the same, whether the body descend through ABC or through FC, which is also again the same as by descending through GC. Consequently it will have the same velocity at D, by descending through the planes AB, BC, CD, as by descending through the plane GD; supposing no obstruction to the motion by the body impinging on the planes at B and C: and this again, is the same velocity as by descending through the same perpendicular height ED.

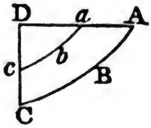
Corol. 1. If the lines ABCD, &c. be supposed indefinitely small, they will form a curve line, which will be the path of the body; from which it appears that a body acquires also the same velocity in descending along any curve, as in falling perpendicularly through the same height.

Corol. 2. Hence also, bodies acquire the same velocity by descending from the same height, whether they descend perpendicularly, or down any planes, or down any curve or curves. And if the velocities be equal, at any one height, they will be equal at all other equal heights. Therefore the velocity acquired by descending down any lines or curves, are as the square roots of the perpendicular heights.

Corol. 3. And a body, after its descent through any curve, will acquire a velocity which will carry it to the same height through an equal curve, or through any other curve, either by running up the smooth concave side, or by being retained in the curve by a string, and vibrating like a pendulum: Also, the velocities will be equal, at all equal altitudes; and the ascent and descent will be performed in the same time, if the curves be the same.

200. PROP. The times in which bodies descend through similar parts of similar curves, ABC, *abc*, placed alike, are as the square roots of their lengths.

That is, the time in AC is to the time in *ac*, as $\sqrt{AC}$ to $\sqrt{ac}$.



For, as the curves are similar, they may be considered as made up of an equal number of corresponding parts, which are every where, each to each, proportional to the whole. And as they are placed alike, the corresponding small similar parts will also be parallel to each other. But the time of describing each of these pairs of corresponding parallel parts, by art. 194, *cor.* 1, are as the square roots of their lengths, which, by the suppositions, are as $\sqrt{AC}$ to $\sqrt{ac}$, the roots of the whole curves. Therefore, the whole times are in the same ratio of $\sqrt{AC}$ to $\sqrt{ac}$.

Corol. 1. Because the axes DC, D*c*, of similar curves, are as the lengths of the similar parts AC, *ac*; therefore the times of descent in the curves AC, *ac*, are as $\sqrt{DC}$ to $\sqrt{Dc}$, or the square roots of their axes.

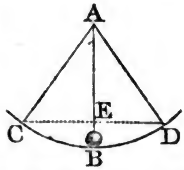
Corol. 2. As it is the same thing, whether the bodies run down the smooth concave side of the curves, or be made to describe those curves by vibrating like a pendulum, the lengths being DC, D*c*; therefore the times of the vibration

of pendulums, in similar arcs of any curves, are as the square roots of the lengths of the pendulums.

SCHOLIUM.

201. Having, in the last corollary, mentioned the pendulum, it may not be improper here to add some remarks concerning it.

A simple pendulum consists of a small ball, or other heavy body B, hung by a fine string or thread, moveable about a centre A, and describing the arc CBD; by which vibration the same motions happen to this heavy body, as would happen to any body descending by its gravity along the spherical superficies CBD, if

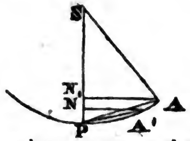


that superficies were perfectly hard and smooth. If the pendulum be carried to the situation AC, and then let fall, the ball in descending will describe the arc CB; and in the point B it will have that velocity which is acquired by descending through CB, or by a body falling freely through EB. This velocity will be sufficient to cause the ball to ascend through an equal arc BD, to the same height D from whence it fell at C; having there lost all its motion, it will again begin to descend by its own gravity; and in the lowest point B it will acquire the same velocity as before; which will cause it to re-ascend to C: and thus, by ascending and descending, it will perform continual vibrations in the circumference CBD. and if the motions of pendulums met with no resistance from the air, and if there were no friction at the centre of motion A, the vibrations of pendulums would never cease. But from these obstructions, though small, it happens, that the velocity of the ball in the point B is a little diminished in every vibration; and consequently it does not return precisely to the same points C or D, but the arcs described continually become shorter and shorter, till at length they are insensible; unless the motion be assisted by a mechanical contrivance. as in clocks, called a maintaining power.

Our present investigations relate to the simple pendulum, above described: the consideration of compound pendulums requires the previous knowledge of the centre of oscillation.

202. Prop. When a pendulum vibrates in a circular arc, the velocities acquired in the lowest point, are as the chords of the semi-arcs described.

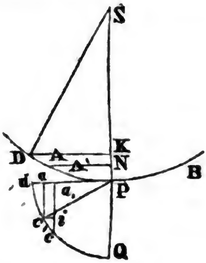
For, the velocity at P of a body that has descended through any arc AP, is equal to the velocity at P of a body that has fallen freely through the versed-sine NP (art. 199, cor. 2.)



Hence, velocity at P after descent through arc AP, is to velocity at P after descent through arc A′P, as $\sqrt{NP}$ to $\sqrt{N'P}$, that is (Geom. th. 87) as chord AP to chord A′P.

Corol. If, therefore, we would impart to a body a given velocity, v, we have only to compute the height NP, such that $NP = \frac{v^2}{2g} = \frac{v^2}{64\frac{1}{3}\text{ feet}}$, and through the point N draw the horizontal line NA; then AA′P an arc (of any circle passing through P) is one, through which when a body has fallen it will have acquired the proposed velocity. This is extremely useful in experiments on collision.

203. **Prop.** To investigate the time of vibration of a pendulum of given length, in an indefinitely small arc.



Now, in estimating the time of an oscillation in an indefinitely small circular arc, let it be recollected that the excess of such an arc above its chord, being incomparably less than itself, may be neglected; so that we may consider the square of such an arc (like that of its chord, Geom. th. 87) an equal to the rectangle under the versed-sine and the diameter.

Indeed, if instead of indefinitely small arcs we took arcs of 40′ or 50′, and compared the respective differences of their squares and those of their chords, we should find that the error would not exceed the 29000dth part of either results.

Thus, $\text{arc}^2\, 50' - \text{arc}^2\, 40' = 145444^2 - 116355^2$
$= 261799 \times 29089$,
while $\text{chord}^2\, 50' - \text{chord}^2\, 40' = 145442' - 116354^2$
$= 261796 \times 29088$.

Let, then, DPB represent such a very short oscillation of a pendulum whose length, l, is SP, S being the point of suspension.

$$\text{Then, versin. KP} = \text{arc}^2\ \text{DP} \div 2l$$
$$\text{versin. NP} = \text{arc}^2\ \text{AP} \div 2l.$$

Their diff. $\text{KN} = \frac{\text{BP}^2 - \text{AP}^2}{2l}$; which is the altitude through which a body must fall to acquire the velocity at A. Putting this value of the altitude in the usual expression for falling bodies, $v = \sqrt{(2gs)}$, it becomes $v = \sqrt{\left(2g \cdot \frac{\text{DP}^2 - \text{AP}^2}{2l}\right)}$

$= \sqrt{\frac{g}{l}} \cdot \sqrt{(DP^2 - AP^2)}$. This will be the velocity with which the pendulum will describe an exceedingly minute portion of the arc, such as AA′.

Draw, horizontally, dP = arc DP; with dP as radius describe the quadrilateral arc $dcc'Q$; make da = DA, aa' = AA′, and draw ac, $a'c'$, parallel to PQ.

Then, vel. at A

$$= \sqrt{\frac{g}{l}} \cdot \sqrt{(DP^2 - AP^2)} = \sqrt{\frac{g}{l}} \cdot \sqrt{(dP^2 - aP^2)} = ac\sqrt{\frac{g}{l}}.$$

But, since time of describing a space as AA′ = aa', is inversely as the velocity, or $t = \frac{s}{v}$, we have

$$\text{time through AA}'(\text{or } aa') = \frac{aa'}{ac}\sqrt{\frac{l}{g}} = \frac{cc'}{cP}\sqrt{\frac{l}{g}},$$

$$(\text{because by sim. tri. } \frac{aa'}{ac} = \frac{cc'}{cP}).$$

The same reasoning applies for every minute successive portion, such as AA′, of the semi-arc described by the pendulum: and when the ball has descended from D to P, the corresponding arc to dP its equal is the quadrant $dcc'Q$: the expression for the time, therefore, becomes, in that case,

$$t = \frac{dcQ}{PQ}\sqrt{\frac{l}{g}} = \frac{\text{semicircum.}}{\text{diam.}}\sqrt{\frac{l}{g}} = \tfrac{1}{2}\pi\sqrt{\frac{l}{g}}.$$

The time of ascending through PB = PD is, manifestly, equal to the above: therefore, ultimately, the time of complete oscillation through DPB, is,

$$t = \pi\sqrt{\frac{l}{g}}. \quad . \quad . \quad . \quad . \quad (1).$$

Consequently, *the times of oscillation are as the square roots of the lengths of the pendulums*, the force of gravity remaining the same.

204. For the same reason that we have the above equa. when l is the length of the pendulum, and g the lineal measure of the force of gravity, we have $t' = \pi\sqrt{\frac{l'}{g'}}$, in any other place where g' measures the force of gravity, and l' is the length of the pendulum.

Consequently, in general,

$$t : t' :: \sqrt{\frac{l}{g}} : \sqrt{\frac{l'}{g'}}. \quad . \quad . \quad . \quad . \quad (2).$$

If the force of gravity be the same, we have

$$t : t' :: \surd l : \surd l'. \quad . \quad . \quad . \quad . \quad . \quad . (3).$$

If the same pendulum be actuated by different gravitating forces, we have

$$t : t' :: \surd \frac{1}{g} : \surd \frac{1}{g'} :: \surd g' : \surd g. \quad . \quad . \quad . \quad (4).$$

When pendulums oscillate in equal times in different places, we have

$$g : g' :: l : l'. \quad . \quad . \quad . \quad . \quad . \quad . (5).$$

Other theorems may readily be deduced.

205. If either g or l be determined by experiment, the equa. 1 for t will give the other. Thus, if $\frac{1}{2}g$, or the space fallen through by a heavy body in 1″ of time, be found, then this theorem will give the length of the seconds pendulum. Or, if the length of the seconds pendulum be observed by experiment, which is the easier way; this theorem will give g. Now, in the latitude of London, the length of a pendulum which vibrates seconds, has been found to be $39\frac{1}{8}$ inches; and this being written for l in the theorem, it gives $\pi \surd \frac{39\frac{1}{8}}{g} = 1''$: and hence is found $\frac{1}{2}g = \frac{1}{2}\pi^2 l = \frac{1}{2}\pi^2 \times 39\frac{1}{8} = 193{\cdot}07$ inches $= 16\frac{1}{12}$ feet, for the descent of gravity in 1″; which it has also been found to be very exactly, by many accurate experiments, $l = \frac{1}{2}g \times {\cdot}20264$; $\frac{1}{2}g = l \times 4{\cdot}9348$.

SCHOLIUM.

206. Hence is found the length of a pendulum that shall make any number of vibrations in a given time. Or, the number of vibrations that shall be made by a pendulum of a given length. Thus, suppose it were required to find the length of a half-seconds pendulum, or a quarter-seconds pendulum; that is, a pendulum to vibrate twice in a second, or 4 times in a second. Then, since the time of vibration is as the square root of the length,

therefore $1 : \frac{1}{2} :: \surd 39\frac{1}{8} : \surd l$,

or $1 : \frac{1}{4} :: \quad 39\frac{1}{8} : \frac{39\frac{1}{8}}{4} = 9\frac{3}{4}$ inches nearly, the length of the half-seconds pendulum.

And $1 : \frac{1}{16} :: 39\frac{1}{8} : 2\frac{7}{16}$ inches, the length of the quarter-seconds pendulum.

Again, if it were required to find how many vibrations a pendulum of 80 inches long will make in a minute. Here

$$\surd 80 : \surd 39\tfrac{1}{8} :: 60'' \text{ or } 1' : 60 \surd \frac{39\frac{1}{8}}{80} = 7\tfrac{1}{2} \surd 31{\cdot}3 = - -$$

$41{\cdot}95987$, or almost 42 vibrations in a minute.

207. For military men it is a good practice to have a portable pendulum, made of painted tape with a brass bob at the end, so that the whole, except the bob, may be rolled up within a box, and the whole enclosed in a shagreen case. The tape is marked 200, 190, 180, 170, 160, &c. 80, 75, 70, 65, 60, at points, which being assumed respectively as points of suspension, the pendulum will make 200, 190, &c. down to 60 vibrations in a minute. Such a portable pendulum is highly useful in experiments relative to falling bodies, the velocity of sound, &c.

For the comparison of the times of oscillation in indefinitely small arcs of circles, in finite arcs of circles, and in cycloidal arcs, the student may turn to probs. 13 and 14, in Practical Exercises on Forces, and prob. 42, in Promiscuous Exercises near the end of this volume.

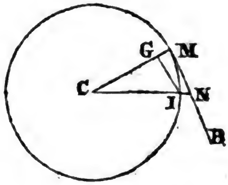
CENTRAL FORCES.

208. *Def.* 1. *Centripetal force* is a force which tends constantly to solicit or to impel a body towards a certain fixed point or centre.

2. *Centrifugal force* is that by which it would recede from such a centre, were it not prevented by the centripetal force.

3. These two forces are, jointly, called *central* forces.

209. PROP. If a body, M, drawn continually towards a fixed point, C, by a constant force, φ, and projected in a direction, MB, perpendicular to CM, describe the circumference of a circle about the centre C, the central force φ, is to the weight of the body, as the altitude due to the velocity of projection, is to half the radius CM.



Let v be the velocity of projection in the tangent MB, and r the radius CM. Independently of the action of the central force, the body would describe, along MB, during the very small time t, a space $MN = tv$, and would recede from the point C by the quantity IN, which may, without error, be regarded as equal to GM, when the arc MI is exceedingly small. If, therefore, the body, instead of moving in the tangent, were kept in the circumference by the central

force φ, its operation in the time t, would (art. 130) be equal to $\frac{1}{2}\varphi t^2$, and at the same time $=$ MG. But by the nature of the circle MG $= \frac{\text{MI}^2}{2r} = \frac{\text{MN}^2}{2r}$ (in an extremely small arc) $= \frac{t^2v^2}{2r}$, by the above.

Making, therefore, $\frac{1}{2}\varphi t^2 = \frac{t^2v^2}{2r}$, it reduces to

$$\varphi = \frac{v^2}{r} \quad . \quad . \quad . \quad . \quad (1).$$

Putting a for the altitude due to the velocity v, since (by art. 154) $v^2 = 2ag$, we have $\varphi = \frac{2ag}{r}$; whence there results.

$$\varphi : g :: a : \tfrac{1}{2}r.$$

Thus far, we have, in reality, considered only the unit of mass; but, if we multiply the first two terms of the above proportion by the mass of the body, the whole will still remain a correct proportion, and the general result may be thus enunciated: viz.

The centripetal force of any body, if it be free, or its centrifugal force, if it be retained to the centre C, by a thread (or otherwise), is to the weight of that body, as the height due to the velocity v, is to the half of the radius CM.

210. Hence, it appears that, so long as φ and r remain constant, the velocity v will be constant.

211. If both members of the equation 1 be multiplied by the mass M of the body, and we put F to represent the centrifugal force of that mass, we shall have $\text{F} = \frac{\text{M}v^2}{r}$. In like manner, if F′ is the centrifugal force of another body which revolves with the velocity v' in a circle whose radius is r', we shall have

$$\text{F} : \text{F}' :: \frac{v^2}{r} : \frac{v'^2}{r'} \quad . \quad . \quad . \quad . \quad (2).$$

212. If T and T′ denote the times of revolution of the two bodies, because $v = \frac{2\pi r}{\text{T}}$, and $v' = \frac{2\pi r'}{\text{T}'}$, we have

$$\text{F} : \text{F}' :: \frac{r}{\text{T}^2} : \frac{r'}{\text{T}'^2} \quad . \quad . \quad . \quad . \quad (3).$$

213. If the times of revolution are equal, we shall have

$$\text{F} : \text{F}' :: r : r' \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

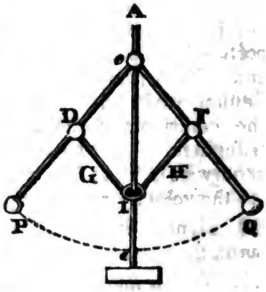
214. And, if we assume $T^2 : T'^2 :: r^3 : r'^3$, as in the planetary motions, the proportion (3) will become

$$F : F' :: r'^2 : r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

SCHOLIUM.

215. The subject of central forces is too extensive and momentous to be adequately pursued here. The student may consult the treatises of mechanics by *Gregory* and *Poisson*, and those on fluxions by *Simpson*, *Dealtry*, &c. We shall simply present in this place, one example connected with practical mechanics.

Exam. Investigate the characteristic property of a conical pendulum applied as a regulator or *governor* to steam-engines, &c.



This contrivance will be readily comprehended from the marginal figure, where A*a* is a vertical shaft capable of turning freely upon the sole *a*. *c*D, *c*F, are two bars which move freely upon the centre *c*, and carry at their lower extremities two equal weights, P, Q: the bars *c*D, *c*F, are united, by a proper articulation, to the bars G, H, which latter are attached to a ring, I, capable of sliding up and down the vertical shaft, A*a*. When this shaft and connected apparatus are made to revolve, in virtue of the centrifugal force, the balls P, Q, fly out more and more from A*a*, as the rotatory velocity increases: if, on the contrary, the rotatory velocity slackens, the balls descend and approach A*a*. The ring I *ascends* in the former case, *descends* in the latter: and a lever connected with I may be made to correct appropriately, the energy of the moving power. Thus, in the steam-engine, the ring may be made to act on the valve by which the steam is admitted into the cylinder; to augment its opening when the motion is slackening, and reciprocally diminish it when the motion is accelerated.

The construction is, often, so modified that the flying out of the balls causes the ring I to be depressed, and *vice versa*; but the general principle is the same. If FQ = FI = DP = DI, then I, P, Q, are always in some one horizontal plane: but that is not essential to the construction.

Now, let t denote the time of one revolution of the shaft, x the variable horizontal distance of each ball from that shaft, π as usual $= 3{\cdot}141593$: then will the velocity of each ball be $= \frac{2\pi x}{t}$, and (art. 209.) its centrifugal force $= \left(\frac{2\pi x}{t}\right)^2 \div x = \frac{4\pi^2 x}{t^2}$. The balls being operated upon simultaneously by the centrifugal force and the force of gravity, of which one operates horizontally, the other vertically, the resultant of the two forces is, evidently, always in the actual position of the handle CD, CF. It follows, therefore, that the ratio of the gravity to the centrifugal force, is that of cos. ICQ to sin. ICQ, or that of the vertical distance of Q below C to its horizontal distance from A*a*. Call the former d, the latter being x :

$$\text{then } d : x :: g : \frac{4\pi^2 x}{t^2},$$

$$\text{theref. } \frac{gt^2}{4\pi^2 x} = \frac{d}{x} \text{ and } t = 2\pi\sqrt{\frac{d}{g}} = 1{\cdot}10784\sqrt{d}.$$

Hence, the periodic time varies as the square root of the altitude of the conic pendulum, let the radius of the base be what it may.

Hence, also, when ICQ = ICP = 45°, the centrifugal force of each ball is equal to its weight.

ON THE CENTRES OF PERCUSSION, OSCILLATION, AND GYRATION.

216. *The Centre of Percussion* of a body, or a system of bodies, revolving about a point, or axis, is that point, which striking an immoveable object, the whole mass shall not incline to either side, but rest, as it were, in equilibrio, without acting on the centre of suspension.

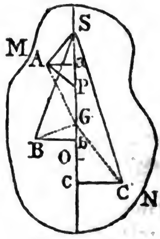
217. *The Centre of Oscillation* is that point, in a body vibrating by its gravity, in which if any body be placed, or if the whole mass be collected, it will perform its vibrations in the same time, and with the same angular velocity, as the whole body, about the same point or axis of suspension.

218. *The Centre of Gyration* is that point, in which if the whole mass be collected, the same angular velocity will

be generated in the same time, by a given force acting at any place, as in the body or system itself.

219. The *angular motion* of a body, or system of bodies, is the motion of a line connecting any point and the centre or axis of motion; and is the same in all parts of the same revolving body. And in different unconnected bodies, each revolving about a centre, the *angular velocity* is as the absolute velocity directly, and as the distance from the centre inversely; so that, if their absolute velocities be as their radii or distances, the angular velocities will be equal.

220. PROP. To find the centre of percussion of a body or system of bodies.



Let the body revolve about an axis passing through any point S in the line SGO, passing through the centres of gravity and percussion, G and O. Let MN be the section of the body, or the plane in which the axis SGO moves. And conceive all the particles of the body to be reduced to this plane, by perpendiculars let fall from them to the plane: a supposition which will not affect the centres G, O, nor the angular motion of the body.

Let A be the place of one of the particles, so reduced; join SA, and draw AP perpendicular to AS, and Aa perpendicular to SGO: then AP will be the direction of A's motion as it revolves about S; and the whole mass being stopped at O, the body A will urge the point P, forward, with a force proportional to its quantity of matter and velocity, or to its matter and distance from the point of suspension S; that is, as $A \cdot SA$; and the efficacy of this force in a direction perpendicular to SO, at the point P, is as $A \cdot Sa$, by similar triangles; also, the effect of this force on the lever, to turn it about O, being as the length of the lever, is as $A \cdot Sa \cdot PO = A \cdot Sa \cdot (SO - SP) = A \cdot Sa \cdot SO - A \cdot Sa \cdot SP = A \cdot Sa \cdot SO - A \cdot SA^2$. In like manner, the forces of B and C, to turn the system about O, are as

$$B \cdot Sb \cdot SO - B \cdot SB^2, \text{ and}$$
$$C \cdot Sc \cdot SO - C \cdot SC^2, \text{ \&c.}$$

But, since the forces on the contrary sides of O destroy one another, by the definition of this force, the sum of the positive parts of these quantities must be equal to the sum of the negative parts,

that is, $A \cdot Sa \cdot SO + B \cdot Sb \cdot SO + C \cdot Sc \cdot SO, \text{ \&c.} = \quad \cdot \quad \cdot$
$$A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2, \text{ \&c; and}$$
$$\text{hence } SO = \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2, \text{ \&c.}}{A \cdot Sa + B \cdot Sb + C \cdot Sc, \text{ \&c.}},$$

which is the distance of the centre of percussion below the axis of motion.

And here it must be observed that, if any of the points A, b, &c. fall on the contrary side of S, the corresponding product A . sa, or B . sb, &c. must be made negative.

221. *Corol.* 1. Since, by art. 105, A + B + C, &c. or the body b × the distance of the centre of gravity, SG, is = A . sa + B . sb + C . SC, &c. which is the denominator of the value of SO; therefore the distance of the centre of percussion, is $\text{SO} = \dfrac{\text{A} \cdot \text{SA}^2 + \text{B} \cdot \text{SB}^2 + \text{C} \cdot \text{SC}^2 \text{ \&c.}}{\text{SG} \times \text{body } b}$.

222. *Corol.* 2. Since, by Geometry, theor. 36, 37,

it is $\text{SA}^2 = \text{SG}^2 + \text{GA}^2 - 2\text{SG} \cdot \text{G}a$,

and $\text{SB}^2 = \text{SG}^2 + \text{GB}^2 + 2\text{SG} \cdot \text{G}b$,

and $\text{SC}^2 = \text{SG}^2 + \text{GC}^2 + 2\text{SG} \cdot \text{GC}$, &c.;

and, by cor. 5, art. 101, the sum of the last terms is nothing, namely, $- 2\text{SG} \cdot \text{G}a + 2\text{SG} \cdot \text{G}b + 2\text{SG} \cdot \text{GC}$ &c. $= 0$; therefore the sum of the others, or A . SA² + B . SB² &c. is $= (\text{A} + \text{B} \text{ \&c.}) \cdot \text{SG}^2 + \text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 + \text{C} \cdot \text{GC}^2$ &c.
or $= b \cdot \text{SG}^2 + \text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 + \text{C} \cdot \text{GC}^2$ &c; which being substituted in the numerator of the foregoing value of SO, gives

$$\text{SO} = \frac{b \cdot \text{SG}^2 + \text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 + \text{\&c.}}{b \cdot \text{SG}},$$

$$\text{or SO} = \text{SG} + \frac{\text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 + \text{C} \cdot \text{GC}^2 \text{ \&c.}}{b \cdot \text{SG}}.$$

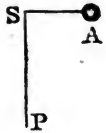
223. *Corol.* 3. Hence the distance of the centre of percussion always exceeds the distance of the centre of gravity, and the excess is always $\text{GO} = \dfrac{\text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 \text{ \&c.}}{b \cdot \text{SG}}$.

224. And hence also, $\text{SG} \cdot \text{GO} = \dfrac{\text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 \text{ \&c.}}{\text{the body } b}$;

that is SG . GO is always the same constant quantity, wherever the point of suspension S is placed; since the point G and the bodies A, B, &c. are constant. Or GO is always reciprocally as SG, that is GO is less, as SG is greater; and consequently the point O rises upwards and approaches towards the point G, as the point S is removed to the greater distance; and they coincide when SG is infinite. But when S coincides with G, then GO is infinite, or O is at an infinite distance.

225. PROP. If a body A, at the distance SA from an axis

passing through S, perpendicular to the plane of the paper, be made to revolve about that axis by any force acting at P in the line SP, perpendicular to the axis of motion: it is required to determine the quantity or matter of another body, Q, which being placed at P, the point where the force acts, it shall be accelerated in the same manner, as when A revolved at the distance SA; and consequently, that the angular velocity of A and Q about S, may be the same in both cases.

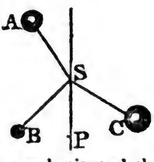


By the nature of the lever, SA : SP :: f : $\frac{SP}{SA} \cdot f$, the effect of the force f, acting at P, on the body at A; that is, the force f acting at P, will have the same effect on the body A, as the force $\frac{SP}{SA}f$, acting directly at the point A. But as ASP revolves altogether about the axis at S, the absolute velocities of the points A and S, or of the bodies A and Q, will be as the radii SA, SP, of the circle described by them. Here then we have two bodies A and Q, which being urged directly by the forces f and $\frac{SP}{SA}f$, acquire velocities which are as SP and SA. And since the motive forces of bodies are as their mass and velocity: therefore . . . . . .

$\frac{SP}{SA}f : f :: A \cdot SA : Q \cdot SP$, and $SP^2 : SA^2 :: A : Q = \frac{SA^2}{SP^2}A$,

which therefore expresses the mass of matter which, being placed at P, would receive the same angular motion from the action of any force at P, as the body A receives. So that the resistance of any body A, to a force acting at any point P, is directly as the square of its distance SA from the axis of motion, and reciprocally as the square of the distance SP of the point where the force acts.

226. *Corol.* 1. Hence the force which accelerates the point P, is to the force of gravity, as $\frac{f \cdot SP^2}{A \cdot SA^2}$ to 1, or as $f \cdot SP^2$ to $A \cdot SA^2$.



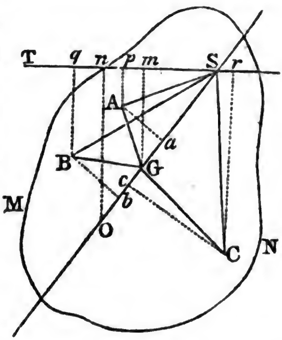
227. *Corol.* 2. If any number of bodies A, B, C, be put in motion, about a fixed axis passing through S, by a force acting at P; the point P will be accelerated in the same manner, and consequently the whole system will have the same angular velocity, if instead of the bodies A, B, C, placed at the distances SA, SB, SC, there be substituted the

bodies $\frac{SA^2}{SP^2}A$, $\frac{SB^2}{SP^2}B$, $\frac{SC^2}{SP^2}C$; these being collected into the point P. And hence, the moving force being f, and the matter moved being $\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SP^2}$: therefore $\frac{f \cdot SP^2}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$ is the accelerating force; which therefore is to the accelerating force of gravity, as $f \cdot SP^2$ to $A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$.

228. *Corol.* 3. The angular velocity of the whole system of bodies, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$. For the absolute velocity of the point P, is as the accelerating force, or directly as the motive force f, and inversely as the mass $\frac{A \cdot SA^2 \,\&c.}{SP^2}$: but the angular velocity is as the absolute velocity directly, and the radius SP inversely; therefore the angular velocity of P, or of the whole system, which is the same thing, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$.

229. PROP. To determine the centre of oscillation of any compound mass, or body MN, or of any system of bodies A, B, C, &c.

Let MN be the plane of vibration, to which let all the matter be reduced, by letting fall perpendiculars from every particle, to this plane. Let G be the centre of gravity, and O the centre of oscillation; through the axis S draw SGO, and the horizontal line Sq; then from every particle A, B, C, &c. let fall perpendiculars Aa, Ap, Bb, Bq, Cc, Cr, to these two lines; and join SA, SB, SC; also, draw Gm, On, perpendicular to Sq. Now the forces of the weights A, B, C, to turn the body about the axis, are $A \cdot Sp$, $B \cdot Sq$, $- C \cdot Sr$; therefore, by cor. 3, art. 228, the angular



motion generated by all these forces is $\frac{A \cdot Sp + B \cdot Sq - C \cdot Sr}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$.

Also, the angular veloc. any particle p, placed in o, generates in the system, by its weight, is $\frac{p \cdot sn}{p \cdot so^2}$ or $\frac{sn}{so^2}$, or $\frac{sm}{sg \cdot so}$, because of the similar triangles sgm, son. But, by the problem, the vibrations are performed alike in both cases, and therefore these two expressions must by equal to each other, that is, $\frac{sm}{sg \cdot so} = \frac{A \cdot sp + B \cdot sq - C \cdot sr}{A \cdot sA^2 + B \cdot sB^2 + C \cdot sC^2}$; and hence $so = \frac{sm}{sg} \times \frac{A \cdot sA^2 + B \cdot sB^2 + C \cdot sC^2}{A \cdot sp + B \cdot sq - C \cdot sr}$.

But, by cor. 2, art. 105, the sum $A \cdot sp + B \cdot sq - C \cdot sr = (A + B + C) \cdot sm$; therefore the distance so = . . .

$$\frac{A \cdot sA^2 + B \cdot sB^2 + C \cdot sC^2}{sg \cdot (A + B + C)} = \frac{A \cdot sA^2 + B \cdot sB^2 + C \cdot sC^2}{A \cdot sa + B \cdot sb + C \cdot sc}$$

by art. 107, which is the distance of the centre of oscillation o, below the axis of suspension; where any of the products $A \cdot sa$, $B \cdot sb$, must be negative, when a, b, &c. lie on the other side of s. So that this is the same expression as that for the distance of the centre of percussion, found in art. 220.

Hence it appears, that the centres of percussion and of oscillation, are in the very same point. And therefore the properties in all the corollaries there found for the former, are to be here understood of the latter; and it will be necessary to mark them carefully, as they are of great practical utility.

230. *Corol.* 1. If p be any particle of a body b, and d its distance from the axis of motion s; also G, O the centres of gravity and oscillation. Then the distance of the centre of oscillation of the body, from the axis of motion, is . . .

$$so = \frac{\text{sum of all the } pd^2}{sg \times \text{the body } b}.$$

231. *Corol.* 2. If b denote the matter in any compound body, whose centres of gravity and oscillation are G and O; the body P, which being placed at P, where the force acts as in the last proposition, and which receives the same motion from that force as the compound body b, is $P = \frac{sg \cdot so}{sp^2} \cdot b$.

For, by corol. 2, art. 222, this body P is = ... $\frac{A \cdot sA^2 + B \cdot sB^2 + C \cdot sC^2}{sp^2}$. But, by corol. 1, art. 221,

$SG \cdot SO \cdot b = A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$; therefore $P = \frac{SG \cdot SO}{SP} \cdot b$.

SCHOLIUM.

232. By the method of Fluxions, the centre of oscillation, for a regular body, will be found from cor. 1. But for an irregular one; suspend it at the given point; and hang up also a simple pendulum of such a length, that making them both vibrate, they may keep time together. Then the length of the simple pendulum is equal to the distance of the centre of oscillation of the body, below the point of suspension.

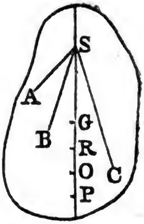
233. Or it will be still better found thus: Suspend the body very freely by the given point, and make it vibrate in very small arcs, counting the number of vibrations it makes in any time, as a minute, by a good stop watch; and let that number of vibrations made in a minute be called n: Then shall the distance of the centre of oscillation, be $SO = \frac{140850}{nn}$ inches. For, the length of the pendulum vibrating seconds, or 60 times in a minute, being $39\frac{1}{8}$ inches; and the lengths of pendulums being reciprocally as the square of the number of vibrations made in the same time; therefore . . . .

$n^2 : 60^2 :: 39\frac{1}{8} : \frac{60^2 \times 39\frac{1}{8}}{nn} = \frac{140850}{nn}$: the length of the pendulum which vibrates n times in a minute, or the distance of the centre of oscillation below the axis of motion.

Or, $SO = 39\frac{1}{8}\, t^2$, in inches, t being the time of one oscillation in a very small arc.

234. The foregoing determination of the point into which all the matter of a body being collected, it shall oscillate in the same manner as before, only respects the case in which the body is put in motion by the gravity of its own particles, and the point is the centre of oscillation: but when the body is put in motion by some other extraneous force, instead of its gravity, and made to rotate instead of oscillate, then the point is different from the former, and is called the Centre of Gyration; which is determined in the following manner:

235. PROP. To determine the centre of gyration of a compound body or of a system of bodies.



Let R be the centre of gyration, or the point into which all the particles A, B, C, &c. being collected, it shall receive the same angular motion from a force f acting at P, as the whole system receives.

Now, by cor. 3, art. 228, the angular velocity generated in the system by the force f, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 \text{ \&c.}}$, and by the same, the angular velocity of the system placed in R, is $\frac{f \cdot SP}{(A + B + C \text{ \&c.}) \cdot SR^2}$: then, by making these two expressions equal to each other, the equation gives $SR = \sqrt{\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A + B + C}}$, for the distance of the centre of gyration below the axis of motion.

236. *Corol.* 1. Because $A \cdot SA^2 + B \cdot SB^2$ &c. $= SG \cdot SO \cdot W$, where G is the centre of gravity, O the centre of oscillation, and W the weight of body $A + B + C$ &c.; therefore $SR^2 = SG \cdot SO$; that is, the distance of the centre of gyration, from the point of suspension, is a mean proportional between those of gravity and oscillation.

237. *Corol.* 2. If p denote any particle of a body W, at d distance from the axis of motion; then SR^2

$$= \frac{\text{sum of all the } pd^2}{\text{body W}}.$$

Or, if ρ be put for SR, the distance of the centre of gyration from the point of suspension, $W\rho^2 = A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2 +$ &c. sum of all the pd^2.

SCHOLIUM.

238. By means of the theory of the centre of gyration, and the values of ρ thence deduced in the note to prop. 2, under the heading "Maximum in Machines" in a subsequent part of this volume, the phænomena of rotatory motion become connected with those of accelerating forces: for then, if a weight or other moving power P act at a radius r to give rotation to a body, weight W, and dist. of centre of gyration from axis of motion $= \rho$, we shall have for the accelerating force, the expression

$$f = \frac{Pr^2}{Pr^2 + W\rho^2};$$

and consequently for the space descended by the actuating

weight or power P, in a given time t, we shall have the usual formula

$$s = \frac{1}{2}fgt^2,$$

introducing the above value of f.

239. For applications of these formulæ and their obvious modifications, as they are exceedingly useful in rotatory motions, the student may solve the following problems.

Problems illustrative of the Principle of the Centre of Gyration.

1. Suppose a cylinder that weighs 100lbs. to turn upon a horizontal axis, and imagine motion to be communicated by a weight of 10lbs. attached to a cord which coils upon the surface of the cylinder: how far will that weight descend in 10 seconds? Ans. 268 055 f.

2. Required the actuating weight such that when attached in the same way to the same cylinder, it shall descend $16\frac{1}{12}$ feet in 3 seconds. $P = \frac{\frac{1}{2}sw}{gt^2 - s} = 6\frac{1}{4}$.

3. Another cylinder, which weighs 200lbs, is actuated in like manner by a weight of 30lbs. How far will the weight descend in 6 seconds? Ans. 133·6 feet.

4. Suppose the actuating weight to be 30 pounds; and that it descends through 48 feet in 2 seconds, what is the weight of the cylinder? Ans. $20\frac{4}{11}$ lbs.

5. Suppose a cylinder that weighs 20lbs. to have a weight of 30lbs. actuating it, by means of a cord coiled about the surface of the cylinder; what velocity will the descending weight have acquired at the end of the first second? Ans. $24\frac{1}{8}$.

6. Of what weight will the axis be relieved in the case of the last example, when the system is completely in motion? Ans. $22\frac{1}{2}$lbs.

7. A sphere, w, whose radius is three feet, and weight 500lbs. turns upon a horizontal axis, being put in motion by a weight of 20lbs. acting by means of a string that goes over a wheel whose radius is half a foot. How long will the weight, P, be in descending 50 feet? Ans. $33\frac{1}{2}''$.

8. Of what weight will the axle be relieved as soon as motion is commenced? Ans. $\frac{20}{361}$lbs.

9. If in example seventh the radius of the wheel be equal to that of the sphere, what ratio will the accelerating force bear to that of gravity?

10. A paraboloid, w, whose weight is 200lbs. and radius of base 20 inches, is put in motion upon a horizontal axis by a weight P of 15lbs. acting by a cord that passes over a wheel whose radius is 6 inches. After P has descended for 10 seconds, suppose it to reach a horizontal plane and cease to act, then how many revolutions would the paraboloid make in a minute?

BALLISTIC PENDULUM.

240. PROP. To explain the construction of the Ballistic Pendulum, and show its use in determining the velocity with which a cannon or other ball strikes it.

The ballistic pendulum is a heavy block of wood MN, suspended vertically by a strong horizontal iron axis at S, to which it is connected by a firm iron stem. This problem is the application of the preceding articles, and was invented by Mr. Robins, to determine the initial velocities of military projectiles; a circumstance very useful in that science; and it is the best method yet known for determining them with any degree of accuracy.

Let G, R, O, be the centres of gravity, gyration, and oscillation, as determined by the foregoing propositions; and let P be the point where the ball strikes the face of the pendulum; the momentum of which, or the product of its weight and velocity, is expressed by the force f, acting at P, in the foregoing propositions. Now,

Put $p =$ the whole weight of the pendulum,
$b =$ the weight of the ball,
$g =$ SG the distance of the centre of gravity,
$o =$ SO the distance of the centre of oscillation,
$r =$ SR $= \sqrt{go}$ the distance of centre of gyration,
$i =$ SP the distance of the point of impact,
$v =$ the velocity of the ball,
$u =$ that of the point of impact P,
$c =$ chord of the arc described by O.

By art. 235, if the mass p be placed all at R, the pendulum will receive the same motion from the blow in the point

P : and as $SP^2 : SR^2 :: p : \frac{SR^2}{SP^2} \cdot p$ or $\frac{r^2}{i^2}p$ or $\frac{go}{ii}p$, (art. 236), the mass which being placed at P, the pendulum will still receive the same motion as before. Here then are two quantities of matter, namely, b and $\frac{go}{ii}p$, the former moving with the velocity v, and striking the latter at rest; to determine their common velocity u, with which they will jointly proceed forward together after the stroke. In which case, by the law of the impact of non-elastic bodies, we have $\frac{go}{ii}p + b : b :: v : u$, and therefore $v = \frac{bii + gop}{bii}u$ the velocity of the ball in terms of u, the velocity of the point P, and the known dimensions and weights of the bodies.

But now to determine the value of u, we must have recourse to the angle through which the pendulum vibrates; for when the pendulum descends again to the vertical position, it will have acquired the same velocity with which it began to ascend, and by the laws of falling bodies, the velocity of the centre of oscillation is such as a heavy body would acquire by freely falling through the versed sine of the arc described by the same centre O. But the chord of that arc is c, and its radius is o; and, by the nature of the circle, the chord is a mean proportional between the versed sine and diameter, therefore $2o : c :: c : \frac{cc}{2o}$, the versed sine of the arc described by O. Then, by the laws of falling bodies $\sqrt{16\frac{1}{12}} : \sqrt{\frac{cc}{2o}} :: 32\frac{1}{6} : c\sqrt{\frac{2a}{o}}$, the velocity acquired by the point O in descending through the arc whose chord is c, where $a = 16\frac{1}{12}$ feet: and therefore $o : i :: c\sqrt{\frac{2a}{o}} : \frac{ci}{o}\sqrt{\frac{2a}{o}}$, which is the velocity u, of the point P.

Then, by substituting this value for u, the velocity of the ball, before found, becomes $v = \frac{bii + gop}{bio} \times c\sqrt{\frac{2a}{o}}$. So that the velocity of the ball is directly as the chord of the arc described by the pendulum in its vibration.

SCHOLIUM.

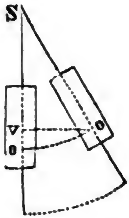
241. In the foregoing solution, the change in the centre of oscillation is omitted, which is caused by the ball lodging in the point P. But the allowance for that small change, and that of some other small quantities, may be seen in my Tracts, where all the circumstances of this method are treated at full length.

For an example in numbers, suppose the weights and dimensions to be as follow: namely,

$p = 570$lb.	Then
$b = 18$oz. $1\frac{1}{2}$dr $= 1\cdot131$lb.	$\frac{bii+gop}{bio}\times c = \frac{1\cdot131\times94\cdot3^2+78\frac{1}{2}\times84\frac{7}{8}\times570}{1\cdot131\times94\frac{3}{16}\times84\frac{7}{8}}$
$g = 78\frac{1}{2}$ inc.	
$o = 84\frac{7}{8}$ inc. $= 7\cdot065$ feet	$\times\frac{18\cdot73}{12} = 656\cdot56.$
$i = 94\frac{3}{16}$ inc.	And $\sqrt{\frac{2a}{o}} = \sqrt{\frac{32\frac{1}{6}}{7\cdot065}} = \sqrt{\frac{193}{42\cdot39}} = 2\cdot1337.$
$c = 18\cdot73$ inc.	

Therefore $656\cdot56\times2\cdot1337$, or 1401 feet, is the velocity, per second, with which the ball moved when it struck the pendulum.

242. When the impact is made upon the centre of oscillation, the computation becomes simplified.



In that case, since the whole mass, p, of the pendulum, may be regarded as concentered at o, and the ball, b, strikes that point, we shall have $bv = (b+p)v'$; v being the velocity of the ball before the impact, and v' that of the ball and pendulum together, after the impact. Now, if the centre of oscillation o, after the blow, describes the arc oo′, before the motion is destroyed, the velocity v' will be equal to that acquired by falling through the versed sine vo, of the arc oo′ or angle s to the radius so. But, if the time t of a very minute oscillation of the pendulum be known or inferred from that in an ascertained arc, we have (art. 233), so $= 39\frac{1}{8}t^2$ inches $= 3\frac{25}{96}t^2$ feet.

$$\text{Hence vo} = \text{so nat. versin s},$$
$$= 3\cdot2604\tfrac{1}{6}t^2 \text{ versin s},$$
$$\text{and (art. 154) } v' = \sqrt{(64\tfrac{1}{3}\times3\cdot2604\tfrac{1}{6}t^2 \text{ versin s})}$$
$$= 14\cdot48286t\sqrt{\text{versin s}}.$$

$$\text{Conseq. } v = \frac{(b+p)v'}{b} = \frac{b+p}{b}\,.\,14\cdot48286t\sqrt{\text{versin s}}.$$

This mode of computation, with a slight and obvious change, applies to qu. 48 of the Practical Exercises in Natural Philosophy.

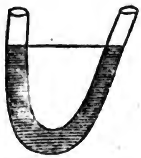
OF HYDROSTATICS.

243. Hydrostatics is the science which treats of the pressure, or weight, and equilibrium of water and other fluids, especially those that are non-elastic.

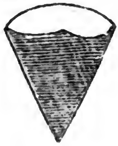
244. A fluid is elastic, when it can be reduced into a less bulk by compression, and which restores itself to its former bulk again when the pressure is removed; as air. And it is non-elastic, when it is not compressible or expansible, as water, &c.

245. Prop. If any part of a fluid be raised higher than the rest, by any force, and then left to itself; the higher parts will descend to the lower places, and the fluid will not rest, till its surface be quite even and level.

For, the parts of a fluid being easily moveable every way, the higher parts will descend by their superior gravity, and raise the lower parts, till the whole come to rest in a level or horizontal plane.



Corol. 1. Hence, water that communicates with other water, by means of a close canal or pipe, will stand at the same height in both places. Like as water in the two legs of a syphon.



Corol. 2. For the same reason, if a fluid gravitate towards a centre; it will dispose itself into a spherical figure, the centre of which is the centre of force. Like the sea in respect of the earth.

246. Prop. When a fluid is at rest in a vessel, the base of which is parallel to the horizon; equal parts of the base are equally pressed by the fluid.

For, on every equal part of this base there is an equal column of the fluid supported by it. And as all the columns are of equal height, by the last proposition they are of equal

weight, and therefore they press the base equally; that is, equal parts of the base sustain an equal pressure.

Corol. 1. All parts of the fluid press equally at the same depth. For, if a plane parallel to the horizon be conceived to be drawn at that depth; then the pressure being the same in any part of that plane, by the proposition, therefore the parts of the fluid, instead of the plane, sustain the same pressure at the same depth.

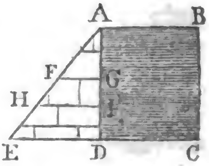
Corol. 2. The pressure of the fluid at any depth, is as the depth of the fluid. For the pressure is as the weight, and the weight is as the height of the fluid.

Corol. 3. The pressure of the fluid on any horizontal surface or plane, is equal to the weight of a column of the fluid, whose base is equal to that plane, and altitude is its depth below the upper surface of the fluid.

247. PROP. When a fluid is pressed by its own weight, or by any other force; at any point it presses equally, in all directions whatever.

This arises from the nature of fluidity, by which it yields to any force in any direction. If it cannot recede from any force applied, it will press against other parts of the fluid in the direction of that force. And the pressure in all directions will be the same: for if it were less in any part, the fluid would move that way, till the pressure be equal every way.

Corol. 1. In a vessel containing a fluid; the pressure is the same against the bottom, as against the sides, or even upwards at the same depth.



Corol. 2. Hence, and from the last proposition, if ABCD be a vessel of water, and there be taken, in the base produced, DE, to represent the pressure at the bottom; joining AE, and drawing any parallels to the base, as FG, HI; then shall FG represent the pressure at the depth AG, and HI the pressure at the depth AI, and so on; because the parallels FG, HI, ED, by sim. triangles, are as the depths AG, AI, AD: which are as the pressures, by the proposition.

And hence the sum of all the FG, HI, &c. or the area of the triangle ADE, is as the pressure against all the points G, I,

&c. that is, against the line AD. But as every point in the line CD is pressed with a force as DE, and that thence the pressure on the whole line CD is as the rectangle ED . DC, while that against the side is as the triangle ADE or $\frac{1}{2}$DA . DE; therefore the pressure on the horizontal line DC, is to the pressure against the vertcal line DA, as DC to $\frac{1}{2}$DA. And hence, if the vessel be an upright rectangular one, the pressure on the bottom, or whole weight of the fluid, is to the pressure against one side, as the base is to half that side. Therefore the weight of the fluid is to the pressure against all the four upright sides, as the base is to half the upright surface. And the same holds true also in any upright vessel, whatever the sides be, or in a cylindrical vessel. Or, in the cylinder, the weight of the fluid is to the pressure against the upright surface, as the radius of the base is to double the altitude.

Also, when the rectangular prism becomes a cube, it appears that the weight of the fluid on the base, is double the pressure against one of the upright sides, or half the pressure against the whole upright surface.

Corol. 3. The pressure of a fluid against any upright surface, as the gate of a sluice or canal, is equal to half the weight of a column of the fluid whose base is equal to the surface pressed, and its altitude the same as the altitude of that surface. For the pressure on a horizontal base equal to the upright surface, is equal to that column; and the pressure on the upright surface, is but half that on the base, of the same area.

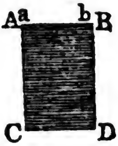
So that, if b denote the breadth, and d the depth of such a gate or upright surface; then the pressure against it, is equal to the weight of the fluid whose magnitude is $\frac{1}{2}bd^2 = \frac{1}{2}$AB . AD^2. Hence, if the fluid be water, a cubic foot of which weighs 1000 ounces, or $62\frac{1}{2}$ pounds; and if the depth AD be 12 feet, the breadth AB 20 feet; then the content, or $\frac{1}{2}$AB . AD^2, is 1440 feet; and the pressure is 1440000 ounces, or 90000 pounds, or $40\frac{1}{5}$ tons weight nearly.

248. **Prop.** The pressure of a fluid on a surface any way immersed in it, whether perpendicular, or horizontal, or oblique, is equal to the weight of a column of the fluid, whose base is equal to the surface pressed, and its altitude equal to the depth of the centre of gravity of the surface pressed below the top or surface of the fluid.

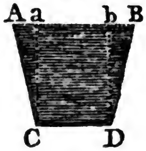
For, conceive the surface pressed to be divided into innumerable sections parallel to the horizon; and let s denote any one of those horizontal sections, also d its distance or depth below the top surface of the fluid. Then, by art. 246,

cor. 3, the pressure of the fluid on the section is equal to the weight of *ds*; consequently the total pressure on the whole surface is equal to all the weights *ds*. But, if *b* denote the whole surface pressed, and *g* the depth of its centre of gravity below the top of the fluid; then, by art. 108. *bg* is equal to the sum of all the *ds*. Consequently the whole pressure of the fluid on the body or surface *b*, is equal to the weight of the bulk *bg* of the fluid, that is, of the column whose base is the given surface *b*, and its height is *g* the depth of the centre of gravity in the fluid.

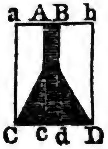
249. PROP. The pressure of a fluid, on the base of the vessel in which it is contained, is as the base and perpendicular altitude; whatever be the figure of the vessel that contains it.



If the sides of the base be upright, so that it be a prism of a uniform width throughout, then the case is evident; for then the base supports the whole fluid, and the pressure is just equal to the weight of the fluid.



But if the vessel be wider at top than bottom; then the bottom sustains, or is pressed by, only the part contained within the upright lines *ac*, *b*D; because the parts AC*a*, BD*b* are supported by the sides AC, BD; and those parts have no other effect on the part *ab*DC than keeping it in its position, by the lateral pressure against *ac* and *b*D, which does not alter its perpendicular pressure downwards. And thus the pressure on the bottom is less than the weight of the contained fluid.



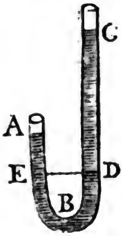
And if the vessel be widest at bottom; then the bottom is still pressed with a weight which is equal to that of the whole upright column *ab*DC. For, as the parts of the fluid are in equilibrio, all the parts have an equal pressure at the same depth; so that the parts within C*c* and *d*D press equally as those in *cd*, and therefore equally the same as if the sides of the vessel had gone upright to *a* and *b*, the defect of fluid in the parts AC*a* and BD*b* being exactly compensated by the downward pressure or resistance of the sides AC and BD against the contiguous fluid. And thus the pressure on the base may be made to exceed the weight of the contained fluid, in any proportion whatever.

So that, in general, be the vessels of any figure whatever, regular or irregular, upright or sloping, or variously wide and narrow in different parts, if the bases and perpendicular

altitudes be but equal, the bases always sustain the same pressure. And as that pressure, in the regular upright vessel, is the whole column of the fluid, which is as the base and altitude; therefore the pressure in all figures is in that same ratio.

Corol. 1. Hence, when the heights are equal, the pressures are as the bases. And when the bases are equal, the pressure is as the height. But when both the heights and bases are equal, the pressures are equal in all, though their contents be ever so different.

Corol. 2. The pressure on the base of any vessel is the same as on that of a cylinder, of an equal base and height.

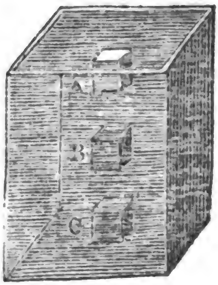


Corol. 3. If there be an inverted syphon, or bent tube, ABC, containing two different fluids CD, ABD, that balance each other, or rest in equilibrio; then their heights in the two legs, AE, CD, above the point of meeting, will be reciprocally as their densities.

For if they do not meet at the bottom, the part BD balances the part BE, and therefore the part CD balances the part AE; that is, the weight of CD is equal to the weight of AE. And as the surface at D is the same, where they act against each other, therefore AE : CD : : density of CD : density of AE.

So, if CD be water, and AE quicksilver, which is near 14 times heavier; then CD will be $= 14$AE; that is, if AE be 1 inch, CD will be 14 inches; if AE be 2 inches, CD will be 28 inches; and so on.

250. PROP. If a body be immersed in a fluid of the same density or specific gravity; it will rest in any place where it is put. But a body of greater density will sink; and one of a less density will rise to the top, and float.



The body being of the same density, or of the same weight with the like bulk of the fluid, will press the fluid under it, just as much as if its space were filled with the fluid itself. The pressure then all around it will be the same as if the fluid were in its place; consequently there is no force, neither upward nor downward, to put the body out of its place. And therefore it will remain wherever it is put.

But if the body be lighter; its

pressure downward will be less than before; and less than the water upward at the same depth; therefore the greater force will overcome the less, and push the body upward to A.

And if the body be heavier than the fluid, the pressure downward will be greater than the fluid at the same depth; therefore the greater force will prevail, and carry the body down to the bottom at C.

Corol. 1. A body immersed in a fluid, loses as much weight, as an equal bulk of the fluid weighs. And the fluid gains the same weight. Thus, if the body be of equal density with the fluid, it loses all its weight, and so requires no force but the fluid to sustain it. If it be heavier, its weight in the water will be only the difference between its own weight and the weight of the same bulk of water; and it requires a force to sustain it just equal to that difference. But if it be lighter, it requires a force equal to the same difference of weights to keep it from rising up in the fluid.

Corol. 2. The weights lost, by immerging the same body in different fluids, are as the specific gravities of the fluids. And bodies of equal weight, but different bulk, lose, in the same fluid, weights which are reciprocally as the specific gravities of the bodies, or directly as their bulks.

Corol. 3. The whole weight of a body which will float in a fluid, is equal to the weight of as much of the fluid, as the immersed part of the body displaces when it floats. For the pressure under the floating body, is just the same as so much of the fluid as is equal to the immersed part; and therefore the weights are the same.

Corol. 4. Hence the magnitude of the whole body, is to the magnitude of the part immersed, as the specific gravity of the fluid, is to that of the body. For, in bodies of equal weight, the densities, or specific gravities, are reciprocally as their magnitudes.

Corol. 5. And because, when the weight of a body taken in a fluid, is subtracted from its weight out of the fluid, the difference is the weight of an equal bulk of the fluid; this therefore is to its weight in the air, as the specific gravity of the fluid is to that of the body.

Therefore, if W be the weight of a body in air,
w its weight in water, or any fluid,
S the specific gravity of the body, and
s the specific gravity of the fluid;
then $W - w : W :: s : S$, which proportion will give either of those specific gravities, the one from the other.

Thus $\text{s} = \frac{\text{w}}{\text{w} - w}\, s$, the specific gravity of the body ;

and $s = \frac{\text{w} - w}{\text{w}}\, \text{s}$, the specific gravity of the fluid.

So that the specific gravities of bodies, are as their weights in the air directly, and their loss in the same fluid inversely.

Corol. 6. And hence, for two bodies connected together, or mixed together into one compound, of different specific gravities, we have the following equations, denoting their weights and specific gravities, as below, viz.

H = weight of the heavier body in air,
h = weight of the same in water, } s its spec. gravity ;
L = weight of the lighter body in air,
l = weight of the same in water, } s its spec. gravity ;
C = weight of the compound in air,
c = weight of the same in water, } f its spec. gravity ;
w = the secific gravity of water. Then,

1st, $(\text{H} - h)\,\text{s} = \text{H}w$,
2d, $(\text{L} - l)\,s = \text{L}w$,
3d, $(\text{C} - c)f = \text{C}w$,
4th, $\text{H} + \text{L} = \text{C}$,
5th, $h + l = c$,
6th, $\frac{\text{H}}{\text{s}} + \frac{\text{L}}{s} = \frac{\text{C}}{f}$.

From which equations may be found any of the above quantities, in terms of the rest.

Thus, from one of the first three equations, is found the specific gravity of any body, as $s = \frac{\text{L}w}{\text{L} - l}$, by dividing the absolute weight of the body by its loss in water, and multiplying by the specific gravity of water.

But if the body L be lighter than water ; then l will be negative, and we must divide by $\text{L} + l$ instead of $\text{L} - l$, and to find l we must have recourse to the compound mass C ; and because, from the 4th and 5th equations, $\text{L} - l = \overline{\text{C} - c} - \overline{\text{H} - h}$, therefore $s = \frac{\text{L}w}{(\text{C} - c) - (\text{H} - h)}$; that is, divide the absolute weight of the light body, by the difference between the losses in water, of the compound and heavier body, and multiply by the specific gravity of water. Or thus, $s = \frac{\text{s}f\text{L}}{\text{C}\text{s} - \text{H}f}$, as found from the last equation.

Also if it were required to find the quantities of two ingredients mixed in a compound, the 4th and 6th equations would give their values as follows, viz.

$$\text{H} = \frac{(f - s)\,\text{s}}{(\text{s} - s)\,f}\,\text{C}, \text{ and } \text{L} = \frac{(\text{s} - f)\,s}{(\text{s} - s)\,f}\,\text{C},$$

the quantities of the two ingredients H and L, in the compound C. And so for any other demand.

PROP. To find the specific gravity of a body.

251. CASE I.—*When the body is heavier than water:* weigh it both in water and out of water, and take the difference, which will be the weight lost in water. Then, by corol. 6, art. 250, $s = \frac{\text{B}w}{\text{B} - b}$, where B is the weight of the body out of water, b its weight in water, s its specific gravity, and w the specific gravity of water. That is,

As the weight lost in water,
Is to the whole or absolute weight,
So is the specific gravity of water,
To the specific gravity of the body.*

EXAMPLE. If a piece of stone weigh 10lb, but in water only $6\frac{3}{4}$lb, required its specific gravity, that of water being 1000? Ans. 3077.

252. CASE II.—*When the body is lighter than water, so that it will not sink:* annex to it a piece of another body, heavier than water, so that the mass compounded of the two may sink together. Weigh the denser body and the compound mass, separately, both in water, and out of it; then find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater. Then say, by proportion,

As the last remainder,
Is to the weight of the light body in air,
So is the specific gravity of water,
To the specific gravity of the body.

That is, the specific gravity is $s = \frac{\text{L}w}{(\text{C} - c) - (\text{H} - h)}$, by cor. 6, art. 250.

EXAMPLE. Suppose a piece of elm weighs 15lb. in air; and that a piece of copper, which weighs 18lb. in air and 16lb. in water, is affixed to it, and that the compound weighs 6lb. in water; required the specific gravity of the elm? Ans. 600.

253. CASE III.—*For a fluid of any sort.*—Take a piece of a body of known specific gravity; weigh it both in and out of the fluid, finding the loss of weight by taking the difference of the two; then say,

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* In the Lectures on Natural Philosophy, in the Royal Mil. Academy, Coates's Hydrostatic *steelyard* is employed for this purpose. It is an improvement upon the one described in Gregory's Mathematics for Practical Men.

As the whole or absolute weight,
Is to the loss of weight,
So is the specific gravity of the solid,
To the specific gravity of the fluid.

That is, the spec. grav. $w = \frac{\text{B} - b}{\text{B}} s$, by cor. 6, art. 250.

EXAMPLE. A piece of cast iron weighed 34·61 ounces in a fluid, and 40 ounces out of it; of what specific gravity is that fluid? Ans. 1000.

254. PROP. To find the quantities of two ingredients in a given compound.

Take the three differences of every pair of the three specific gravities, namely, the specific gravities of the compound and each ingredient; and multiply each specific gravity by the difference of the other two. Then say, by proportion,

As the greatest product,
Is to the whole weight of the compound,
So is each of the other two products,
To the weights of the two ingredients.

That is, $\text{H} = \frac{(f - s)\text{S}}{(\text{S} - s)f}\text{C}$ = the one, and $\text{L} = \frac{(s - f)\text{S}}{(\text{S} - s)f}\text{C}$, the other, by cor. 6, art. 250.

EXAMPLE. A composition of 112lb. being made of tin and copper, whose specific gravity is found to be 8784; required the quantity of each ingredient, the specific gravity of tin being 7320, and that of copper 9000?

Answer, there is 100lb. of copper and consequently 12lb. of tin. } in the composition.

SCHOLIUM.

255. The specific gravities of several sorts of matter, as found from experiments, are expressed by the numbers annexed to their names in the following Tables.

TABLES OF SPECIFIC GRAVITIES.

SOLIDS.

Platina - -	20,722	Rhodium - :	11,000
Gold, pure, hammered	19,362	Virgin Silver -	10,744
Guinea of George III.	17,629	Shilling of George III.	10,534
Tungsten - -	17,600	Bismuth, molten -	9,822
Mercury, at 32° Fahr.	13,598	Copper, wiredrawn	8,878
Lead - -	11,352	Red Copper, molten	8,788
Palladium - -	11,300	Molybdena - -	8,611

Arsenic - -	8,308	Jasper - -	2,710
Nickel, molten -	8,279	Coral - - -	2,680
Uranium - -	8,100	Rock Crystal -	2,653
Steel - from 7,767 to	7,816	English Pebble -	2,619
Cobalt, molten -	7,812	Limpid Feldspar -	2,564
Bar Iron - -	7,788	Glass, green -	2,642
Pure Cornish Tin -	7,291	—— white - -	2,892
Do. hardened -	7,299	—— bottle - -	2,733
Cast Iron - -	7,207	Porcelaine, China -	2,385
Zinc - - -	6,862	———— Limoges -	2,341
Antimony - -	6,712	Native Sulphur -	2,033
Tellurium - -	6,115	Ivory - - -	1,917
Chromium - -	5,900	Alabaster - -	1,874
Spar, heavy - -	4,430	Alum - -	1,720
Jargon of Ceylon -	4,416	Copal, opaque -	1,140
Oriental Ruby -	4,283	Sodium - -	973
Sapphire, Oriental -	3,994	Oak, heart of, -	950
Do. Brazilian -	3,131	Gunpowder, about -	937
Oriental Topaz -	4,019	Ice - -	930
Oriental Beryl -	3,549	Potassium - -	866
Diamond from 3,501 to	3,531	Beech - -	852
English Flint-Glass	3,329	Ash - -	845
Tourmalin - -	3,155	Apple-Tree - -	793
Asbestus - -	2,996	Orange-Wood -	705
Marble, green, Campan.	2,742	Pear-Tree - -	661
——— Parian -	2,837	Linden-Tree -	604
——— Norwegian -	2,728	Cypress - -	598
——— green, Egyptian	2,668	Cedar - -	561
Emerald - -	2,775	Fir - -	550
Pearl - - -	2,752	Poplar - -	383
Chalk, British -	2,784	Cork - -	240

LIQUIDS.

Sulphuric Acid -	1,841	Olive Oil - -	915
Nitrous Acid -	1,550	Muriatic Ether -	874
Water from the Dead Sea	1,240	Oil of Turpentine -	870
Nitric Acid - -	1,218	Liquid Bitumen -	848
Sea-Water - -	1,026	Alcohol, absolute -	792
Milk - - -	1,030	Sulphuric Ether -	716
Distilled Water -	1,000	Air at the Earth's Surface, about - -	$1\frac{1}{4}$
Wine of Bourdeaux	994		
Burgundy Wine -	991		

*** Since a cubic foot of water at the temperature 40° Fahrenheit, weighs 1000 ounces avoirdupois, or $62\frac{1}{2}$ pounds, the numbers in the preceding Tables exhibit very nearly the

respective weights of a cubic foot of the several substances tabulated.

256. PROP. To find the magnitude of any body, from its weight.

As the tabular specific gravity of the body,
Is to its weight in avoirdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, respectively.

EXAM. 1. Required the content of an irregular block of green marble, which weighs 1 cwt. or 112lb?

Ans. 1160·6 cubic inches.

EXAM. 2. How many cubic inches of gunpowder are there in 1lb. weight? Ans. $29\frac{1}{2}$ cubic inches nearly.

EXAM. 3. How many cubic feet are there in a ton weight of dry oak? Spec. grav. 925. Ans. $38\frac{138}{185}$ cubic feet.

257. PROP. To find the weight of a body from its magnitude.

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is the tabular specific gravity,
To the weight of the body.

EXAM. 1. Required the weight of a block of marble, whose length is 63 feet, and breadth and thickness each 12 feet; being the dimensions of one of the stones in the walls of Balbeck?

Ans. $683\frac{4}{10}$ ton, which is nearly equal to the burden of an East-India ship.

EXAM. 2. What is the weight of 1 pint, ale measure, of gunpowder? Ans. 19 oz. nearly.

EXAM. 3. What is the weight of a block of dry oak, which measures 10 feet in length, 3 feet broad, and $2\frac{1}{2}$ feet deep or thick? Ans. $4335\frac{15}{16}$lb.

BUOYANCY OF PONTOONS.

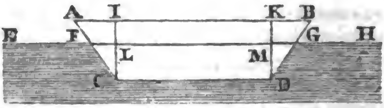
GENERAL SCHOLIUM.

258. The principles established in art. 250 have an interesting application to military men, in the use of pontoons, and the buoyancy by which they become serviceable in the

construction of temporary bridges. When the dimensions, magnitude, and weight of a pontoon are known, that weight can readily be deducted from the weight of an equal bulk of water, and the remainder is evidently the weight which the pontoon will carry before it will sink.

Pontoons as usually constructed, are prisms whose vertical sections are equal trapezoids, as exhibited in the marginal figure.

Suppose $AB = L$
$CD = l$
$AI = KB = \frac{1}{2}(L - l) = \delta$
$CI = D$



Uniform width of the pontoon $= b$: all in feet and parts. Suppose also $CL = d$, depth of the part immersed; $w =$ weight in avoirdupois pounds of the water displaced; and $c = 62\frac{1}{2}$ lbs. weight of a cubic foot of rain water. Then, by the following expressions, which are left for the student to investigate, d may be found when w and the rest are given, and w may be found when d and the rest are given; also the maximum value of w.

1. $w = bcd\left(l + \frac{d\delta}{D}\right)$

2. w when a max. $= bcD\,(l + \delta) = \frac{1}{2}\,bcD\,(L + l)$

3. $d = \sqrt{\left[\frac{D}{\delta}\left(\frac{w}{bc} + \frac{l^2 D}{4\delta}\right)\right]} - \frac{lD}{2\delta}.$

Ex. 1. Given $AB = 21\frac{1}{2}$ feet, $CD = 17\frac{1}{6}$ feet, $CI = 2\frac{1}{4}$ feet, $b = 4\frac{3}{4}$ feet. Required the weight of the pontoon and its load, when it is immersed to the depth CL, of $1\frac{1}{2}$ feet.

Ans. $8287\frac{3}{4}$lbs. nearly.

Ex. 2. Suppose the weight of such a pontoon to be 900lbs. what is the greatest weight it will carry? Ans. $12014\frac{1}{18}$lbs.

Ex. 3. Suppose the weight of the above pontoon and its load to be 6000lbs, how deep will it sink in water?

Ans. $1{\cdot}08872\,f = 13{\cdot}064$ inches.

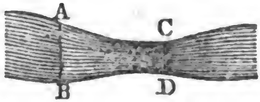
HYDRAULICS OR HYDRODYNAMICS.

259. Hydraulics or Hydrodynamics is that part of mechanical science which relates to the motion of fluids, and the forces with which they act upon bodies against which they strike, or which move in them.

This is a very extensive subject: but we shall here give only a few elementary propositions.

260. PROP. If a fluid run through a canal or river, or pipe of various widths, always filling it; the velocity of the fluid in different parts of it, AB, CD, will be reciprocally as the transverse sections in those parts.

That is, veloc. at A : veloc. at C : : CD : AB; where AB and CD denote, not the diameters at A and B, but the areas or sections there.



For, as the channel is always equally full, the quantity of water running through AB is equal to the quantity running through CD, in the same time; that is, the column through AB is equal to the column through CD, in the same time; or AB $\times$ length of its column $=$ CD $\times$ length of its column; therefore AB : CD : : length of column through CD : length of column through AB. But the uniform velocity of the water, is as the space run over, or length of the columns; therefore AB : CD : : velocity through CD : velocity through AB.

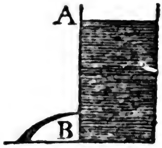
261. *Corol.* Hence, by observing the velocity at any place AB, the quantity of water discharged in a second, or any other time, will be found, namely, by multiplying the section AB by the velocity there.

But if the channel be not a close pipe or tunnel, kept always full, but an open canal or river; then the velocity in all parts of the section will not be the same, because the velocity towards the bottom and sides will be diminished by the friction against the bed or channel; and therefore a medium among the three ought to be taken. So, if the velocity at the top be - 100 feet per minute,
that at the bottom - 60
and that at the sides - 50

3)210 sum;

dividing their sum by 3, gives 70 for the mean velocity, which is to be multiplied by the section, to give the quantity discharged in a minute: and in many cases still greater accuracy will be necessary in determining the mean.

262. PROP. The velocity with which a fluid runs out by a hole in the bottom or side of a vessel, is equal to that which is generated by gravity through the height of the water above the hole; that is, the velocity of a heavy body acquired by falling freely through the height AB.



DIVIDE the altitude AB into a great number of very small parts, each being 1, their number a, or $a =$ the altitude AB.

Now, by art. 246, the pressure of the fluid against the hole B, by which the motion is generated, is equal to the weight of the column of fluid above it, that is the column whose height is AB or a, and base the area of the hole B. Therefore the pressure on the hole, or small part of the fluid 1, is to its weight, or the natural force of gravity, as a to 1. But, by art. 127, the velocities generated in the same body in any time, are as those forces; and because gravity generates the velocity 2 in descending through the small space 1, therefore $1 : a :: 2 : 2a$, the velocity generated by the pressure of the column of fluid in the same time. But $2a$ is also, by corol. 1, art. 132, the velocity generated by gravity in descending through a or AB. That is, the velocity of the issuing water, is equal to that which is acquired by a body in falling through the height AB.

The same otherwise.

The momenta, or quantities of motion, generated in two given bodies, by the same force, acting during the same or an equal time, are equal. And the force in this case, is the weight of the superincumbent column of the fluid over the hole. Let then the one body to be moved, be that column itself, expressed by ah, where a denotes the altitude AB, and h the area of the hole; and the other body is the column of the fluid that runs out uniformly in one second suppose, with the middle or medium velocity of that interval of time, which is $\frac{1}{2}hv$, if v be the whole velocity required. Then the mass $\frac{1}{2}hv$, with the velocity v, gives the quantity of motion $\frac{1}{2}hv \times v$, or $\frac{1}{2}hv^2$, generated in one second, in the spouting water: also g, or $32\frac{1}{6}$ feet, is the velocity generated in the mass ah, during the same interval of one second; consequently $ah \times g$, or ahg, is the motion generated in the column ah in the same time of one second. But as these two momenta must be equal, this gives $\frac{1}{2}hv^2 = ahg$: hence then $v^2 = 2ag$, and $v = \sqrt{2ag}$, for the value of the velocity sought; which therefore is exactly the same as the velocity generated by the gravity in falling through the space a, or the whole height of the fluid*.

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* In this investigation the author uses the whole momentum $ah \times 2g$, which is generated in one second by the gravity of the mass ah; but he

For example, if the fluid were air, of the whole height of the atmosphere, supposed uniform, which is about $5\frac{1}{4}$ miles, or 27720 feet $= a$. Then $\sqrt{2ag} = 2\sqrt{(27720 \times 16\frac{1}{12})} =$ 1335 feet $= v$ the velocity, that is, the velocity with which common air would rush into a vacuum.

263. *Corol.* 1. The velocity, and quantity run out, at different depths, are as the square roots of the depths. For the velocity acquired in falling through AB, is as $\sqrt{\text{AB}}$.

264. *Corol.* 2. The fluid spouts out with the same velocity, whether it be downward or upward, or sideways; because the pressure of fluids is the same in all directions, at the same depth. And therefore, if an adjutage be turned upward, the jet will ascend to the height of the surface of the water in the vessel. And this is confirmed by experience, by which it is found that jets really ascend nearly to the height of the reservoir, abating a small quantity only, for the friction against the sides, and some resistance from the air and from the oblique motion of the fluid in the hole.

265. *Corol.* 3. The quantity run out in any time, is equal to a column or prism, whose base is the area of the hole, and its length the space described in that time by the velocity acquired by falling through the altitude of the fluid. And the quantity is the same, whatever be the figure of the orifice, if it is of the same area.

Therefore, if a denote the altitude of the fluid,
and h the area of the orifice,
also $\frac{1}{2}g = 16\frac{1}{12}$ feet, or 193 inches;
then $2h\sqrt{\frac{1}{2}ag}$ will be the quantity of water discharged in a second of time; or nearly $8\frac{1}{48}\, h\sqrt{a}$ cubic feet, when a and h are taken in feet.

So, for example, if the height a be 25 inches, and the orifice $h = 1$ square inch; then $2h\sqrt{\frac{1}{2}ag} = 2\sqrt{25 \times 193} =$ 139 cubic inches, which is the quantity that would be discharged per second.

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does not use the whole momentum $hv \times v$, which is also generated by the same force in the same time, instead of which he uses only half the the latter momentum; on this account his solution appears to be more erroneous. The two momenta $ah \times 2g$ and $hv \times v$, produced in one second by the same force, ought to be equal, which gives $v^2 = 2ag$, instead of the equation $v^2 = 4ag$ as found by Dr. Hutton. *Ed.*

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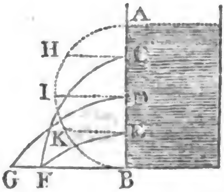
266. When the orifice is in the side of the vessel, then the velocity is different in the different parts of the hole, being less in the upper parts of it than in the lower. However, when the hole is but small, the difference is inconsiderable, and the altitude may be estimated from the centre to obtain the mean velocity. But when the orifice is pretty large, then the mean velocity is to be more accurately computed by other principles, given in the next proposition.

267. It is not to be expected that experiments, as to the quantity of water run out, will exactly agree with this theory, both on account of the resistance of the air, the resistance of the water against the sides of the orifice, and the oblique motion of the particles of the water in entering it. For, it is not merely the particles situated immediately in the column over the hole, which enter it and issue forth, as if that column only were in motion; but also particles from all the surrounding parts of the fluid, which is in a commotion quite around; and the particles thus entering the hole in all directions, strike against each other, and impede one another's motion: from which it happens, that it is the particles in the centre of the hole only that issue out with the whole velocity due to the entire height of the fluid, while the other particles towards the sides of the orifices pass out with decreased velocities; and hence the medium velocity through the orifice, is somewhat less than that of a single body only, urged with the same pressure of the superincumbent column of the fluid. And experiments on the quantity of water discharged through aperatures, show that the quantity must be diminished, by those causes, rather more than the fourth part, when the orifice is small, or such as to make the mean velocity nearly equal to that in a body falling through $\frac{1}{2}$ the height of the fluid above the orifice. If the velocity be taken as that due to the whole altitude above the orifice, then instead of the area of the orifice, the area of the contracted vein at a small distance from it must be taken. See Gregory's Mechanics and Bossut's Hydrodynamique.

268. Experiments have also been made on the extent to which the spout of water ranges on a horizontal plane, and compared with the theory, by calculating it as a projectile discharged with the velocity acquired by descending through the height of the fluid. For, when the aperture is in the side of the vessel, the fluid spouts out horizontally with a uniform velocity, which, combined with the perpendicular

velocity from the action of gravity, causes the jet to form the curve of a parabola. Then the distances to which the jet will spout on the horizontal plane BG, will be as the roots of the rectangles of the segments AC . CB, AD . DB, AE . EB. For the spaces BF, BG, are as the times and horizontal velocities; but the velocity is as $\sqrt{AC}$; and the time of the fall, which is the same as the time of moving, is as $\sqrt{CB}$; therefore the distance BF is as $\sqrt{AC . CB}$; and the distance BG as $\sqrt{AD . DB}$. And hence, if two holes are made equidistant from the top and bottom, they will project the water to the same distance; for if AC = EB, then the rectangle AC . CB is equal the rectangle AE . EB: which makes BF the same for both. Or, if on the diameter AB a semicircle be described; then, because the squares of the ordinates CH, DI, EK are equal to the rectangles AC . EB, &c.; therefore the distances BF, BG are as the ordinates CH, DI. And hence also it follows, that the projection from the middle point D will be farthest, for DI is the greatest ordinate.



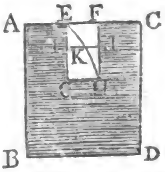
These are the *proportions* of the distances; but for the absolute distances, it will be thus. The velocity through any hole C, is such as will carry the water horizontally through a space equal to 2AC in the time of falling through AC: but, after quitting the hole, it describes a parabola, and comes to F in the time a body will fall through CB; and to find this distance, since the times are as the roots of the spaces, therefore $\sqrt{AC} : \sqrt{CB} :: 2AC : 2\sqrt{AC . CB} = 2CH = BF$, the space ranged on the horizontal plane. And the greatest range $BG = 2DI$, or $2AD$, or equal to AB.

And as these ranges answer very nearly to the experiments, this confirms the theory, as to the velocity assigned.

269. PROP. If a notch or slit EH in form of a parallelogram, be cut in the side of a vessel, full of water, AD; the quantity of water flowing through it, will be $\frac{2}{3}$ of the quantity flowing through an equal orifice, placed at the whole depth EG, or at the base GH, in the same time; it being supposed that the vessel is always kept full.

For the velocity at GH is to the velocity at IL, as $\sqrt{EG}$ to $\sqrt{EI}$; that is, as GH or IL to IK, the ordinate of a parabola EKH, whose axis is EG. Therefore the sum of the velocities at all the points I, is to as many times the velociy at G,

as the sum of all the ordinates IK, to the sum of all the IL's; namely, as the area of the parabola EGH, is to the area EGHF; that is, the quantity running through the notch EH, is to the quantity running through an equal horizontal area placed at GH, as EGHKE, to EGHF, or as 2 to 3; the area of a parabola being $\frac{2}{3}$ of its circumscribing parallelogram.



Corol. 1. The mean velocity of the water in the notch, is equal to $\frac{2}{3}$ of that at GH.

Corol. 2. The quantity flowing through the hole IGHL, is to that which would flow through an equal orifice placed as low as GH, as the parabolic frustum IGHK, is to the rectangle IGHL. This appears from the demonstration.

OF PNEUMATICS.

270. PNEUMATICS is the science which treats of the properties of air, or elastic fluids.

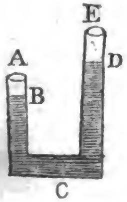
271. PROP. Air is a fluid body; which surrounds the earth, and gravitates on all parts of its surface.

These properties of air are proved by experience.—That it is a fluid, is evident from its easily yielding to any the least force impressed on it, without making a sensible resistance.

But when it is moved briskly, by any means, as by a fan or a pair of bellows; or when any body is moved very briskly through it; in these cases we become sensible of it as a body, by the resistance it makes in such motions, and also by its impelling or blowing away any light substances. So that, being capable of resisting or moving other bodies, by its impulse, it must itself be a body, and be heavy, like all other bodies, in proportion to the matter it contains; and therefore it will press on all bodies that are placed under it.

Also, as it is a fluid, it spreads itself all over on the earth; and, like other fluids, it gravitates and presses every where on the earth's surface.

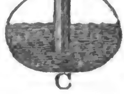
272. The gravity and pressure of the air are also evident from many experiments. Thus, for instance, if water, or quicksilver, be poured into the tube ACE, and the air be suffered to press on it, in both ends of the tube, the fluid will rest at the same height in both legs: but if the air be drawn out of one end as E, by any means; then the air pressing on the other end A, will press down the fluid in this leg at B, and raise it up in the other to D, as much higher than at B, as the pressure of the air is equal to. From which it appears, not only that the air does really press, but also how much the intensity of that pressure is equal to. And this is the principle of the barometer.



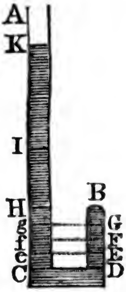
273. PROP. The air is also an elastic fluid, being condensible and expansible: and the law it observes is this, that its density and elasticity are proportional to the force or weight which compresses it.

This property of the air is proved by many experiments. Thus, if the handle of a syringe be pushed inward, it will condense the inclosed air into less space, thereby showing its condensibility. But the included air, thus condensed, is felt to act strongly against the hand, resisting the force compressing it more and more; and, on withdrawing the hand, the handle is pushed back again to where it was at first. Which shows that the air is elastic.

274. Again, fill a strong bottle half full of water; then insert a small glass tube into it, putting its lower end down near to the bottom, and cementing it very close round the mouth of the bottle. Then, if air be strongly injected through the pipe, as by blowing with the mouth or otherwise, it will pass through the water from the lower end, ascending into the parts before occupied with air at B, and the whole mass of air become there condensed, because the water is not compressible into a less space. But, on removing the force which injected the air at A, the water will begin to rise from thence in a jet, being pushed up the pipe by the increased elasticity of the air B, by which it presses on the surface of the water, and forces it through the pipe, till as much be expelled as there was air forced in; when the air at B will be reduced to the same density as at first, and, the balance being restored, the jet will cease.



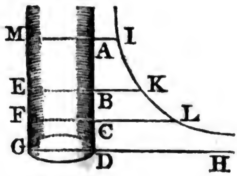
275. Likewise, if into a jar of water AB, be inverted an empty glass tumbler CD, or such-like, the mouth downward; the water will enter it, and partly fill it, but not near so high as the water in the jar, compressing and condensing the air into a less space in the upper parts C, and causing the glass to make a sensible resistance to the hand in pushing it down. Then, on removing the hand, the elasticity of the internal condensed air throws the glass up again. All these showing that the air is condensible and elastic.



276. Again, to show the relation of the elasticity to the condensation: take a long crooked glass tube, equally wide throughout, or at least in the part BD, and open at A, but close at the other end B. Pour in a little quicksilver at A, just to cover the bottom to the bend at CD, and to stop the communication between the external air and the air in BD. Then pour in more quicksilver, and mark the corresponding heights at which it stands in the two legs: so, when it rises to H in the open leg AC, let it rise to E in the close one, reducing its included air from the natural bulk BD to the contracted space BE, by the pressure of the column H*e*; and when the quicksilver stands at I and K, in the open leg, let it rise to F and G in the other, reducing the air to the respective spaces BF, BG, by the weights of the columns I*f*, K*g*. Then it is always found, within moderate limits, that the condensations and elasticities are as the compressing weights and columns of the quicksilver, and the atmosphere together. So, if the natural bulk of the air BD be compressed into the spaces BE, BF, BG, which are $\frac{3}{4}$, $\frac{2}{4}$, $\frac{1}{4}$ of BD, or as the numbers 3, 2, 1; then the atmosphere, together with the corresponding columns H*e*, I*f*, K*g*, are also found to be in the same proportion reciprocally, viz. as $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{1}$, or as the numbers 2, 3, 6. And then $He = \frac{1}{2}A$, $If, = A$, and $Kg = 3A$; where A is the weight of the atmosphere. Which show that the condensations are directly as the compressing forces. And the elasticities are in the same ratio, since the columns in AC are sustained by the elasticities in BD.

From the foregoing principles may be deduced many useful remarks, as in the following corollaries, viz.

277. *Corol.* 1. The space in which any quantity of air is confined, is reciprocally as the force that compresses it. So,



the forces which confine a quantity of air in the cylindrical spaces AG, BG, CG, are reciprocally as the same, or reciprocally as the heights AD, BD, CD. And therefore if to the two perpendicular lines DA, DH, as asymptotes, the hyperbola IKL be described, and the ordinates AI, BK, CL be drawn; then the forces which confine the air in the spaces AG, BG, CG, will be directly as the corresponding ordinates AI, BK, CL, since these are reciprocally as the abscisses AD, BD, CD, by the nature of the hyperbola.

Corol. 2. All the air near the earth is in a state of compression, by the weight of the incumbent atmosphere.

Corol. 3. The air is denser near the earth, than in high places; or denser at the foot of a mountain, than at the top of it. And the higher above the earth the less dense it is.

Corol. 4. The spring or elasticity of the air, is equal to the weight of the atmosphere above it; and they will produce the same effects; since they always sustain and balance each other.

Corol. 5. If the density of the air be increased, preserving the same heat or temperature, its spring or elasticity is also increased, and in the same proportion.

Corol. 6. By the pressure and gravity of the atmosphere, on the surface of fluids, the fluids are made to rise in any pipes or vessels, when the spring or pressure within is decreased or taken off.

278. PROP. Heat increases the elasticity of the air, and cold diminishes it. Or, heat expands, and cold condenses the air.

This property is also proved by experience.

Thus, tie a bladder very close with some air in it; and lay it before the fire: then as it warms it will more and more distend the bladder, and at last burst it, if the heat be continued, and increased high enough. But if the bladder be removed from the fire, as it cools it will contract again, as before. And it was on this principle that the first air-balloons were made by Montgolfier: for, by heating the air within them, by a fire beneath, the hot air distends them to a size which occupies a space in the atmosphere, whose weight of common air exceeds that of the balloon.

Also, if a cup or glass, with a little air in it, be inverted

into a vessel of water; and the whole be heated over the fire, or otherwise; the air in the top will expand till it fill the glass, and expel the water out of it; and part of the air itself will follow, by continuing or increasing the heat.

Many other experiments, to the same effect, might be adduced, all proving the properties mentioned in the proposition.

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279. So that, when the force of the elasticity of air is considered, regard must be had to its heat or temperature; the same quantity of air being more or less elastic, as its heat is more or less. And it has been found, by experiment, that the elasticity is increased by the 435th part, for each degree of heat, of which there are 180, between the freezing and boiling heat of water, in Fahrenheit's thermometer.

N. B. Water expands about the $\frac{2}{56000}$ part, with each degree of heat. (Sir Geo. Shuckburgh, Philos. Trans. 1777, p. 560, &c.)

Also, the

Spec. grav. of	air	1·201 or $1\frac{1}{5}$	when the barom. is 29·5,
	water	1000	and the therm. is 55°
	mercury	13592	which are their mean heights in this country.
Or thus,	air	1·222 or $1\frac{2}{9}$	when the barom. is 30,
	water	1000	and thermometer 55.
	mercury	13600	

280. Prop. The weight or pressure of the atmosphere, on any base at the earth's surface, is equal to the weight of a column of quicksilver, of the same base, and the height of which is between 28 and 31 inches.

This is proved by the barometer, an instrument which measures the pressure of the air, and which is described below (art. 302). For, at some seasons, and in some places, the air sustains and balances a column of mercury, of about 28 inches: but at other times it balances a column of 29, or 30, or near 31 inches high; seldom in the extremes 28 or 31, but commonly about the means 29 or 30. This variation depends partly on the different degrees of heat in the air near the surface of the earth, and partly on the commotions and changes in the atmosphere, from winds and other causes, by which it is accumulated in some places, and depressed in others, being thereby rendered denser and heavier, or rarer and lighter; which changes in its state are almost

continually happening in any one place. But the medium state is commonly about $29\frac{1}{2}$ or 30 inches.

281. *Corol.* 1. Hence the pressure of the atmosphere on every square inch at the earth's surface, at a medium, is very near 15 pounds avoirdupois, or rather $14\frac{3}{4}$ pounds. For, a cubic foot of mercury, weighing 13600 ounces nearly, an inch of it will weigh 7·866 or almost 8 ounces, or nearly half a pound, which is the weight of the atmosphere for every inch of the barometer on a base of a square inch; and therefore 30 inches, or the medium height, weighs very near $14\frac{3}{4}$ pounds.

282. *Corol.* 2. Hence also the weight or pressure of the atmosphere, is equal to that of a column of water from 32 to 35 feet high, or on a medium 33 or 34 feet high. For, water and quicksilver are in weight nearly as 1 to 13·6; so that the atmosphere will balance a column of water 13·6 times as high as one of quicksilver; consequently

13·6 times 28 inches = 381 inches, or $31\frac{3}{4}$ feet,
13·6 times 29 inches = 394 inches, or $32\frac{5}{6}$ feet,
13·6 times 30 inches = 408 inches, or 34 feet,
13·6 times 31 inches = 422 inches, or $35\frac{1}{6}$ feet.

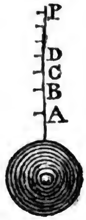
And hence a common sucking pump (art. 292) will not raise water higher than about 33 or 34 feet. And a siphon will not run, if the perpendicular height of the top of it be more than about 33 or 34 feet (art. 291).

283. *Corol.* 3. If the air were of the same uniform density at every height up to the top of the atmosphere, as at the surface of the earth; its height would be about $5\frac{1}{4}$ miles at a medium. For, the weights of the same bulk of air and water, are nearly as 1·222 to 1000; therefore as 1·222 : 1000 : : $33\frac{3}{4}$ feet : 27600 feet, or $5\frac{1}{4}$ miles nearly. And so high the atmosphere would be, if it were *homogeneous*, or all of uniform density, like water. But, instead of that, from its expansive and elastic quality, it becomes continually more and more rare, the farther above the earth, in a certain proportion, which will be treated of below, as also the method of measuring heights by the barometer, which depends on it.

284. *Corol.* 4. From this proposition and the last it follows, that the height is always the same, of a *homogeneous atmosphere* above any place, which shall be all of the uniform density with the air there, and of equal weight or pressure with the real height of the atmosphere above that place, whether it be at the same place, at different times, or at any different places or heights above the earth; and

that height is always about $5\frac{1}{4}$ miles, or 27600 feet, as above found. For, as the density varies in exact proportion to the weight of the column, therefore it requires a column of the same height in all cases, to make the respective weights or pressures. Thus, if W and w be the weights of atmosphere above any places, D and d their densities, and H and h the heights of the uniform columns, of the same densities and weights; then $H \times D = W$, and $h \times d = w$; therefore $\frac{W}{D}$ or H is equal to $\frac{w}{d}$ or h: the temperature being the same.

285. PROP. With regard to the atmosphere, at different heights above the earth, this law obtains that when the heights increase in arithmetical progression, the densities decrease in geometrical progression.



Let the indefinite perpendicular line AP, erected on the earth, be conceived to be divided into a great number of very small equal parts, A, B, C, D, &c. forming so many thin strata of air in the atmosphere, all of different density, gradually decreasing from the greatest at A: then the density of the several strata A, B, C, D, &c. will be in geometrical progression decreasing.

For, as the strata A, B, C, &c. are all of equal thickness, the quantity of matter in each of them, is as the density there; but the density in any one, being as the compressing force, is as the weight or quantity of all the matter from that place upward to the top of the atmosphere; therefore the quantity of matter in each stratum, is also as the whole quantity from that place upward. Now, if from the whole weight at any place as B, the weight or quantity in the stratum B be subtracted, the remainder is the weight at the next stratum C; that is, from each weight subtracting a part which is proportional to itself, leaves the next weight; or, which is the same thing, from each density subtracting a part which is proportional to itself, leaves the next density. But when any quantities are continually diminished by parts which are proportional to themselves, the remainders form a series of continued proportionals: consequently these densities are in geometrical progression.

Thus, if the first density be D, and from each be taken its nth part; there will then remain its $\frac{n-1}{n}$ part, or the $\frac{m}{n}$ part, putting m for $n - 1$; and therefore the series of den-

sities will be D, $\frac{m}{n}$ D, $\frac{m^2}{n^2}$ D, $\frac{m^3}{n^3}$ D, $\frac{m^4}{n^4}$ D, &c. the common ratio of the series being that of n to m.

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286. Because the terms of an arithmetical series, are proportional to the logarithms of the terms of a geometrical series: therefore different altitudes above the earth's surface, are as the logarithms of the densities, or of the weights of air, at those altitudes.

So that, if D denote the density at the altitude A,
and d - the density at the altitude a;
then A being as the log. of D, and a as the log. of d,

the dif. of alt. A — a will be as the log. D — log. d, or log. $\frac{D}{d}$.

And if A = 0, or D the density at the surface of the earth; then any altitude above the surface a, is as the log. of $\frac{D}{d}$.

Or, in general, the log. of $\frac{D}{d}$ is as the altitude of the one place above the other, whether the lower place be at the surface of the earth, or any where else.

And from this property is derived the method of determining the heights of mountains and other eminences, by the barometer, which (art. 302) is an instrument that measures the pressure or density of the air at any place. For, by taking, with this instrument, the pressure or density, at the foot of a hill for instance, and again at the top of it, the difference of the logarithms of these two pressures, or the logarithm of their quotient, will be as the difference of altitude, or as the height of the hill; supposing the temperatures of the air to be the same at both places, and the gravity of air not altered by the different distances from the earth's centre.

287. But as this formula expresses only the relations between different altitudes with respect to their densities, recourse must be had to some experiment, to obtain the real altitude which corresponds to any given density, or the density which corresponds to a given altitude. And there are various experiments by which this may be done. The first, and most natural, is that which results from the known specific gravity of air, with respect to the whole pressure of the atmosphere on the surface of the earth. Now, as the altitude a is always as log. $\frac{D}{d}$; assume h so that $a = h \times$ log. $\frac{D}{d}$,

where h will be of one constant value for all altitudes; and to determine that value, let a case be taken in which we know the altitude a corresponding to a known density d; as for instance, take $a = 1$ foot, or 1 inch, or some such small altitude; then, because the density D may be measured by the pressure of the atmosphere, or the uniform column of 27600 feet, when the temperature is 55°; therefore 27600 feet will denote the density D at the lower place, and 27599 the less density d at 1 foot above it; consequently $1 = h \times \log. \frac{27600}{27599}$;

which, by the nature of logarithms, is nearly $= h \times \frac{\cdot 43429448}{27600}$

$= \frac{h}{63551}$ nearly; and hence $h = 63551$ feet; which gives, for any altitude in general, this theorem, viz. $a = 63551 \times \log. \frac{D}{d}$, or $= 63551 \times \log. \frac{M}{m}$ feet, or $10592 \times \log. \frac{M}{m}$ fathoms: where M is the column of mercury which is equal to the pressure or weight of the atmosphere at the bottom, and m that at the top of the altitude a; and where M and m may be taken in any measure, either feet or inches, &c.

288. Note, that this formula is adapted to the mean temperature of the air 55°. But, for every degree of temperature different from this, in the medium between the temperatures at the top and bottom of the altitude a, that altitude will vary by its 435th part; which must be added, when that medium exceeds 55°, otherwise subtracted.

Note, also, that a column of 30 inches of mercury varies its length by about the $\frac{1}{320}$ part of an inch for every degree of heat, or rather $\frac{1}{9600}$ of the whole volume.

289. But the formula may be rendered much more convenient for use, by reducing the factor 10592 to 10000, by changing the temperature proportionally from 55°; thus, as the diff. 592 is the 18th part of the whole factor 10592; and as 18 is the 24th part of 435; therefore the corresponding change of temperature is 24°, which reduces the 55° to 31°. So that the formula is, $a = 10000 \times \log. \frac{M}{m}$ fathoms, when the temperature is 31 degrees; and for every degree above that, the result is to be increased by so many times its 435th part.

290. Taking, instead of the logarithms, the first term of the logarithmic series, we have $55000 \cdot \frac{B-b}{B+b}$, for the altitude

in feet : B and b, being the heights of the barometrical columns observed at the bottom and top of the hill. This formula is for the mean temperature 55°, and is easily remembered because the effective figures of the co-efficient are also 55. The reductions for any other temperature are the same as in the logarithmic rule.

EXAM. 1. To find the height of a hill when the pressure of the atmosphere is equal to 29·68 inches of mercury at the bottom, and 25·28 at the top; the mean temperature being 50°? Ans. 4378 feet, or 730 fathoms.

EXAM. 2. To find the height of a hill when the atmosphere weighs 29·45 inches of mercury at the bottom, and 26·82 at the top, the mean temperature being 33°?
Ans. 2385 feet, or 397½ fathoms.

EXAM. 3. At what altitude is the density of the atmosphere only the 4th part of what it is at the earth's surface?
Ans. 6020 fathoms.

By the weight and pressure of the atmosphere, the effect and operations of pneumatic engines may be accounted for, and explained; such as siphons, pumps, barometers, &c.; of which it will be proper here to give a brief description.

OF THE SIPHON.

291. A Siphon, or Syphon, is any bent tube, having its two legs either of equal or of unequal length.

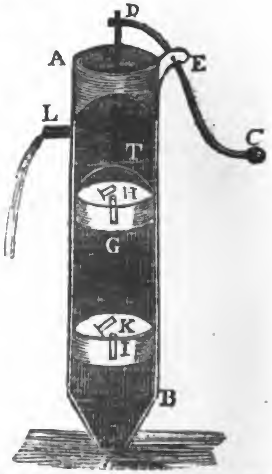
If it be filled with water, and then inverted, with the two open ends downward, and held level in that position; the water will remain suspended in it, if the two legs be equal. For the atmosphere will press equally on the surface of the water in each end, and support them, if they are not more than 34 feet high; and the legs being equal, the water in them is an exact counterpoise by their equal weights; so that the one has no power to move more than the other; and they are both supported by the atmosphere.

But if now the siphon be a little inclined to one side, so that the orifice of one end be lower than that of the other; or if the legs be of unequal length, which is the same thing;

then the equilibrium is destroyed, and the water will all descend out by the lower end, and rise up in the higher. For, the air pressing equally, but the two ends weighing unequally, a motion must commence where the power is greatest, and so continue till all the water has run out by the lower end. And if the shorter leg be immersed into a vessel of water, and the siphon be set running as above, it will continue to run till all the water be exhausted from the vessel, or at least as low as that end of the siphon. Or, it may be set running without filling the siphon as above, by only inverting it, with its shorter leg into the vessel of water; then, with the mouth applied to the lower orifice A, suck out the air; and the water will presently follow, being forced up into the siphon by the pressure of the air on the water in the vessel.

If a siphon be fixed in a vessel of water capable of rotation upon a vertical axis, and the orifice be lateral instead of at the bottom of the pipe, the reaction may be advantageously employed as a motive force. This is the principle of Mr. Busby's *Hydraulic Orrery*.

OF THE PUMP.



292. There are three sorts of pumps; the Sucking, the Lifting, and the Forcing Pump. By the first, water can be raised only to about 33 feet, viz. by the pressure of the atmosphere; but by the others, to any height; but then they require more apparatus and power.

The annexed figure represents a common sucking pump. AB is the barrel of the pump, being a hollow cylinder, made of metal, and smooth within, or of wood for very common purposes. CD is the handle, moveable about the pin E, by moving the end C up and down. DF an iron rod turning about a pin D, which connects it to the

end of the handle. This rod is fixed to the piston, bucket, or sucker, FG, by which this is moved up and down within the barrel, which it must fit very tight and close, that no air or water may pass between the piston and the sides of the barrel; and for this purpose it is commonly armed with leather. The piston is made hollow, or it has a perforation through it, the orifice of which is covered by a valve H opening upwards. I is a plug firmly fixed in the lower part of the barrel, also perforated, and covered by a valve K opening upwards.

293. When the pump is first to be worked, and the water is below the plug I; raise the end C of the handle, then the piston descending, compresses the air in HI, which by its spring shuts fast the valve K, and pushes up the valve H, and so enters into the barrel above the piston. Then putting the end C of the handle down again, raises the piston or sucker, which lifts up with it the column of air above it, the external atmosphere by its pressure keeping the valve H shut: the air in the barrel being thus exhausted, or rarefied, is no longer a counterpoise to that which presses on the surface of the water in the well; this is forced up the pipe, and through the valve K, into the barrel of the pump. Then pushing the piston down again into this water, now in the barrel, its weight shuts the lower valve K, and its resistance forces up the valve of the piston, and enters the upper part of the barrel, above the piston. Then, the bucket being raised, lifts up with it the water which had passed above its valve, and it runs out by the cock L; and taking off the weight below it, the pressure of the external atmosphere on the water in the well again forces it up through the pipe and lower valve close to the piston, all the way as it ascends, thus keeping the barrel always full of water. And thus, by repeating the strokes of the piston, a continued discharge is made at the cock L.

294. There is a farther limitation of the operation, than that which relates to the 33 feet. If the elastic force of the air within the tube joined to the weight of water in the tube equal the pressure of the atmosphere, the water cannot rise in the pump. To prevent this, the product of the stroke of the piston into 33 must always exceed the square of half the greatest altitude of the piston above the surface of the water in the well. Otherwise diminish the diameter of the sucking-pipe proportionally.

OF THE AIR-PUMP.

295. Nearly on the same principles as the water-pump, is the invention of the air-pump, by which the air is drawn out of any vessel, like as water is drawn out by the former. A brass barrel is bored and polished truly cylindrical, and exactly fitted with a turned piston, so that no air can pass by the sides of it, and furnished with a proper valve opening upward. Then, by lifting up the piston, the air in the close vessel below it follows the piston, and fills the barrel; and being thus diffused through a larger space than before, when it occupied the vessel or receiver only, but not the barrel, it is made rarer than it was before, in proportion as the capacity of the barrel and receiver together exceeds the receiver alone. Another stroke of the piston exhausts another barrel of this now rarer air, which again rarefies it in the same proportion as before. And so on, for any number of strokes of the piston, still exhausting in the same geometrical progression, of which the ratio is that which the capacity of the receiver and barrel together exceeds the receiver, till this is exhausted to any proposed degree, or as far as the nature of the machine is capable of performing; which happens when the elasticity of the included air is so far diminished, by rarefying, that it is too feeble to push up the valve of the piston, and escape.

296. From the nature of this exhausting, in geometrical progression, we may easily find how much the air in the receiver is rarefied by any number of strokes of the piston: or what number of such strokes is necessary, to exhaust the receiver to any given degree. Thus, if the capacity of the receiver and barrel together, be to that of the receiver alone, as c to r, and 1 denote the natural density of the air at first; then.

$c : r :: 1 : \frac{r}{c}$, the density after 1 stroke of the piston,

$c : r :: \frac{r}{c} : \frac{r^2}{c^2}$, the density after two strokes,

$c : r :: \frac{r^2}{c^2} : \frac{r^3}{c^3}$, the density after three strokes,

&c.; and $\frac{r^n}{c^n}$, the density after n strokes.

So, if the barrel be equal to $\frac{1}{4}$ of the receiver; then $c : r ::$ $5 : 4$; and $\frac{4^n}{5^n} = 0 \cdot 8^n$ is $= d$ the density after n turns. And

if n be 20, then $0{\cdot}8^{20} = {\cdot}0115$ is the density of the included air after 20 strokes of the piston; which being the $86\frac{7}{10}$ part of 1, or the first density, it follows that the air is $86\frac{7}{10}$ times rarefied by the 20 strokes.

297. Or, if it were required to find the number of strokes necessary to rarefy the air any number of times; because $\frac{r^n}{c^n}$ is = the proposed density d; therefore, taking the logarithms, $n \times \log. \frac{r}{c} = \log. d$, and $n = \frac{\log. d}{\mathrm{l}.\, r - \mathrm{l}.\, c}$, the number of strokes required. So if r be $\frac{4}{5}$ of c, and it be required to rarefy the air 100 times: then $d = \frac{1}{100}$ or $\cdot 01$; and hence $n = \frac{\log. 100}{\mathrm{l}.\, 5 - \mathrm{l}.\, 4.} = 20\frac{2}{3}$ nearly. So that in $20\frac{2}{3}$ strokes the air will be rarefied 100 times.

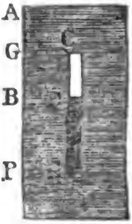
OF THE DIVING BELL AND CONDENSING MACHINE.

298. On the same principles too depend the operations and effect of the Condensing Engine, by which air may be condensed to any degree, instead of rarefied as in the air-pump. And, like as the air-pump rarefies the air, by extracting always one barrel of air after another; so, by this other machine, the air is condensed, by throwing in or adding always one barrel of air after another; which it is evident may be done by only turning the valves of the piston and barrel, that is, making them to open the contrary way, and working the piston in the same manner; so that, as they both open upward or outward in the air-pump, or rarefier, they will both open downward or inward in the condenser.

299. And on the same principles, namely, of the compression and elasticity of the air, depends the use of the Diving Bell, which is a large vessel, in which a person descends to the bottom of the sea, the open end of the vessel being downward; only in this case the air is not condensed by forcing more of it into the same space, as in the condensing engine; but by compressing the same quantity of air into a less space in the bell, by increasing always the force which compresses it.

300. If a vessel of any sort be inverted into water, and pushed or let down to any depth; then by the pressure of the water some of it will ascend into the vessel, but not so high as the water without, and will compress the air into less space, according to the difference between the heights of the internal and external water; and the density and elastic force of the air will be increased in the same proportion as its space in the vessel is diminished.

So, if the tube CE be inverted, and pushed down into water, till the external water exceed the internal, by the height AB, and the air of the tube be reduced to the space CD; then that air is pressed both by a column of water of the height AB, and by the whole atmosphere which presses on the upper surface of the water; consequently the space CD is to the whole space CE, as the weight of the atmosphere, is to the weights both of the atmosphere and the column of water AB. So that, if AB be about 34 feet, which is equal to the force of the atmosphere, then CD will be equal to $\frac{1}{2}$CE; but if AB be double of that, or 68 feet, then CD will be $\frac{1}{3}$CE; and so on. And hence, by knowing the depth AF, to which the vessel is sunk, we can easily find the point D, to which the water will rise within it at any time. For let the weight of the atmosphere at that time be equal to that of 34 feet of water; also, let the depth AF be 20 feet, and the length of the tube CE 4 feet; then, putting the height of the internal water DE $= x$,



it is $34 + AB : 34 :: CE : CD$,
that is $34 + AF - DE : 34 :: CE : CE - DE$,
or $54 - x : 34 :: 4 : 4 - x$;

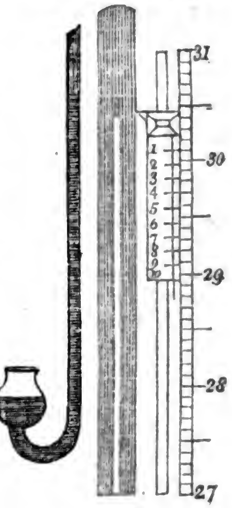
hence, multiplying extremes and means, $216 - 58x + x^2 = 136$, and the root is $x = \sqrt{2}$ very nearly $= 1{\cdot}414$ of a foot, or 17 inches nearly; being the height DE to which the water will rise within the tube.

301. But if the vessel be not equally wide throughout, but of any other shape, as of a bell-like form, such as is used in diving; then the altitudes will not observe the proportion above, but the spaces or bulks only will accord with that proportion, namely, $34 + AB : 34 ::$ capacity CKL : capacity CHI, if it be common or fresh-water; and $33 + AB : 33 ::$ capacity CKL :

capacity CHI, if it be sea-water. From which proportion, the height DE may be found, when the nature and shape of the vessel or bell CKL are known.

OF THE BAROMETER.

302. THE Barometer is an instrument for measuring the pressure of the atmosphere, and elasticity of the air, at any time. It is commonly made of a glass tube, of near 3 feet long, close at one end, and filled with mercury. When the tube is full, by stopping the open end with the finger, then inverting the tube, and immersing that end with the finger into a bason of quicksilver, on removing the finger from the orifice, the fluid in the tube will descend into the bason, till what remains in the tube be of the same weight with a column of the atmosphere, which is commonly between 28 and 31 inches of quicksilver; and leaving an entire vacuum in the upper end of the tube above the mercury. For, as the upper end of the tube is quite void of air, there is no pressure downwards but from the column of quicksilver, and therefore that will be an exact balance to the counter pressure of the whole column of atmosphere, acting on the orifice of the tube by the quicksilver in the bason. The upper 3 inches of the tube, namely, from 28 to 31 inches, have a scale attached to them, divided into inches, tenths, and hundredths, for measuring the length of the column at all times, by observing which division of the scale the top of the quicksilver is opposite to; as it ascends and descends within these limits, according to the state of the atmosphere.



The weight of the quicksilver in the tube, above that in the bason, is at all times equal to the weight or pressure of the column of atmosphere above it, and of the same base with the tube; and hence the weight of it may at all times be computed; being nearly at the rate of half a pound avoirdupois for every inch of quicksilver in the tube, on every square inch of base; or more exactly it is $\frac{59}{120}$ of

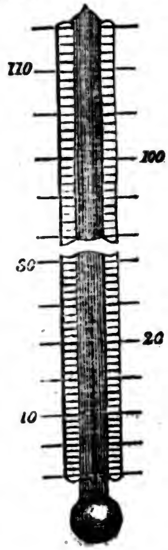
a pound on the square inch, for every inch in the altitude of the quicksilver weighs just $\frac{59}{120}$ lb, or nearly $\frac{1}{2}$ a pound, in the mean temperature of 55° of heat. And consequently, when the barometer stands at 30 inches, or $2\frac{1}{2}$ feet high, which is nearly the medium or standard height, the whole pressure of the atmosphere is equal to $14\frac{3}{4}$ pounds, on every square inch of the base: and so in proportion for other heights.

Barometers are now constructed so as to be susceptible of convenient motion from place to place without derangement; thus facilitating the pneumatic method of determining the heights of hills, &c.

OF THE THERMOMETER.

303. The Thermometer is an instrument for measuring the temperature of the air, as to heat and cold.

It is found by experience, that all bodies expand by heat, and contract by cold: and since the expansion is, to a certain extent, uniform, the degrees of expansion become the measure of the degrees of heat. Fluids are more convenient for this purpose than solids: and quicksilver or mercury is now most commonly used for it. A very fine glass tube, having a pretty large hollow ball at the bottom, is filled about half way up with quicksilver: the whole being then heated very hot till the quicksilver rise quite to the top, the top is then hermetically sealed, so as perfectly to exclude all communication with the outward air. Then, in cooling, the quicksilver contracts, and consequently its surface descends in the tube, till it come to a certain point, correspondent to the temperature or heat of the air. And when the weather becomes warmer, the quicksilver expands, and its surface rises in the tube; and again contracts and descends when the weather becomes cooler. So that, by placing a scale of any divisions against the side of the tube, it will show the degrees of heat by the expansion and contraction of the quicksilver in the tube; observing at what division of the scale the top of



the quicksilver stands. The method of preparing the scale, as used in England, is thus :—Bring the thermometer into the temperature of freezing, by immersing the ball in water just freezing, or in ice just thawing, the latter is best, and mark the scale where the mercury then stands, for the point of freezing. Next, immerge it in boiling water; and the quicksilver will rise to a certain height in the tube; which mark also on the scale, for the boiling point, or the heat of boiling water. Then the distance between these two points, is divided into 180 equal divisions, or degrees; and the like equal degrees are also continued to any extent below the freezing point, and above the boiling point. The divisions are then numbered as follows, namely, at the freezing point is set the number 32, and consequently 212 at the boiling point; and all the other numbers in their order.

This division of the scale is commonly called *Fahrenheit's*. According to this division, 55 is at the mean temperature of the air in this country; and it is in this temperature, and in an atmosphere which sustains a column of 30 inches of quicksilver in the barometer, that all measures and specific gravities are taken, unless when otherwise mentioned; and in this temperature and pressure, the relative weights, or specific gravities of air, water, and quicksilver, are as

$$\left.\begin{array}{r} 1\frac{2}{9} \text{ for air,} \\ 1000 \text{ for water,} \\ 13600 \text{ for mercury;} \end{array}\right\}$$ these also are the weights of a cubic foot of each, in avoirdupois ounces, in that state of the barometer and thermometer. For other states of the thermometer, each of these bodies expands or contracts according to the following rate, with each degree of heat, viz.

Air about - $\frac{1}{435}$ part of its bulk,
Water about $\frac{1}{5000}$ part of its bulk,
Mercury about $\frac{1}{9600}$ part of its bulk.

Another division is that of 100 equal degrees between the freezing and the boiling points, the 0 or *zero* being at the former. This is called the *centigrade* thermometer. It is now very common to put Fahrenheit's division on the left of the tube, and the centigrade division on the right.

ON THE MEASUREMENT OF ALTITUDES BY THE BAROMETER AND THERMOMETER.

304. From the principles laid down in arts. 286 to 289, concerning the measuring of altitudes by the barometer, and the foregoing descriptions of the barometer and thermometer,

we may now collect together the precepts for the practice of such measurements, which are as follow :

First. Observe the height of the barometer at the bottom of any height, or depth, intended to be measured ; with the temperature of the quicksilver, by means of a thermometer attached to the barometer, and also the temperature of the air in the shade by a detatched thermometer.

Secondly. Let the same thing be done also at the top of the said height or depth, and at the same time, or as near the same time as may be. And let those altitudes of barometer be reduced to the same temperature, if it be thought necessary, by correcting either the one or the other, that is, augment the height of the mercury in the colder temperature, or diminish that in the warmer, by its $\frac{1}{10000}$ part for every degree of difference of the two.

Thirdly. Take the difference of the common logarithms of the two heights of the barometer, corrected as above if necessary, cutting off 3 figures next the right hand for decimals, when the log-tables go to 7 figures, or cut off only 2 figures when the tables go to 6 places, and so on ; or in general remove the decimal point 4 places more towards the right hand, those on the left hand being fathoms in whole numbers.

Fourthly. Correct the number last found for the difference of temperature of the air, as follows : take half the sum of the two temperatures, for the mean one ; and for every degree which this differs from the temperature 31°, take so many times the $\frac{1}{435}$ part of the fathoms above found, and add them if the mean temperature be above 31°, but subtract them if the mean temperature be below 31° ; and the sum or difference will be the true altitude in fathoms : or, being multiplied by 6, it will be the altitude in feet.

EXAM. 1. Let the state of the barometers and thermometers be as follows ; to find the altitude, viz.

Barom.	Thermom. attach.	Thermom. detach.	
Lower 29·68	57	57	Ans. the alt. is 720 fathoms.
Upper 25·28	43	42	

EXAM. 2. To find the altitude, when the state of the barometers and thermometers is as follows, viz.

Barom.	Thermom. attach.	Thermom. detach.	
Lower 29·45	38	31	Ans. the alt. is $409\frac{8}{13}$ fathoms, or 2458 feet.
Upper 26·82	41	35	

This is a highly useful method within certain limits; but is by no means susceptible of that degree of accuracy which many have imputed to it.

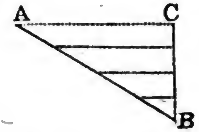
ON THE RESISTANCE OF FLUIDS, WITH THEIR FORCES AND ACTIONS ON BODIES.

305. Prop. If any body move through a fluid at rest, or the fluid move against the body at rest; the force or resistance of the fluid against the body, will be as the square of the velocity and the density of the fluid. That is, $R \propto dv^2$.

For, the force or resistance is as the quantity of matter or particles struck, and the velocity with which they are struck. But the quantity or number of particles struck in any time, are as the velocity and the density of the fluid. Therefore the resistance, or force of the fluid, is as the density and square of the velocity.

306. *Corol.* 1. The resistance to any plane, is also more or less, as the plane is greater or less; and therefore the resistance on any plane, is as the area of the plane a, the density of the medium, and the square of the velocity. That is, $R \propto adv^2$.

307. *Corol.* 2. If the motion be not perpendicular, but oblique to the plane, or to the face of the body; then the resistance, in the direction of motion, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination of the plane to the direction of the motion, or as the cube of radius to the cube of the sine of that angle. So that $R \propto adv^2s^3$, putting $1 =$ radius, and $s =$ sine of the angle of inclination CAB.



For, if AB be the plane, AC the direction of motion, and BC perpendicular to AC; then no more particles meet the plane than what meet the perpendicular, BC, and therefore their number is diminished as AB to BC, or as 1 to s. But the force to each particle, striking the plane obliquely in the direction CA, is also diminished as AB to BC, or as 1 to s: therefore the resistance, which is perpendicular to the face of the plane is as 1^2 to s^2. But again, this resistance in the direction perpendicular to the face of the plane, is to that in the direction AC, by the

parallelogram of forces, as AB to BC, or as 1 to s. Consequently, on all these accounts, the resistance to the plane when moving perpendicular to its face, is to that when moving obliquely, as 1^3 to s^3, or 1 to s^3. That is, the resistance in the direction of the motion, is diminished as 1 to s^3, or in the triplicate ratio of radius to the sine of inclination.

308. PROP. The real resistance to a plane, from a fluid acting in a direction perpendicular to its face, is equal to the weight of a column of the fluid, whose base is the plane, and altitude equal to that which is due to the velocity of the motion, or through which a heavy body must fall to acquire that velocity.

The resistance to the plane moving through a fluid, is the same as the force of the fluid in motion with the same velocity, on the plane at rest. But the force of the fluid in motion, is equal to the weight or pressure which generates that motion; and this is equal to the weight or pressure of a column of the fluid, whose base is the area of the plane, and its altitude that which is due to the velocity.

309. *Corol.* 1. If a denote the area of the plane, v the velocity, n the density or specific gravity of the fluid, and $\frac{1}{2}g = 16\frac{1}{12}$ feet, or 193 inches. Then, the altitude due to the velocity v being $\frac{v^2}{2g}$, therefore $a \times n \times \frac{v^2}{2g} = \frac{anv^2}{2g}$ will be the whole resistance, or motive force R.

310. *Corol.* 2. If the direction of motion be not perpendicular to the face of the plane, but oblique to it, in any angle, whose sine is s. Then the resistance to the plane will be $\frac{anv^2s^3}{2g}$.

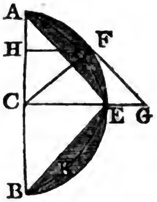
311. *Corol.* 3. Also, if w denote the weight of the body, whose plane face a is resisted by the absolute force R; then the retarding force f, or $\frac{\text{R}}{w}$ will be $\frac{anv^2s^3}{2gw}$.

312. *Corol.* 4. And if the body be a cylinder, whose face or end is a, and radius r, moving in the direction of its axis; because then $s = 1$, and $a = \pi r^2$, where $\pi = 3{\cdot}141593$; then $\frac{\pi nv^2r^2}{2g}$ will be the resisting force R, and $\frac{\pi nv^2r^2}{2gw}$ the retarding force f.

313. *Corol.* 5. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face an elliptic

section, or a conical surface, or any other figure every where equally inclined to the axis, or direction of motion, the sine of inclination being s; then, the number of particles of the fluid striking the face being still the same, but the force of each, opposed to the direction of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force R would be $\frac{\pi n r^2 v^2 s^2}{2g}$.

314. PROP. The resistance to a sphere moving through a fluid, is but half the resistance to its great circle, or to the end of a cylinder of the same diameter, moving with an equal velocity.



Let AFEB be half the sphere, moving in the direction CEG. Describe the paraboloid AIEKB on the same base. Let any particle of the medium meet the semicircle in F, to which draw the tangent FG, the radius FC, and the ordinate FIH. Then the force of any particle on the surface at F, is to its force on the base at H, as the square of the sine of the angle G, or its equal the angle FCH, to the square of radius, that is, as HF^2 to CF^2. Therefore the force of all the particles, or the whole fluid, on the whole surface, is to its force on the circle of the base, as all the HF^2 to as many times CF^2. But CF^2 is $= CA^2 = AC \cdot CB$, and $HF^2 = AH \cdot HB$ by the nature of the circle: also, $AH \cdot HB : AC \cdot CB :: HI : CE$ by th. 2, parabola; consequently the force on the spherical surface is to the force on its circular base, as all the HI's to as many CE's, that is, as the content of the paraboloid to the content of its circumscribed cylinder, namely, as 1 to 2 (th. 18, parab.)

315. *Corol.* Hence, the resistance to the sphere is $R = \frac{\pi n v^2 r^2}{4g}$, being the half of that of a cylinder of the same diameter. For example, a 9lb. iron ball, whose diameter is 4 inches, when moving through the air with a velocity of 1600 feet per second, would meet a resistance which is equal to a weight of $132\frac{2}{3}$lb. over and above the pressure of the atmosphere, for want of the counterpoise behind the ball.

OF THE WEIGHT AND DIMENSIONS OF BALLS AND SHELLS.

THE weight and dimensions of Balls and Shells might be found from the problems given under the head of specific gravity. But they may be found still easier by means of the experimental weight of a ball of a given size, from the known proportion of similar figures, namely, as the cubes of their diameters, or like linear dimensions.

PROBLEM I.

To find the Weight of an Iron Ball, from its Diameter.

An iron ball of 4 inches diameter weighs 9lb. and the weights being as the cubes of the diameters, it will be, as 64 (which is the cube of 4) is to 9 its weight, so is the cube of the diameter of any other ball, to its weight. Or, take $\frac{9}{64}$ of the cube of the diameter, for the weight. Or, take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that again, add the two together, for the weight.

EXAM. 1. The diameter of an iron shot being 6·7 inches, required its weight? Ans. 42·294lb.

EXAM. 2. What is the weight of an iron ball, whose diameter is 5·54 inches? Ans. 24lb. nearly.

PROBLEM II.

To find the Weight of a Leaden Ball.

A leaden ball of 1 inch diameter weighs $\frac{3}{14}$ of a lb, ; therefore as the cube of 1 is to $\frac{3}{14}$, or as 14 is to 3, so is the cube of the diameter of a leaden ball, to its weight. Or, take $\frac{3}{14}$ of the cube of the diameter, for the weight, nearly.

EXAM. 1. Required the weight of a leaden ball of 6·6 inches diameter? Ans. 61·606lb.

EXAM. 2. What is the weight of a leaden ball of 5·30 inches diameter? Ans. 32lb. nearly.

EXAM. 3. How many shot, each $\frac{1}{40}$ of an inch diameter, may be made out of 10lb. of lead? Ans. 2986667.

PROBLEM II.

To find the Diameter of an Iron Ball.

Multiply the weight by $7\frac{1}{9}$, and the cube root of the product will be the diameter.

EXAM. 1. Required the diameter of a 42lb. iron ball?
Ans. 6·685 inches.

EXAM. 2. What is the diameter of a 24lb. iron ball?
Ans. 5·54 inches.

PROBLEM IV.

To find the Diameter of a Leaden Ball.

Multiply the weight by 14, and divide the product by 3; then the cube root of the quotient will be the diameter.

EXAM. 1. Required the diameter of a 64lb. leaden ball?
Ans. 6·684 inches.

EXAM. 2. What is the diameter of an 8lb. leaden ball?
Ans. 3·343 inches.

PROBLEM V.

To find the Weight of an Iron Shell.

Take $\frac{9}{64}$ of the difference of the cubes of the external and internal diameter, for the weight of the shell.

That is, from the cube of the external diameter, take the cube of the internal diameter, multiply the remainder by 9, and divide the product by 64.

EXAM. 1. The outside diameter of an iron shell being 12·8, and the inside diameter 9·1 inches; required its weight?
Ans. 188·941lb.

EXAM. 2. What is the weight of an iron shell, whose external and internal diameters are 9·8 and 7 inches?
Ans. $84\frac{1}{8}$lb.

PROBLEM VI.

To find how much Powder will fill a Shell.

Divide the cube of the internal diameter, in inches, by 57·3, for the lbs. of powder*.

EXAM. 1. How much powder will fill a shell whose internal diameter is 9·1 inches? Ans. $13\frac{2}{13}$lb. nearly.

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* This and the following are only approximative rules, founded upon the supposition that, at a medium, 30 cubic inches of gunpowder weigh a pound. Of 18 different kinds of gunpowder used in the Royal Laboratory, Woolwich, the weights vary from 58lb. 1oz. to 49lb. 13oz. per cubic foot, and the specific gravities, consequently, from 929 to 727. The specific gravity of French gunpowder usually lies between narrower limits; viz. those of 944 and 897.

Exam. 2. How much powder will fill a shell whose internal diameter is 7 inches? Ans. 6lb.

PROBLEM VII.

To find how much Powder will fill a Rectangular Box.

Find the content of the box in inches, by multiplying the length, breadth, and depth altogether. Then divide by 30 for the pounds of powder.

Exam. 1. Required what quantity of powder will fill a box, the length being 15 inches, the breadth 12, and the depth 10 inches? Ans. 60lb.

Exam. 2. How much powder will fill a cubical box whose side is 12 inches? Ans. $57\frac{3}{5}$lb.

PROBLEM VIII.

To find how much Powder will fill a Cylinder.

Multiply the square of the diameter by the length, then divide by 38·2 for the pounds of powder.

Exam. 1. How much powder will the cylinder hold, whose diameter is 10 inches, and length 20 inches?
Ans. $52\frac{1}{3}$lb. nearly.

Exam. 2. How much powder can be contained in the cylinder whose diameter is 4 inches, and length 12 inches?
Ans. $5\frac{5}{191}$lb.

PROBLEM IX.

To find the Size of a Shell to contain a given Weight of Powder.

Multiply the pounds of powder by 57·3, and the cube root of the product will be the diameter in inches.

Exam. 1. What is the diameter of a shell that will hold $13\frac{1}{6}$lb. of powder? Ans. 9·1 inches.

Exam. 2. What is the diameter of a shell to contain 6lb. of powder? Ans. 7 inches.

PROBLEM X.

To find the Size of a Cubical Box, to contain a given Weight of Powder.

Multiply the weight in pounds by 30, and the cube root of the product will be the side of the box in inches.

EXAM. 1. Required the side of a cubical box, to hold 50lb. of gunpowder? Ans. 11·44 inches.

EXAM. 2. Required the side of a cubical box, to hold 400lb. of gunpowder? Ans. 22·89 inches.

PROBLEM XI.

To find what Length of a Cylinder will be filled by a given Weight of Gunpowder.

Multiply the weight in pounds by 38·2, and divide the product by the square of the diameter in inches, for the length.

EXAM. 1. What length of a 36-pounder gun, of $6\frac{2}{3}$ inches diameter, will be filled with 12lb. of gunpowder?

Ans. 10·314 inches.

EXAM. 2. What length of a cylinder, of 8 inches diameter, may be filled with 20lb. of powder? Ans. $11\frac{15}{16}$ inches.

OF THE PILING OF BALLS AND SHELLS.

IRON Balls and Shells are commonly piled by horizontal courses, either in a pyramidical or in a wedge-like form; the base being either an equilateral triangle, or a square, or a rectangle. In the triangle and square, the pile finishes in a single ball; but in the rectangle, it finishes in a single row of balls, like an edge.

In triangular and square piles, the number of horizontal rows, or courses, is always equal to the number of balls in one side of the bottom row. And in rectangular piles, the number of rows is equal to the number of balls in the breadth of the bottom row. Also, the number in the top row, or edge, is one more than the difference between the length and breadth of the bottom row. A rule or two on this subject have been given in the first volume: the substance of them is repeated here, with a few additional rules.

PROBLEM I.

To find the Number of Balls in a Triangular Pile.

Multiply continually together the number of balls in one side of the bottom row, and that number increased by 1, also the same number increased by 2; then $\frac{1}{6}$ of the last product will be the answer.

That is, $\frac{1}{6}n \cdot (n+1) \cdot (n+2)$ is the number or sum, where n is the number in the bottom row.

Exam. 1. Required the number of balls in a triangular pile, each side of the base containing 30 balls? Ans. 4960.

Exam. 2. How many balls are in the triangular pile, each side of the base containing 20? Ans. 1540.

PROBLEM II.

To find the Number of Balls in a Square Pile.

Multiply continually together the number in one side of the bottom course, that number increased by 1, and double the same number increased by 1; then $\frac{1}{6}$ of the last product will be the answer.

That is, $\frac{1}{6}n \cdot (n+1) \cdot (2n+1)$ is the number.

Exam. 1. How many balls are in a square pile of 30 rows? Ans. 9455.

Exam. 2. How many balls are in a square pile of 20 rows? Ans. 2870.

PROBLEM III.

To find the Number of Balls in a Rectangular Pile.

From 3 times the number in the length of the base row, subtract one less than the breadth of the same, multiply the remainder by the same breadth, and the product by one more than the same; and divide by 6 for the answer.

That is, $\frac{1}{6}b \cdot (b+1) \cdot (3l - b + 1)$ is the number; where l is the length, and b the breadth of the lowest course.

Note.—In all the piles the breadth of the bottom is equal to the number of courses. And in the oblong or rectangular pile, the top row is one more than the difference between the length and breadth of the bottom.

Exam. 1. Required the number of balls in a rectangular pile, the length and breadth of the base row being 46 and 15? Ans. 4960.

Exam. 2. How many shot are in a rectangular complete pile, the length of the bottom course being 59, and its breadth 20? Ans. 11060.

PROBLEM IV.

To find the Number of Balls in an Incomplete Pile.

From the number in the whole pile, considered as com-

plete, subtract the number in the upper pile which is wanting at the top, both computed by the rule for their proper form; and the remainder will be the number in the frustum, or incomplete pile.

Exam. 1. To find the number of shot in the incomplete triangular pile, one side of the bottom course being 40, and the top course 20? Ans. 10150.

Exam. 2. How many shot are in the incomplete triangular pile, the side of the base being 24, and of the top 8? Ans. 2516.

Exam. 3. How many balls are in the incomplete square pile, the side of the base being 24, and of the top 8? Ans. 4760.

Exam. 4. How many shot are in the incomplete rectangular pile, of 12 courses, the length and breadth of the base being 40 and 20? Ans. 6146.

OF DISTANCES BY THE VELOCITY OF SOUND.

From various experiments recently made, with great care, by the present editor of this volume, it has been found that sound flies through the air uniformly at the rate of about 1110 feet per second, when the air is quiescent, and at a medium temperature. At the temperature of freezing, or a little below, the velocity is 1100 feet; at the temperature of 75°, on Fahrenheit's thermometer, the velocity is about 1120. The approximate velocity under different temperatures may be found, by adding to 1100, *half a foot*, for every degree, on Fahrenheit's thermometer, above the freezing point. The mean velocity may be taken at 370 yards per second; or a mile in $4\frac{3}{4}$ seconds.

Hence, multiplying any time employed by sound in moving, by 370, will give the corresponding space in yards. Or, dividing any space in yards by 370, will give the time which sound will occupy in passing uniformly over that space.

If the wind blow briskly, as at the rate of from 20 to 60 feet per second, in the direction in which the sound moves, the velocity of the sound will be proportionably augmented: if the direction of the wind is opposed to that of the sound, the difference of their velocities must be employed.

Note.—The time for the passage of sound in the interval between seeing the flash of a gun, or lightning, and hearing

the report, may be observed by a watch, or a small pendulum. Or, it may be observed by the beats of the pulse in the wrist, counting, on an average, about 70 to a minute for persons in moderate health, or $5\frac{5}{8}$ pulsations to a mile; and more or less according to circumstances.

Exam. 1. After observing a flash of lightning, it was 12 seconds before the thunder was heard; required the distance of the cloud from whence it came? Ans. 2·52 miles.

Exam. 2. How long, after firing the Tower guns, may the report be heard at Shooter's-Hill, supposing the distance to be 8 miles in a straight line? Ans. $38\frac{3}{5}$ seconds.

Exam. 3. After observing the firing of a large cannon at a distance, it was 7 seconds before the report was heard; what was its distance? Ans. 1·47 mile.

Exam. 4. Perceiving a man at a distance hewing down a tree with an axe, I remarked that 4 of my pulsations passed between seeing him strike and hearing the report of the blow; what was the distance between us, allowing 70 pulses to a minute?

Exam. 5. How far off was the cloud from which thunder issued, whose report was 5 pulsations after the flash of lightning; counting 75 to a minute?

Exam. 6. If I see the flash of a cannon, fired by a ship in distress at sea, and hear the report 33 seconds after, how far is she off?

PRACTICAL EXERCISES IN MECHANICS, STATICS, HYDROSTATICS, SOUND, MOTION, GRAVITY, PROJECTILES, AND OTHER BRANCHES OF NATURAL PHILOSOPHY.

Question 1. Required the weight of a cast iron ball of 3 inches diameter, supposing the weight of a cubic inch of the metal to be 0·258lb. avoirdupoise. Ans. 3·64739lb.

Quest. 2. To determine the weight of a hollow spherical iron shell, 5 inches in diameter, the thickness of the metal being one inch. Ans. 13·78lb.

Quest. 3. Being one day ordered to observe how far a battery of cannon was from me, I counted, by my watch, 17 seconds between the time of seeing the flash and hearing the report; what then was the distance? Ans. $3\frac{5}{8}$ miles.

QUEST. 4. It is proposed to determine the proportional quantities of matter in the earth and moon; the density of the former being to that of the latter, as 10 to 7, and their diameters as 7930 to 2160. Ans. as 71 to 1 nearly.

QUEST. 5. What difference is there, in point of weight, between a block of marble, containing 1 cubic foot and a half, and another of brass of the same dimensions?

Ans. 496lb. 14oz.

QUEST. 6. In the walls of Balbeck in Turkey, the ancient Heliopolis, there are three stones laid end to end, now in sight, that measure in length 61 yards; one of which in particular is 21 yards or 63 feet long, 12 feet thick, and 12 feet broad: now if this block be marble, what power would balance it, so as to prepare it for moving?

Ans. $683\frac{7}{16}$ tons, the burden of an East-India ship.

QUEST. 7. The battering-ram of Vespasian weighed, suppose 10,000 pounds; and was moved, let us admit, with such a velocity, by strength of hand, as to pass through 20 feet in one second of time; and this was found sufficient to demolish the walls of Jerusalem. The question is, with what velocity a 32lb. ball must move, to do the same execution?

Ans. 6250 feet.

QUEST. 8. There are two bodies, of which the one contains 25 times the matter of the other, or is 25 times heavier: but the less moves with 1000 times the velocity of the greater; in what proportion then are the momenta, or forces with which they move?

Ans. the less moves with a force 40 times greater.

QUEST. 9. A body, weighing 20lb. is impelled by such a force, as to send it through 100 feet in a second; with what velocity then would a body of 8lb. weight move, if it were impelled by the same force? Ans. 250 feet per second.

QUEST. 10. There are two bodies, the one of which weighs 100lb. the other 60; but the less body is impelled by a force 8 times greater than the other; the proportion of the velocities, with which these bodies move, is required?

Ans. the velocity of the greater to that of the less, as 3 to 40.

QUEST. 11. There are two bodies, the greater contains 8 times the quantity of matter in the less, and is moved with a force 48 times greater: the ratio of the velocities of these two bodies is required?

Ans. the greater is to the less, as 6 to 1.

QUEST. 12. There are two bodies, one of which moves 40 times swifter than the other; but the swifter body has

moved only one minute, whereas the other has been in motion 2 hours: the ratio of the spaces described by these two bodies is required?

Ans. the swifter is to the slower, as 1 to 3.

QUEST. 13. Supposing one body to move 30 times swifter than another, as also the swifter to move 12 minutes, the other only 1: what difference will there be between the spaces described by them, supposing the last has moved 5 feet?

Ans. 1795 feet.

QUEST. 14. There are two bodies, the one of which has passed over 50 miles, the other only 5; and the first had moved with 5 times the celerity of the second; what is the ratio of the times they have been in describing those spaces?

Ans. as 2 to 1.

QUEST. 15. What weight will a man be able to raise, who presses with the force of a hundred and a half, on the end of an equipoised handspike, 100 inches long, meeting with a convenient prop exactly $7\frac{1}{2}$ inches from the lower end of the machine? Ans. 2072lb.

QUEST. 16. A weight of $1\frac{1}{2}$ lb. laid on the shoulder of a man, is no greater burden to him than its absolute weight, or 24 ounces: what difference will he feel between the said weight applied near his elbow, at 12 inches from the shoulder, and in the palm of his hand, 28 inches from the same; and how much more must his muscles then draw, to support it at right angles, that is, having his arm stretched right out?

Ans. 24lb. avoirdupois.

QUEST. 17. What weight hung on at 70 inches from the centre of motion of a steel-yard, will balance a small gun of $9\frac{1}{2}$ cwt. freely suspended at 2 inches distance from the said centre on the contrary side? Ans. $30\frac{2}{5}$lb.

QUEST. 18. It is proposed to divide the beam of a steel-yard, or to find the points of division where the weights of 1, 2, 3, 4, &c. lb. on the one side, will just balance a constant weight of 95lb. at the distance of 2 inches on the other side of the fulcrum; the weight of the beam being 10lb. and its whole length 36 inches?

Ans. 30, 15, 10, $7\frac{1}{2}$, 6, 5, $4\frac{2}{7}$, $3\frac{3}{4}$, $3\frac{1}{3}$, 3, $2\frac{8}{11}$, $2\frac{1}{2}$, &c.

QUEST. 19. Two men carrying a burden of 200lb. weight between them, hung on a pole, the ends of which rest on their shoulders; how much of this load is borne by each man, the weight hanging 6 inches from the middle, and the whole length of the pole being 4 feet?

Ans. 125lb. and 75lb.

QUEST. 20. If, in a pair of scales, a body weigh 90lb. in one scale, and only 40lb. in the other; required its true weight, and the proportion of the lengths of the two arms of the balance beam, on each side of the point of suspension?

Ans. the weight 60lb. and the proportion 3 to 2.

QUEST. 21. To find the weight of a beam of timber, or other body, by means of a man's own weight, or any other weight. For instance, a piece of tapering timber, 24 feet long, being laid over a prop, or the edge of another beam, is found to balance itself when the prop is 13 feet from the less end; but removing the prop a foot nearer to the said end, it takes a man's weight of 210lb, standing on the less end, to hold it in equilibrium. Required the weight of the tree? Ans. 2520lb.

QUEST. 22. If AB be a cane or walking-stick, 40 inches long, suspended by a string SD fastened to the middle point D: now a body being hung on at E, 6 inches distance from D, is balanced by a weight of 2lb, hung on at the larger end A; but removing the body to F, one inch nearer to D, the 2lb, weight on the other side is moved to G, within 8 inches of D, before the cane will rest in equilibrio. Required the weight of the body? Ans. 24lb.

QUEST. 23. If AB, BC be two inclined planes, of the lengths of 30 and 40 inches, and moveable about the joint at B; what will be the ratio of two weights P, Q, in equilibrio on the planes, in all positions of them: and what will be the altitude BD of the angle B above the horizontal plane AC, when this is 50 inches long?

Ans. BD = 24; and P to Q as AB to BC, or as 3 to 4.

QUEST. 24. A lever, of 6 feet long, is fixed at right angles in a screw, whose threads are one inch asunder, so that the lever turns just once round in raising or depressing the screw one inch. If then this lever be urged by a weight or force of 50lb. with what force will the screw press?

Ans. $22619\frac{1}{2}$lb.

QUEST. 25. If a man can draw a weight of 150lb. up the side of a perpendicular wall, of 20 feet high; what weight will he be able to raise along a smooth plank of 30 feet long, laid aslope from the top of the wall? Ans. 225lb.

QUEST. 26. If a force of 150lb. be applied on the head of a rectangular wedge, its thickness being 2 inches, and the length of its side 12 inches; what weight will it raise or balance perpendicular to its side? Ans. 900lb.

QUEST. 27. If a round pillar of 30 feet diameter be raised

on a plane, inclined to the horizon in an angle of 75°, or the shaft inclining 15 degrees out of the perpendicular; what length will it bear before it overset?

Ans. $30(2 + \sqrt{3})$ or 111·9615 feet.

QUEST. 28. If the greatest angle at which a bank of natural earth will stand, be 45°; it is proposed to determine what thickness an upright wall of stone must be made throughout, just to support a bank of 12 feet high: the specific gravity of the stone being to that of earth, as 5 to 4.

Ans. $\frac{48}{5}\sqrt{\frac{1}{5}}$, or 4·29325 feet.

QUEST. 29. If the stone wall be made like a wedge, or having its upright section a triangle, tapering to a point at top, but its side next the bank of earth perpendicular to the horizon; what is its thickness at the bottom, so as to support the same bank? Ans. $12\sqrt{\frac{1}{5}}$, or 5·36656 feet.

QUEST. 30. But if the earth will only stand at an angle of 30 degrees to the horizontal line; it is required to determine the thickness of wall in both the preceding cases?

Ans. the breadth of the rectangle $12\sqrt{\frac{1}{5}}$, or 5·36656.
but the base of the triangular bank $12\sqrt{\frac{3}{10}}$, or 6·57267.

QUEST. 31. To find the thickness of an upright rectangular wall, necessary to support a body of water; the water being 10 feet deep, and the wall 12 feet high; also the specific gravity of the wall to that of the water, as 11 to 7.

Ans. 4·204374 feet.

QUEST. 32. To determine the thickness of the wall at the bottom, when the section of it is triangular, and the altitudes as before. Ans. 5·1492865 feet.

QUEST. 33. Supposing the distance of the earth from the sun to be 95 millions of miles; I would know at what distance from him another body must be placed, so as to receive light and heat quadruple to that of the earth.

Ans. at half the distance, or $47\frac{1}{2}$ millions.

QUEST. 34. The distance between the earth and the sun being accounted 95 millions of miles, and between Jupiter and the sun 495 millions; the degree of light and heat received by Jupiter, compared with that of the earth, is required?

Ans. $\frac{361}{9801}$, or nearly $\frac{1}{27}$ of the earth's light and heat.

QUEST. 35. A certain body on the surface of the earth weighs a cwt., or 112lb.; the question is, whither this body must be carried, that it may weigh only 10lb.?

Ans. either at 3·3466 semi-diameters, or $\frac{5}{56}$ of a semi-diameter, from the centre.

QUEST. 36. If a body weigh 1 pound, or 16 ounces, on the surface of the earth; what will its weight be at 50 miles above it, taking the earth's diameter at 7930 miles?

Ans. 15 oz. $9\frac{5}{8}$ dr. nearly.

QUEST. 37. Whereabouts, in the line between the earth and moon, is their common centre of gravity; supposing the earth's diameter to be 7930 miles, and the moon's 2160; also the density of the former to that of the latter, as 99 to 68, or as 10 to 7 nearly, and their mean distance 30 of the earth's diameters?

Ans. at $\frac{105}{251}$ parts of a diameter from the earth's centre, or $\frac{41}{502}$ parts of a diameter, or 648 miles below the surface.

QUEST. 38. Whereabouts, between the earth and moon, are their attractions equal to each other? Or where must another body be placed, so as to remain suspended in equilibrio, not being more attracted to the one than to the other, or having no tendency to fall either way? Their dimensions being as in the last question.

Ans. From the earth's centre $26\frac{9}{11}$ } of the earth's
From the moon's centre $3\frac{2}{11}$ } diameters.

QUEST. 39. Suppose a stone dropped into an abyss should be stopped at the end of the 11th second after its delivery: what space would it have gone through? Ans. $1946\frac{1}{12}$ feet.

QUEST. 40. If a heavy body be observed to fall through 100 feet in the last second of time, from what height did it fall, and how long was it in motion?

Ans. time $3\frac{235}{386}$ sec. and height $209\frac{4273}{9264}$ feet.

QUEST. 41. A stone being let fall into a well, it was observed that, after being dropped, it was ten seconds before the sound of the fall at the bottom reached the ear. What is the depth of the well? Ans. 1270 feet nearly.

QUEST. 42. It is proposed to determine the length of a pendulum vibrating seconds, in the latitude of London, where a heavy body falls through $16\frac{1}{12}$ feet in the first second of time? Ans. 39·11 inches.

By experiment this length is found to be $39\frac{1}{8}$ inches.

QUEST. 43. What is the length of a pendulum vibrating in 2 seconds; also in half a second, and in a quarter second?

Ans. the 2 second pendulum $156\frac{1}{2}$
the $\frac{1}{2}$ second pendulum $9\frac{25}{32}$
the $\frac{1}{4}$ second pendulum $2\frac{57}{128}$ inches.

Quest. 44. What difference will there be in the number of vibrations, made by a pendulum of 6 inches long, and another of 12 inches long, in an hour's time? Ans. $2692\frac{1}{2}$.

Quest. 45. Observed that while a stone was descending, to measure the depth of a well, a string and plummet, that from the point of suspension, or the place where it was held, to the centre of oscillation, measured just 18 inches, had made 8 vibrations, when the sound from the bottom returned. What was the depth of the well? Ans. 412·61 feet.

Quest. 46. If a ball vibrate in the arch of a circle, 10 degrees on each side of the perpendicular; or a ball roll down the lowest 10 degrees of the arch; required the velocity at the lowest point? the radius of the circle, or length of the pendulum, being 20 feet. Ans. 4·4213 feet per second.

Quest. 47. If a ball descend down a smooth inclined plane, whose length is 100 feet, and altitude 10 feet; how long will it be in descending, and what will be the last velocity?

Ans. the veloc. 25·364 feet per sec. and time 7·8852 sec.

Quest. 48. If a cannon ball, of 1lb. weight, be fired against a pendulous block of wood, and, striking the centre of oscillation, cause it to vibrate an arc whose chord is 30 inches; the radius of that arc, or distance from the axis to the lowest point of the pendulum, being 118 inches, and the pendulum vibrating in small arcs 40 oscillations per minute. Required the velocity of the ball, and the velocity of the centre of oscillation of the pendulum, at the lowest point of the arc; the whole weight of the pendulum being 500lb.

Ans. veloc. ball 1956·6054 feet per sec.
and veloc. cent. oscil. 3·9054 feet per sec.

Quest. 49. How deep will a cube of oak sink in common water; each side of the cube being 1 foot, Spec. grav.$=$925? Ans. $11\frac{1}{10}$ inches.

Quest. 50. How deep will a globe of oak sink in water; the diameter being 1 foot? Ans. 9·9867 inches.

Quest. 51. If a cube of wood, floating in common water, have three inches of it dry above the water, and $4\frac{8}{103}$ inches dry when in sea-water; it is proposed to determine the magnitude of the cube, and what sort of wood it is made of?

Ans. the wood is oak, and each side 40 inches.

Quest. 52. An irregular piece of lead ore weighs, in air

12 ounces, but in water only 7; and another fragment weighs in air $14\frac{1}{2}$ ounces, but in water only 9; required their comparative densities, or specific gravities?

Ans. as 145 to 132.

QUEST. 53. An irregular fragment of glass, in the scale, weighs 171 grains, and another of magnet 102 grains; but in water the first fetches up no more than 120 grains, and the other 79: what then will their specific gravities turn out to be?

Ans. glass to magnet as 3933 to 5202,
or nearly as 10 to 13.

QUEST. 54. Hiero, king of Sicily, ordered his jeweller to make him a crown, containing 63 ounces of gold. The workmen thought that substituting part silver was only a proper perquisite: which being suspected, Archimedes was appointed to examine it; who, on putting it into a vessel of water, found it raised the fluid 8·2245 cubic inches: and having discovered that the inch of gold more critically weighed 10·36 ounces, and that of silver but 5·85 ounces, he found by calculation what part of the king's gold had been changed. And you are desired to repeat the process.

Ans. 28·8 ounces.

QUEST. 55. Supposing the cubic inch of common glass weigh 1·4921 ounces troy, the same of sea-water ·59542, and of brandy ·5368; then a seaman having a gallon of this liquor in a glass bottle, which weighs 3·84lb. out of water, and, to conceal it from the officers of the customs, throws it overboard. It is proposed to determine, if it will sink, how much force will just buoy it up?

Ans. 14·1496 ounces.

QUEST. 56. Another person has half an anker of brandy, of the same specific gravity as in the last qestion; the wood of the cask suppose measures $\frac{1}{8}$ of a cubic foot; it is proposed to assign what quantity of lead is just requisite to keep the cask and liquor under water?

Ans. 89·743 ounces.

QUEST. 57. Suppose, by measurement, it be found that a man-of-war, with its ordnance, rigging, and appointments, sinks so deep as to displace 50000 cubic feet of fresh water; what is the whole weight of the vessel?

Ans. $1395\frac{5}{16}$ tons.

QUEST. 58. It is required to determine what would be the height of the atmosphere, if it were every where of the same density as at the surface of the earth, when the quicksilver in the barometer stands at 30 inches; and also, what would be the height of a water barometer at the same time?

Ans. height of the air $28636\frac{4}{11}$ feet, or 5·4235 miles,
height of water 35 feet.

QUEST. 59. With what velocity would each of those three fluids, viz. quicksilver, water, and air, issue through a small orifice in the bottom of vessels, of the respective heights of 30 inches, 35 feet, and 5·5240 miles, estimating the pressure by the whole altitudes, and the air rushing into a vacuum ?

Ans. the veloc. of quicksilver 12·681 feet.
the veloc. of water - 47·447
the veloc. of air - 1369·8

QUEST. 60. A very large vessel of 10 feet high (no matter what shape) being kept constantly full of water, by a large supplying cock at the top ; if 9 small circular holes, each $\frac{1}{5}$ of an inch diameter, be opened in its perpendicular side at every foot of the depth : it is required to determine the several distances to which they will spout on the horizontal plane of the base, and the quantity of water discharged by all of them in 10 minutes ?

Ans. the distances are
$\sqrt{36}$ or 6·00000
$\sqrt{64}$ - 8·00000
$\sqrt{84}$ - 9·16515
$\sqrt{96}$ - 9·79796
$\sqrt{100}$ - 10·00000
$\sqrt{96}$ - 9·79796
$\sqrt{84}$ - 9·16515
$\sqrt{64}$ - 8·00000
$\sqrt{36}$ - 6·00000

and the quantity discharged in 10 min. 123·8849 gallons.

Note. In this solution, the velocity of the water is supposed to be equal to that which is acquired by a heavy body in falling through the whole height of the water above the orifice, and that it is the same in every part of the holes.

QUEST. 61. If the inner axis of a hollow globe of copper, exhausted of air, be 100 feet ; what thickness must it be of, that it may just float in the air ?

Ans. ·02688 of an inch thick.

QUEST. 62. If a spherical balloon of copper, of $\frac{1}{100}$ of an inch thick, have its cavity of 100 feet diameter, and be filled with inflammable air, of $\frac{1}{10}$ of the gravity of common air, what weight will just balance it, and prevent it from rising up into the atmosphere ? Ans. 21273lb.

QUEST. 63. If a glass tube, 36 inches long, close at top, be sunk perpendicularly into water, till its lower or open

end be 30 inches below the surface of the water; how high will the water rise within the tube, the quicksilver in the common barometer at the same time standing at $29\frac{1}{2}$ inches?

Ans. 2·26545 inches.

Quest. 64. If a diving bell, of the form of a parabolic conoid, be let down into the sea to the several depths of 5, 10, 15, and 20 fathoms; it is required to assign the respective heights to which the water will rise within it: its axis and the diameter of its base being each 8 feet, and the quicksilver in the barometer standing at 30·9 inches?

Ans.	at 5 fathoms deep the water rises	2·03546 feet.
	at 10 - - -	3·06393
	at 15 - - -	3·70267
	at 20 - - -	4·14658

THE DOCTRINE OF FLUXIONS.

DEFINITIONS AND PRINCIPLES.

Art. 1. In the Doctrine of Fluxions, magnitudes or quantities of all kinds are considered, not as made up of a number of small parts, but as generated by continued motion, by means of which they increase or decrease. As, a line by the motion of a point; a surface by the motion of a line; and a solid by the motion of a surface. So likewise, time may be considered as represented by a line, increasing uniformly by the motion of a point. And quantities of all kinds whatever, which are capable of increase and decrease, may in like manner be represented by geometrical magnitudes, conceived to be generated by motion. Indeed, notwithstanding all that has been advanced to the contrary, this seems the most natural, as well as the simplest, way of conducting the higher investigations; since it is impossible to conceive a geometrical magnitude to be brought into existence, or to change its magnitude, figure, or place, without motion.

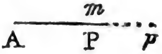
2. Any quantity thus generated, and variable, is called a Fluent, or a Flowing Quantity. And the rate or proportion according to which any flowing quantity increases, at any position or instant, is the Fluxion of the said quantity, at that position or instant: and it is proportional to the magnitude by which the flowing quantity would be uniformly increased

in a given time, with the generating celerity uniformly continued during that time.

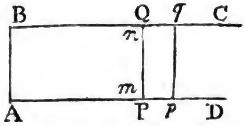
3. The small quantities that are actually generated, produced, or described, in any small given time, and by any continued motion, either uniform or variable, are called Increments.

4. Hence, if the motion of increase be uniform, by which increments are generated, the increments will in that case be proportional, or equal, to the measures of the fluxions: but if the motion of increase be accelerated, the increment so generated, in a given finite time, will exceed the fluxion: and if it be a decreasing motion, the increment, so generated, will be less than the fluxion. But if the time be indefinitely small, so that the motion be considered as uniform for that instant; then these nascent increments will always be proportional, or equal, to the fluxions, and may be substituted instead of them, in any calculation.

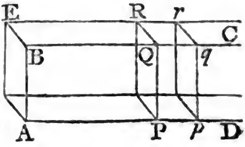
5. To illustrate these definitions: Suppose a point m be conceived to move from the position A, and to generate a line AP, by a motion any how regulated; and suppose the celerity of the point m, at any position P, to be such as would, if from thence it should become or continue uniform, be sufficient to cause the point to describe, or pass uniformly over, the distance Pp, in the given time allowed for the fluxion: then will the said line Pp represent the fluxion of the fluent, or flowing line, AP, at that position.



6. Again, suppose the right line mn to move from the position AB, continually parallel to itself, with any continued motion, so as to generate the fluent or flowing rectangle ABQP, while the point m describes the line AP: also, let the distance Pp be taken, as before, to express the fluxion of the line or base AP; and complete the rectangle PQqp. Then, like as Pp is the fluxion of the line AP, so is Pq the fluxion of the flowing parallelogram AQ; both these fluxions, or increments, being uniformly described in the same time.

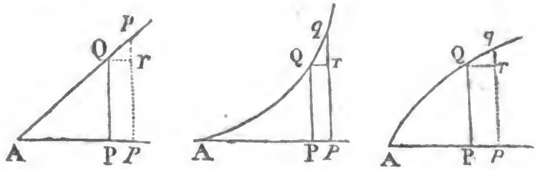


7. In like manner, if the solid AERP be conceived to be generated by the plane PQR, moving from the position ABE, always parallel to itself, along the line AD; and if Pp denote the fluxion of the line AP: Then, like as the



rectangle PQqp, or PQ $\times$ Pp, denotes the fluxion of the flowing rectangle ABQP, so also shall the fluxion of the variable solid, or prism ABERQP, be denoted by the prism PQRrqp, or the plane PR $\times$ Pp. And, in both these last two cases, it appears that the fluxion of the generated rectangle, or prism, is equal to the product of the generating line, or plane, drawn into the fluxion of the line along which it moves.

8. Hitherto the generating line, or plane, has been considered as of a constant and invariable magnitude; in which case the fluent, or quantity generated, is a rectangle, or a prism, the former being described by the motion of a line, and the latter by the motion of a plane. So, in like manner, are other figures, whether plane or solid, conceived to be described by the motion of a Variable Magnitude, whether it be a line or a plane. Thus, let a variable line PQ be carried by a parallel motion along AP; or while a point P is carried along, and describes the line AP, suppose another point



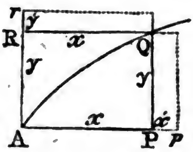
Q to be carried by a motion perpendicular to the former, and to describe the line PQ: let pq be another position of PQ, indefinitely near to the former; and draw Qr parallel to AP. Now in this case there are several fluents, or flowing quantities, with their respective fluxions; namely, the line or fluent AP, the fluxion of which is Pp or Qr; the line or fluent PQ, the fluxion of which is rq; the curve or oblique line AQ, described by the oblique motion of the point Q, the fluxion of which is Qq; and lastly, the surface APQ, described by the variable line PQ, the fluxion of which is the rectangle PQrp, or PQ $\times$ Pp. In the same manner may any solid be conceived to be described, by the motion of a variable plane parallel to itself, substituting the variable plane for the variable line; in which case the fluxion of the solid, at any position, is represented by the variable plane, at that position, drawn into the fluxion of the line along which it is carried.

9. Hence then it follows in general, that the fluxion of any figure, whether plane or solid, at any position, is equal to the section of it, at that position, drawn into the fluxion of the axis, or line along which the variable section is sup-

THE DIRECT METHOD OF FLUXIONS.

To find the Fluxion of the Product or Rectangle of two Variable Quantities.

15. Let ARQP $= xy$, be the flowing or variable rectangle, generated by two lines PQ and RQ, moving always perpendicular to each other, from the positions AR and AP; denoting the one by x, and the other by y; supposing x and y to be so related, that the curve line AQ may always pass through the intersection Q of those lines, or the opposite angle of the rectangle.



Now, the rectangle consists of the two trilinear spaces APQ, ARQ, of which, the

fluxion of the former is PQ $\times$ Pp, or $y\dot{x}$,

that of the latter is - RQ $\times$ Rr, or $x\dot{y}$, by art. 8;

therefore the sum of the two $\dot{x}y + x\dot{y}$, is the fluxion of the whole rectangle xy or ARQP.

The Same Otherwise.

16. Let the sides of the rectangle x and y, by flowing, become $x + x'$ and $y + y'$: then the product of these two, or $xy + xy' + yx' + x'y'$ will be the new or contemporaneous value of the flowing rectangle PR or xy: subtract the one value from the other, and the remainder, $xy' + yx' + x'y'$, will be the increment generated in the same time as x' or y'; of which the last term $x'y'$ is nothing, or indefinitely small, in respect of the other two terms, because x' and y' are indefinitely small in respect of x and y; which term being therefore omitted, there remains $xy' + yx'$ for the value of the increment; and hence, by substituting $\dot{x}$ and $\dot{y}$ for x' and y', to which they are proportional, there arises $x\dot{y} + y\dot{x}$ for the true value of the fluxion of xy; the same as before.

17. Hence may be easily derived the fluxion of the powers and products of any number of flowing or variable quantities whatever; as of xyz, or $uxyz$, or $vuxyz$, &c. And first, for the fluxion of xyz: put $p = xy$, and the whole given fluent $xyz = q$, or $q = xyz = pz$. Then, taking the fluxions of $q = pz$, by the last article, they are $\dot{q} = \dot{p}z + p\dot{z}$; but $p = xy$, and so $\dot{p} = \dot{x}y + x\dot{y}$ by the same article; substituting therefore these values of p and $\dot{p}$ instead of them, in the value of $\dot{q}$, this becomes $\dot{q} = \dot{x}yz + x\dot{y}z + xy\dot{z}$,

the fluxion of xyz required; which is therefore equal to the sum of the products, arising from the fluxion of each letter, or quantity, multiplied by the product of the other two.

Again, to determine the fluxion of $uxyz$, the continual product of four variable quantities; put this product, namely, $uxyz$, or $qu = r$, where $q = xyz$ as above. Then, taking the fluxions by the last article, $\dot{r} = \dot{q}u + q\dot{u}$; which, by substituting for q and $\dot{q}$ their values as above, becomes - - $\dot{r} = \dot{u}xyz + u\dot{x}yz + ux\dot{y}z + uxy\dot{z}$, the fluxion of $uxyz$ as required: consisting of the fluxion of each quantity, drawn into the products of the other three.

In the very same manner it is found, that the fluxion of $vuxyz$ is $\dot{v}uxyz + v\dot{u}xyz + vu\dot{x}yz + vux\dot{y}z + vuxy\dot{z}$; and so on, for any number of quantities whatever; in which it is always found, that there are as many terms as there are variable quantities in the proposed fluent; and that these terms consist of the fluxion of each variable quantity, multiplied by the product of all the rest of the quantities.

18. Hence is easily derived the fluxion of any power of a variable quantity, as of x^2, or x^3, or x^4, &c. For, in the product or rectangle xy, if $x = y$, then is $xy = xx$ or x^2, and also its fluxion $\dot{x}y + x\dot{y} = \dot{x}x + x\dot{x}$ or $2x\dot{x}$, the fluxion of x^2.

Again, if all the three x, y, z be equal; then is the product of the three $xyz = x^3$; and consequently, its fluxion $\dot{x}yz + x\dot{y}z + xy\dot{z} = \dot{x}xx + x\dot{x}x + xx\dot{x}$ or $3x^2\dot{x}$, the fluxion of x^3.

In the same manner, it will appear that

the fluxion of x^4 is $= 4x^3\dot{x}$, and

the fluxion of x^5 is $= 5x^4\dot{x}$, and, in general,

the fluxion of x^n is $= nx^{n-1}\dot{x}$;

where n is any positive whole number whatever.

That is, the fluxion of any positive integral power, is equal to the fluxion of the root ($\dot{x}$), multiplied by the exponent of the power (n), and by the power of the same root whose index is less by 1, (x^{n-1})*.

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* In the text, the fluxion of the product of two, three, or more, variable quantities is found, and thence, by supposing them to become equal, the fluxions of the square, cube, &c. of a variable quantity, are inferred. Sometimes, the investigation commences with the fluxion of a square, and proceeds thence to that of a rectangle.

Let $x—s$ and x be two states of the same line generated by an equable motion: then, while the line $x—s$ by flowing equably becomes x, its square $(x—s)^2$ will become x^2. That is, while the space s is described equably by the flowing line, the space $x^2—(x—s)^2 = 2sx—s^2$ will be described by the flowing square of that line, and this latter is the space which *would* have been generated in the same time by a certain

And thus, the fluxion of $a + cx$ being $c\dot{x}$,
that of $(a + cx)^2$ is $2c\dot{x} \times (a + cx)$ or $2ac\dot{x} + 2c^2x\dot{x}$,
that of $(a + cx^2)^2$ is $4cx\dot{x} \times (a + cx^2)$ or $4acx\dot{x} + 4c^2x^3\dot{x}$,
that of $(x^2 + y^2)^2$ is $(4x\dot{x} + 4y\dot{y}) \times (x^2 + y^2)$
that of $(x + cy^2)^3$ is $(3\dot{x} + 6cy\dot{y}) \times (x + cy^2)^2$.

19. From the conclusions in the same article, we may also derive the fluxion of any fraction, or the quotient of one variable quantity divided by another, as of

$\frac{x}{y}$. For, put the quotient or fraction $\frac{x}{y} = q$; then multiplying by the denominator, $x = qy$; and, taking the fluxions, $\dot{x} = \dot{q}y + q\dot{y}$, or $\dot{q}y = \dot{x} - q\dot{y}$; and, by division,

$$\dot{q} = \frac{\dot{x}}{y} - \frac{q\dot{y}}{y} = \text{(by substituting the value of } q\text{, or } \frac{x}{y}\text{)},$$

$$\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2} = \frac{\dot{x}y - x\dot{y}}{y^2}, \text{ the fluxion of } \frac{x}{y}, \text{ as required.}$$

That is, the fluxion of any fraction, is equal to the fluxion of the numerator drawn into the denominator, minus the fluxion of the denominator drawn into the numerator, and the remainder divided by the square of the denominator. So that the fluxion of $\frac{ax}{y}$ is $a \times \frac{\dot{x}y - x\dot{y}}{y^2}$ or $\frac{a\dot{x}y - ax\dot{y}}{y^2}$.

20. Hence too is easily derived the fluxion of any negative

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magnitude (whether assignable or not) moving uniformly. Hence, the fluxion of the flowing magnitude $(x-s)$, is to the fluxion of the flowing magnitude $(x-s)^2$, as s to $2sx - s^2$, or as 1 to $2x - s$: and as this must obtain in all possible values of $x-s$, it must obtain in the ultimate state, when $(x-s)$ by flowing, becomes x; and then, s vanishing, the ratio becomes 1 to $2x$. That is, the ratio of the fluxions of x and x^2 is that of 1 to $2x$. Consequently, if $\dot{x}$ denote the fluxion of x, then will $2x\dot{x}$ denote the fluxion of x^2.

The fluxion of the square of a quantity being thus found, that of any product is easily assigned. Thus, to determine the fluxion of the product of xy:

Put $x + y = s$; then $\phi(x + y) = \dot{x} + \dot{y} = \dot{s}$,
also, $x^2 + 2xy + y^2 = s^2$; $\therefore 2xy = s^2 - x^2 - y^2$,

$$\text{and } xy = \tfrac{1}{2}s^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2,$$

$$\begin{aligned} \therefore \text{ by the above } \phi(xy) &= s\dot{s} - x\dot{x} - y\dot{y}, \\ &= s(\dot{x} + \dot{y}) - x\dot{x} - y\dot{y}, \\ &= (x + y)(\dot{x} + \dot{y}) - x\dot{x} - y\dot{y}, \\ &= x\dot{x} + x\dot{y} + y\dot{x} + y\dot{y} - x\dot{x} - y\dot{y}, \\ &= x\dot{y} + y\dot{x}, \end{aligned}$$

agreeing with the result in art. 15.

integer power of a variable quantity, as of x^{-n}, or $\frac{1}{x^n}$, which is the same thing. For here the numerator of the fraction is 1, whose fluxion is nothing; and therefore, by the last article, the fluxion of such a fraction, or negative power, is barely equal to minus the fluxion of the denominator, divided by the square of the said denominator. That is, the fluxion of x^{-n}, or $\frac{1}{x^n}$ is $-\frac{nx^{n-1}\dot{x}}{x^{2n}}$ or $-\frac{n\dot{x}}{x^{n+1}}$ or $-nx^{-n-1}\dot{x}$; or the fluxion of any negative integer power of a variable quantity, as x^{-n}, is equal to the fluxion of the root, multiplied by the exponent of the power, and by the next power less by 1; the same rule as for positive powers.

The same thing is otherwise obtained thus: Put the proposed fraction, or quotient $\frac{1}{x^n} = q$; then is $qx^n = 1$; and, taking the fluxions, we have
$\dot{q}x^n + qnx^{n-1}\dot{x} = 0$; hence $\dot{q}x^n = -qnx^{n-1}\dot{x}$; divide by x^n, then $\dot{q} = -\frac{qn\dot{x}}{x} =$ (by substituting $\frac{1}{x^n}$ for q), $\frac{-n\dot{x}}{x^{n+1}}$ or $= -nx^{-n-1}\dot{x}$; the same as before.

Hence the fluxion of x^{-1} or $\frac{1}{x}$ is $-x^{-2}\dot{x}$, or $-\frac{\dot{x}}{x^2}$,

that of . x^{-2} or $\frac{1}{x^2}$ is $-2x^{-3}\dot{x}$ or $-\frac{2\dot{x}}{x^3}$,

that of . x^{-3} or $\frac{1}{x^3}$ is $-3x^{-4}\dot{x}$ or $-\frac{3\dot{x}}{x^4}$,

that of . ax^{-4} or $\frac{a}{x^4}$ is $-4ax^{-5}\dot{x}$ or $-\frac{4a\dot{x}}{x^5}$,

that of . x^{-n} or $\frac{1}{x^n}$ is $-\frac{n\dot{x}}{x^{n+1}}$,

that of $(a+x)^{-1}$ or $\frac{1}{a+x}$ is $-(a+x)^{-2}\dot{x}$ or $-\frac{\dot{x}}{(a+x)^2}$,

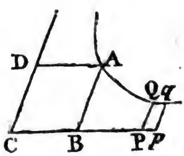
that of $c\,(a+3x^2)^{-2}$ or $\frac{c}{(a+3x^2)^2}$ is $-12cx\dot{x}\times(a+3x^2)^{-3}$,

or $-\frac{12cx\dot{x}}{(a+3x^2)^3}$.

21. Much in the same manner is obtained the fluxion of any fractional power of a fluent quantity, as of $x^{\frac{m}{n}}$, or $\sqrt[n]{x^m}$.

For, put the proposed quantity $x^{\frac{m}{n}} = q$; then, raising each side to the n power, gives $x^m = q^n$;
taking the fluxions, gives $mx^{m-1}\dot{x} = nq^{n-1}\dot{q}$; then dividing by nq^{n-1}, gives $\dot{q} = \frac{mx^{m-1}\dot{x}}{nq^{n-1}} = \frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$.
Which is still the same rule, as before, for finding the fluxion of any power of a fluent quantity, and which therefore is general, whether the exponent be positive or negative, integral or fractional. And hence the fluxion of $ax^{\frac{3}{2}}$ is $\frac{3}{2}ax^{\frac{1}{2}}\dot{x}$;
that of $ax^{\frac{1}{2}}$ is $\frac{1}{2}ax^{\frac{1}{2}-1}\dot{x} = \frac{1}{2}ax^{-\frac{1}{2}}\dot{x} = \frac{a\dot{x}}{2x^{\frac{1}{2}}} = \frac{a\dot{x}}{2\surd x}$; and that of
$\surd(a^2 - x^2)$ or $(a^2 - x^2)^{\frac{1}{2}}$ is $\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}} \times -2x\dot{x} = \frac{-x\dot{x}}{\surd(a^2 - x^2)}$.

22. Having now found out the fluxions of all the ordinary forms of algebraical quantities; it remains to determine those of logarithmic expressions; and also of exponential ones, that is, such powers as have their exponents variable or flowing quantities. And first, for the fluxion of Napier's, or the hyperbolic logarithm.



23. Now, to determine this from the nature of the hyperbolic spaces. Let A be the principal vertex of an hyperbola, having its asymptotes CD, CP, with the ordinates DA, BA, PQ, &c. parallel to them. Then, from the nature of the hyperbola and of logarithms, it is known, that any space ABPQ is the log. of the ratio of CB to CP, to the modulus ABCD. Now, put $1 =$ CB or BA the side of the square or rhombus DB; $m =$ the modulus, or CB $\times$ BA $\times$ sin. C; or area of DB, or sine of the angle C to the radius 1; also the absciss CP $= x$, and the ordinate PQ $= y$. Then, by the nature of the hyperbola, CP $\times$ PQ is always equal to DB, that is, $xy = m$; hence $y = \frac{m}{x}$, and the fluxion of the space, $\dot{x}y$ is $\frac{m\dot{x}}{x} =$ PQqp the fluxion of the log. of x, to the modulus m. And, in the hyperbolic logarithms, the modulus m being 1, therefore $\frac{\dot{x}}{x}$ is the fluxion of the hyp. log. of x; which is therefore equal to the fluxion of the quantity, divided by the quantity itself.

Hence the fluxion of the hyp. log.

of $1 + x$ is $\frac{\dot{x}}{1+x}$,

of $1 - x$ is $\frac{-\dot{x}}{1-x}$,

of $x + z$ is $\frac{\dot{x}+\dot{z}}{x+z}$,

of $\frac{a+x}{a-x}$ is $\frac{\dot{x}(a-x)+\dot{x}(a+x)}{(a-x)^2} \times \frac{a-x}{a+x} = \frac{2a\dot{x}}{a^2-x^2}$,

of ax^n is $\frac{nax^{n-1}\dot{x}}{ax^n} = \frac{n\dot{x}}{x}$.

24. By means of the fluxions of logarithms, are usually determined those of exponential quantities, that is, quantities which have their exponent a flowing or variable letter. These exponentials are of two kinds, namely, when the root is a constant quantity, as e^x, and when the root is variable as well as the exponent, as y^x.

25. In the first case, put the exponential, whose fluxion is to be found, equal to a single variable quantity z, namely, $z = e^x$; then take the logarithm of each, so shall log. $z = x \times$ log. e; take the fluxions of these, so shall $\frac{\dot{z}}{z} = \dot{x} \times$ log. e, by the last article; hence $\dot{z} = z\dot{x} \times$ log. $e = e^x\dot{x} \times$ log. e, which is the fluxion of the proposed quantity e^x or z; and which therefore is equal to the said given quantity drawn into the fluxion of the exponent, and into the log. of the root.

Hence also, the fluxion of $(a + c)^{nx}$ is $(a + c)^{nx} \times n\dot{x} \times$ log. $(a + c)$.

26. In like manner, in the second case, put the given quantity $y^x = z$; then the logarithms give log. $z = x \times$ log. y, and the fluxions give $\frac{\dot{z}}{z} = \dot{x}$. log. $y + x$. $\frac{\dot{y}}{y}$; hence $\dot{z} = z\dot{x}$. log. $y + \frac{zx\dot{y}}{y}$ = (by substituting y^x for z)$y^x\dot{x}$. log. $y + xy^{x-1}\dot{y}$, which is the fluxion of the proposed quantity y^x; and which therefore consists of two terms, of which the one is the fluxion of the given quantity considering the exponent as constant, and the other the fluxion of the same quantity considering the root as constant.

27. The fluxions of the usual trigonometrical quantities, sin. z, cos. z, &c. are easily found by blending these principles with the analytical formulæ at pa. 395, vol. i.

We assume the proportionality of the increments, and of their contemporaneous fluxions, and proceed thus:

To find φ sin. z, we suppose that by a motion of one of the legs including the angle, it becomes $z + x'$ or $z + \dot{z}$. Then φ sin. $z =$ sin. $(z + \dot{z})$ — sin. z. But by equa. 9. p. 395, vol. i. we have

$$\sin.(z + \dot{z}) = \sin. z \,.\, \cos. \dot{z} + \sin. \dot{z} \cos. z.$$

But the sine of an arc indefinitely small does not differ sensibly from that arc itself, nor its cosine differ perceptibly from radius; hence we have sin. $\dot{z} = \dot{z}$, and cos. $\dot{z} = 1$; and therefore sin. $(z + \dot{z}) =$ sin. $z + \dot{z}$ cos. z; whence sin. $(z+\dot{z})$—sin. z, or $\varphi(\sin. z) = \dot{z} \cos. z$, viz. the fluxion of the sine of an arc whose radius is unity, is equal to the product of the fluxion of the arc into the cosine of the same arc.

28. In like manner, the fluxion of cos. z, or cos. $(z + \dot{z})$ — cos. $z =$ cos. z cos. $\dot{z}$ — sin. z sin. $\dot{z}$ — cos. z, or since cos. $(z + \dot{z}) =$ cos. z cos $\dot{z}$ — sin. z sin. $\dot{z}$; therefore, because sin. $\dot{z} = \dot{z}$, and cos. $\dot{z} = 1$, we have φ cos. $z =$ cos. z — $\dot{z}$ sin. z — cos. $z = - \dot{z}$ sin. z, that is, the fluxion of the cosine of an arc, radius being 1, is found by multiplying the fluxion of the arc (taken with a contrary sign) by the sine of the same arc.

29. By means of these two formulæ, many other fluxional expressions may be found, viz.

$$\varphi \cos. mz = - m\dot{z} \sin. mz.$$

$$\varphi \sin. mz = + m\dot{z} \cos. mz.$$

$$\varphi \tan. z = \frac{\dot{z}}{\cos.^2 z} = \dot{z} \sec.^2 z.$$

$$\varphi \operatorname{cotan}. z = - \frac{\dot{z}}{\sin.^2 z} = - \dot{z} \operatorname{cosec}.^2 z.$$

$$\varphi \sec. z = \frac{\dot{z} \sin. z}{\cos.^2 z} = \frac{\dot{z} \tan. z}{\cos. z}.$$

$$\varphi \operatorname{cosec}. z = - \frac{z \cos. z}{\sin.^2 z} = - \frac{\dot{z} \cot. z}{\sin. z}.$$

$$\varphi \sin.^m z = m \sin.^{m-1} z\, \dot{z} \cos. z.$$

$$\varphi \cos.^m z = - m \cos.^{m-1} z\, \dot{z} \sin. z.$$

30. Hence, by the way, will flow this useful practical conclusion, that if x be any arc, then

$$\dot{x} = \frac{\varphi \sin. z}{\cos. z} = \frac{- \varphi \cos. z}{\sin. z} = \cos.^2 z\, \varphi \tan. z.$$

$$= \frac{\varphi \tan. z}{1 + \tan.^2 z} = - \varphi \cot. z \sin.^2 z = \frac{- \varphi \cot. z}{1 + \cot.^2 z}.$$

OF SECOND, THIRD, &c. FLUXIONS.

HAVING explained the manner of considering and determining the first fluxions of flowing or variable quantities ; it remains now to consider those of the higher orders, as second, third, fourth, &c. fluxions.

31. If the rate or celerity with which any flowing quantity changes its magnitude be constant, or the same at every position ; then is the fluxion of it also constantly the same. But if the variation of magnitude be continually changing, either increasing or decreasing ; then will there be a certain degree of fluxion peculiar to every point or position ; and the rate of variation or change in the fluxion, is called the Fluxion of the Fluxion, or the Second Fluxion of the given fluent quantity. In like manner, the variation or fluxion of this second fluxion, is called the Third Fluxion of the first proposed fluent quantity ; and so on.

These orders of fluxions are denoted by the same fluent letter with the corresponding number of points over it : namely, two points for the second fluxion, three points for the third fluxion, four points for the fourth fluxion, and so on. So, the different orders of the fluxion of x, are $\dot{x}$ $\ddot{x}$, $\dddot{x}$, $\ddddot{x}$, &c. ; where each is the fluxion of the one next before it.

32. This description of the higher orders of fluxions may be illustrated by the figures exhibited in art. 8, where, if x denote the absciss AP, and y the ordinate PQ ; and if the ordinate PQ or y flow along the absciss AP or x, with a uniform motion ; then the fluxion of x, namely, $\dot{x} = \text{P}p$ or $\text{Q}r$, is a constant quantity, or $\ddot{x} = 0$, in all the figures. Also, in fig. 1, in which AQ is a right line, $\dot{y} = rq$, or the fluxion of PQ, is a constant quantity, or $\ddot{y} = 0$; for the angle Q, = the angle A, being constant, $\text{Q}r$ is to rq, or $\dot{x}$ to $\dot{y}$, in a constant ratio. But in the 2d fig. rq, or the fluxion of PQ, continually increases more and more ; and in fig. 3 it continually decreases more and more, and therefore in both these cases y has a second fluxion, being positive in fig. 2, but negative in fig. 3. And so on, for the other orders of fluxions.

Thus if, for instance, the nature of the curve be such, that x^3 is every where equal to a^2y ; then, taking the fluxions, it is $a^2\dot{y} = 3x^2\dot{x}$; and, considering $\dot{x}$ always as a constant quantity, and taking always the fluxions, the equations of the several orders of fluxions will be as below, viz.

the 1st fluxions $a^2\dot{y} = 3x^2\dot{x}$,

the 2d fluxions $a^2\ddot{y} = 6x\dot{x}^2$,

the 3d fluxions $a^2\dddot{y} = 6\dot{x}^3$,

the 4th fluxions $a^2\ddddot{y} = 0$,

and all the higher fluxions also $= 0$, or nothing.

Also the higher orders of fluxions are found in the same manner as the lower ones. Thus,

the first fluxion of y^3 is . . . . $3y^2\dot{y}$;

is 2d flux, or the flux. of $3y^2\dot{y}$, considered as the rectangle of $3y^2$, and $\dot{y}$, is . . . . $\Big\}$ $3y^2\ddot{y} + 6y\dot{y}^2$;

and the flux. of this again, or the 3d flux. of y^3, is . . . $\Big\}$ $3y^2\dddot{y} + 18y\dot{y}\ddot{y} + 6\dot{y}^3$.

33. If the function proposed were ax^n, we should find $\varphi\, ax^n = nax^{n-1}\dot{x}$; the factors n, a, and $\dot{x}$ being regarded as constant in the first fluxion $nax^{n-1}\dot{x}$, to obtain the second fluxion it will suffice to make x^{n-1} flow, and to multiply the result by $na\dot{x}$; but $\varphi\, x^{n-1} = (n-1)\,x^{n-2}\dot{x}$; we have, therefore,

$$2\text{nd } \varphi\, ax^n = n\,(n-1)\,ax^{n-2}\,\dot{x}^2.$$
$$3\text{rd } \varphi\, ax^n = n\,(n-1)\,(n-2)\,ax^{n-3}\,\dot{x}^3.$$
$$4\text{th } \varphi\, ax^n = n\,(n-1)\,(n-2)\,(n-3)\,ax^{n-4}\,\dot{x}^4.$$
$$\&\text{c.} \quad = \quad \&\text{c.}$$
$$m\text{th } \varphi\, ax^n = n\,(n-1)\,(n-2)\,\ldots\,(n-m+1)\; a^{n-m}\,\dot{x}^m,$$

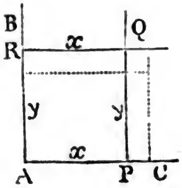
m being supposed not to exceed n, for it is manifest that in the case of n being integral, the function ax^n has only a limited number of fluxions, of which the most elevated is the nth, and which of course is expressed by the formula,

$$n\text{th } \varphi\, ax^n = n\,(n-1)\,(n-2)\,\ldots\,3\,.\,2\,.\,1\,.\,a\dot{x}^n$$

in which state it admits no longer of being put into fluxions, as it contains no variable quantity, or, in other words, its fluxion is equal to zero.

34. In the foregoing articles, it has been supposed that the fluents increase, or that their fluxions are positive; but it often happens that some fluents decrease, and that therefore their fluxions are negative: and whenever this is the case, the sign of the fluxion must be changed, or made contrary to that of the fluent. So, of the rectangle xy, when both x and y increase together, the fluxion is $\dot{x}y + x\dot{y}$: but if one of them, as y, decrease, while the other, x, increases; then, the fluxion of y being $-\dot{y}$, the fluxion of xy will

in that case be $\dot{x}y - x\dot{y}$. This may be illustrated by the annexed rectangle APQR $= xy$, supposed to be generated by the motion of the line PQ from A towards C, and by the motion of the line RQ from B towards A: For, by the motion of PQ, from A towards C, the rectangle is increased, and its fluxion is $+\dot{x}y$; but, by the motion of RQ, from B towards A, the rectangle is decreased, and the fluxion of the decrease is $x\dot{y}$; therefore, taking the fluxion of the decrease from that of the increase, the fluxion of the rectangle xy, when x increases and y decreases, is $\dot{x}y - x\dot{y}$.



35. We may now collect the principal rules, which have been demonstrated in the foregoing articles, for finding the fluxions of all sorts of quantities. And hence,

1st, *For the fluxion of any Power of a flowing quantity.* —Multiply all together the exponent of the power, the fluxion of the root, and the power next less by 1 of the same root.

2d, *For the fluxion of the Rectangle of two quantities.*—Multiply each quantity by the fluxion of the other, and connect the two products together by their proper signs.

3d, *For the fluxion of the Continual Product of any number of flowing quantities.* — Multiply the fluxion of each quantity by the product of all the other quantities, and connect all the products together by their proper signs.

4th, *For the fluxion of a Fraction.*— From the fluxion of the numerator drawn into the denominator, subtract the fluxion of the denominator drawn into the numerator, and divide the result by the square of the denominator.

5th, *Or, the 2d, 3d, and 4th cases may be all included under one, and performed thus.*—Take the fluxion of the given expression as often as there are variable quantities in it, supposing first only one of them variable, and the rest constant; then another variable, and the rest constant; and so on, till they have all in their turns been singly supposed variable; and connect all these fluxions together with their own signs.

6th, *For the fluxion of a Logarithm.*—Divide the fluxion of the quantity by the quantity itself, and multiply the result by the modulus of the system of logarithms.

Note.—The modulus of the hyperbolic logarithms is 1, and the modulus of the common logs., is 0·43429448, &c.

7th, *For the fluxion of an Exponential quantity, having the Root Constant.*—Multiply all together, the given quantity, the fluxion of its exponent, and the hyp. log. of the root.

8th, *For the fluxion of an Exponential quantity, having the Root Variable.*—To the fluxion of the given quantity, found by the 1st rule, as if the root only were variable, add the fluxion of the same quantity found by the 7th rule, as if the exponent only were variable; and the sum will be the fluxion for both of them variable.

Note.—When the given quantity consists of several terms, find the fluxion of each term separately, and connect them all together with their proper signs; also, for the fluxions of trigonometrical formulæ, take the formulæ in arts. 27—30.

36. PRACTICAL EXAMPLES TO EXERCISE THE FOREGOING RULES.

1. The fluxion of axy is
2. The fluxion of $bxyz$ is
3. The fluxion of $cx \times (ax - cy)$ is
4. The fluxion of $x^m y^n$ is
5. The fluxion of $x^m y^n z^r$ is
6. The fluxion of $(x + y) \times (x - y)$ is
7. The fluxion of $2ax^2$ is
8. The fluxion of $2x^3$ is
9. The fluxion of $3x^4 y$ is
10. The fluxion of $4x^{\frac{2}{3}}y^4$ is
11. The fluxion of $ax^2y - x^{\frac{1}{2}}y^3$ is
12. The fluxion of $4x^4 - x^2y + 2byz$ is
13. The fluxion of $\sqrt[n]{x}$ or $^{\frac{1}{n}}$ is
14. The fluxion of $\sqrt[n]{x^m}$ or $x^{\frac{m}{n}}$ is
15. The fluxion of $\dfrac{1}{\sqrt[n]{x^m}}$ or $\dfrac{1}{x^{\frac{m}{n}}}$ or $x^{-\frac{m}{n}}$ is
16. The fluxion of $\sqrt{x}$ or $x^{\frac{1}{2}}$ is

17. The fluxion of $\sqrt[3]{x}$ or $x^{\frac{1}{3}}$ is

18. The fluxion of $\sqrt[3]{x^2}$ or $x^{\frac{2}{3}}$ is

19. The fluxion of $\sqrt{x^3}$ or $x^{\frac{3}{2}}$ is

20. The fluxion of $\sqrt[4]{x^3}$ or $x^{\frac{3}{4}}$ is

21. The fluxion of $\sqrt[3]{x^4}$ or $x^{\frac{4}{3}}$ is

22. The fluxion of $\sqrt{(a^2 + x^2)}$ or $(a^2 + x^2)^{\frac{1}{2}}$ is

23. The fluxion of $\sqrt{(a^2 - x^2)}$ or $(a^2 - x^2)^{\frac{1}{2}}$ is

24. The fluxion of $\sqrt{(2rx - xx)}$ or $(2rx - xx)^{\frac{1}{2}}$ is

25. The fluxion of $\dfrac{1}{\sqrt{(a^2 - x^2)}}$ or $(a^2 - x^2)^{-\frac{1}{2}}$ is

26. The fluxion of $(ax - xx)^{\frac{1}{3}}$ is

27. The fluxion of $2x\sqrt{a^2 \pm x^2}$ is

28. The fluxion of $(a^2 - x^2)^{\frac{3}{2}}$ is

29. The fluxion of $\sqrt{xz}$ or $(xz)^{\frac{1}{2}}$ is

30. The fluxion of $\sqrt{xz - zz}$ or $(xz - zz)^{\frac{1}{2}}$ is

31. The fluxion of $-\dfrac{1}{a\sqrt{x}}$ or $-\dfrac{1}{a}x^{-\frac{1}{2}}$ is

32. The fluxion of $\dfrac{ax^3}{a+x}$ is

33. The fluxion of $\dfrac{x^m}{y^n}$ is

34. The fluxion of $\dfrac{x+y+z}{x+y}$ is

35. The fluxion of $\dfrac{c}{xx}$ is

36. The fluxion of $\dfrac{3x}{a-x}$ is

37. The fluxion of $\dfrac{z}{x+z}$ is

38. The fluxion of $\sqrt[3]{(a + bx + cx^2 + dx^3)}$ is

39. The fluxion of $\sqrt[n]{(a + bx + cx^2 + \text{\&c. to } mx^n)}$ is

40. The fluxion of $\dfrac{axy^2}{z}$ is

41. The fluxion of $\frac{3}{\sqrt{(x^2 - y^2)}}$ is

42. The fluxion of the hyp. log. of ax is

43. The fluxion of the hyp. log. of $1 + x$ is

44. The fluxion of the hyp. log. of $1 - x$ is

45. The fluxion of the hyp. log. of x^2 is

46. The fluxion of the hyp. log. of $\sqrt{x}$ is

47. The fluxion of the hyp. log. of x^m is

48. The fluxion of the hyp. log. of $\frac{2}{x^2}$ is

49. The fluxion of the hyp. log. of $\frac{1+x}{1-x}$ is

50. The fluxion of the hyp. log. of $\frac{1-x}{1+x}$ is

51. The fluxion of c^x is

52. The fluxion of 10^x is

53. The fluxion of $(a + c)^x$ is

54. The fluxion of 100^{xy} is

55. The fluxion of x^x is

56. The fluxion of y^{10x} is

57. The fluxion of x^x is

58. The fluxion of $(xy)^{xz}$ is

59. The fluxion of $x\dot{y}$ is

60. The fluxion of $\dot{x}\dot{y}^2$ is

61. The second fluxion of xy is

62. The second fluxion of xy, when $\dot{x}$ is constant, is

63. The second fluxion of x^n is

64. The third fluxion of x^n, when $\dot{x}$ is constant, is

65. The third fluxion of xy is

THE INVERSE METHOD, OR THE FINDING OF FLUENTS.

37. It has been observed, that a Fluent, or Flowing Quantity, is the variable quantity which is considered as increasing or decreasing. Or, the fluent of a given fluxion, is such a quantity, that its fluxion, found according to the foregoing rules, shall be the same as the fluxion given or proposed.

38. It may be further observed, that Contemporary Fluents, or Contemporary Fluxions, are such as flow together, or for the same time—When contemporary fluents are always equal, or in any constant ratio; then also are their fluxions respectively either equal, or in that same constant ratio. That is, if $x = y$, then is $\dot{x} = \dot{y}$; or if $x : y :: n : 1$, then is $\dot{x} : \dot{y} :: n : 1$; or if $x = ny$, then is $\dot{x} = n\dot{y}$.

39. It is easy to find the fluxions to all the given forms of fluents; but, on the contrary, it is difficult to find the fluents of many given fluxions; and indeed there are numberless cases in which this cannot at all be done, excepting by the quadrature and rectification of curve lines, or by logarithms, or by infinite series. For it is only in certain particular forms and cases that the fluents of given fluxions can be found; there being no method of performing this universally, *a priori*, by a direct investigation, like finding the fluxion of a given fluent quantity. We can only therefore lay down a few rules for such forms of fluxions as we know, from the direct method, belong to such and such kinds of flowing quantities: and these rules, it is evident, much chiefly consist in performing such operations as are the reverse of those by which the fluxions are found of given fluent quantities. The principal cases of which are as follow.

40. *To find the Fluent of a Simple Fluxion; or of that in which there is no variable quantity, and only one fluxional quantity.*

This is done by barely substituting the variable or flowing quantity instead of its fluxion; being the result or reverse of the notation only.—Thus,

The fluent of $a\dot{x}$ is ax.

The fluent of $a\dot{y} + 2\dot{y}$ is $ay + 2y$.

The fluent of $\dot{\overline{\sqrt{a^2 + x^2}}}$ is $\sqrt{a^2 + x^2}$.

41. *When any Power of a flowing quantity is Multiplied by the Fluxion of the Root :*

Then, having substituted, as before, the flowing quantity, for its fluxion, divide the result by the new index of the power. Or, which is the same thing, take out, or divide by, the fluxion of the root ; add 1 to the index of the power ; and divide by the index so increased. Which is the reverse of the 1st rule for finding fluxions.

So if the fluxion proposed be - - $3x^5\dot{x}$.
Leave out, or divide by $\dot{x}$, then it is - $3x^5$;
add 1 to the index, and it is - - $3x^6$;
divide by the index 6, and it is - - $\frac{3}{6}x^6$ or $\frac{1}{2}x^6$,
which is the fluent of the proposed fluxion $3x^5\dot{x}$.

In like manner,

The fluent of $2ax\dot{x}$ is ax^2.

The fluent of $3x^2\dot{x}$ is x^3.

The fluent of $4x^{\frac{1}{2}}\dot{x}$ is $\frac{8}{3}x^{\frac{3}{2}}$.

The fluent of $2y^{\frac{3}{4}}\dot{y}$ is $\frac{8}{7}y^{\frac{7}{4}}$.

The fluent of $az^{\frac{5}{6}}\dot{z}$ is $\frac{6}{11}az^{\frac{11}{6}}$.

The fluent of $x^{\frac{1}{2}}\dot{x} + 3y^{\frac{2}{3}}\dot{y}$ is $\frac{2}{3}x^{\frac{3}{2}} + \frac{9}{5}y^{\frac{5}{3}}$.

The fluent of $x^{n-1}\dot{x}$ is $\frac{1}{n}x^n$.

The fluent of $ny^{n-1}\dot{y}$ is

The fluent of $\frac{\dot{z}}{z^2}$, or $z^{-2}\dot{z}$ is

The fluent of $\frac{a\dot{y}}{y^n}$ is

The fluent of $(a + x)^4\dot{x}$ is

The fluent of $(a^4 + y^4)y^3\dot{y}$ is

The fluent of $(a^3 + z^3)^4z^2\dot{z}$ is

The fluent of $(a^n + x^n)^m x^{n-1}\dot{x}$ is

The fluent of $(a^2 + y^2)^3y\dot{y}$ is

The fluent of $\frac{z\dot{z}}{\sqrt{(a^2 + z^2)}}$ is

The fluent of $\frac{\dot{x}}{\sqrt{(a - x)}}$ is

42. *When the Root under a Vinculum is a Compound Quantity; and the Index of the part or factor Without the Vinculum, increased by* 1, *is some Multiple of that Under the Vinculum:*

Put a single variable letter for the compound root; and substitute its power and fluxion instead of those of the same value, in the given quantity; so will it be reduced to a simpler form, to which the preceding rule can then by applied.

Thus, if the given fluxion be $\dot y = (a^2 + x^2)^{\frac{2}{3}} x^3 \dot x$, where 3, the index of the quantity without the vinculum, increased by 1, making 4, which is just the double of 2, the exponent of x^2 within the vinculum: therefore, putting $z = a^2 + x^2$, thence $x^2 = z - a^2$, the fluxion of which is $2x\dot x = \dot z$; hence then $x^3\dot x = \frac{1}{2}x^2\dot z = \frac{1}{2}\dot z(z - a^2)$, and the given fluxion $\dot y$, or $(a^2 + x^2)^{\frac{2}{3}} x^3 \dot x$, is $= \frac{1}{2} z^{\frac{2}{3}} \dot z (z - a^2)$ or $= \frac{1}{2} z^{\frac{5}{3}} \dot z - \frac{1}{2} a^2 z^{\frac{2}{3}} \dot z$; and hence the fluent y is $= \frac{3}{16} z^{\frac{8}{3}} - \frac{3}{10} a^2 z^{\frac{5}{3}} = 3z^{\frac{5}{3}}(\frac{1}{16} z - \frac{3}{10} a^2)$. Or, by substituting the value of z instead of it, the same fluent is $3\,(a^2+x^2)^{\frac{5}{3}} \times (\frac{1}{16}x^2 - \frac{3}{80}a^2)$, or $\frac{3}{16}(a^2+x^2) \times (x^2 - \frac{3}{5}a^2)$.

In like manner for the following examples.

To find the fluent of $\sqrt{a + cx} \times x^3 \dot x$.

To find the fluent of $(a+cx)^{\frac{3}{4}} x^2 \dot x$.

To find the fluent of $(a+cx^2)^{\frac{1}{3}} \times dx^3 \dot x$.

To find the fluent of $\dfrac{cz\dot z}{\sqrt{a+z}}$ or $(a + z)^{\frac{1}{2}} cz\dot z$.

To find the fluent of $\dfrac{cz^{3n-1}\dot z}{\sqrt{a+z^n}}$ or $(a+z^n)^{-\frac{1}{2}} cz^{3n-1} \dot z$.

To find the fluent of $\dfrac{\dot z\sqrt{a^2-z^2}}{z^6}$ or $(a^2+z^2)^{\frac{1}{2}} z^{-6} \dot z$.

To find the fluent of $\dfrac{\dot x\sqrt{a-x^n}}{x^{\frac{7}{2}n+1}}$ or $(a-x^n)^{-\frac{1}{2}} x^{\frac{7}{2}n+1} \dot x$.

43. *When there are several Terms, involving Two or more Variable Quantities, having the Fluxion of each Multiplied by the other Quantity or Quantities.*

Take the fluent of each term, as if there were only one variable quantity in it, namely, that whose fluxion is contained in it, supposing all the others to be constant in that term ; then, if the fluents of all the terms, so found, be the very same quantity in all of them, that quantity will be the fluent of the whole. Which is the reverse of the 5th rule for finding fluxions : Thus, if the given fluxion be $\dot{x}y + x\dot{y}$, then the fluent of $\dot{x}y$ is xy, supposing y constant : and the fluent of $x\dot{y}$ is also xy, supposing x constant : therefore xy is the required fluent of the given fluxion $\dot{x}y + x\dot{y}$.

In like manner,

The fluent of $\dot{x}yz + x\dot{y}z + xy\dot{z}$ is xyz.

The fluent of $2xy\dot{x} + x^2\dot{y}$ is x^2y.

The fluent of $\frac{1}{2}x^{-\frac{1}{2}}\dot{x}y^2 + 2x^{\frac{1}{2}}y\dot{y}$ is

The fluent of $\dfrac{\dot{x}y - x\dot{y}}{y^2}$ or $\dfrac{\dot{x}}{y} - \dfrac{x\dot{y}}{y^2}$ is

The fluent of $\dfrac{2ax\dot{x}y^{\frac{1}{2}} - \frac{1}{2}ax^2y^{-\frac{1}{2}}\dot{y}}{y}$ or $\dfrac{2ax\dot{x}}{\sqrt{y}} - \dfrac{ax^2\dot{y}}{2y\sqrt{y}}$ is

44. *When the given Fluxional Expression is in this Form* $\dfrac{\dot{x}y - x\dot{y}}{y^2}$*, namely, a fraction, including Two Quantities, being the Fluxion of the former of them drawn into the latter, minus the Fluxion of the latter drawn into the former, and divided by the Square of the latter.*

Then, the fluent is the fraction $\dfrac{x}{y}$, or the former quantity divided by the latter, by the reverse of rule 4, of finding fluxions. That is,

The fluent of $\dfrac{\dot{x}y - x\dot{y}}{y^2}$ is $\dfrac{x}{y}$. And, in like manner,

The fluent of $\dfrac{2x\dot{x}y^2 - 2x^2y\dot{y}}{y^4}$ is $\dfrac{x^2}{y^2}$.

Though, indeed, the examples of this case may be performed by the foregoing one. Thus, the given fluxion $\frac{\dot{x}y - x\dot{y}}{y^2}$ reduces to $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2}$, or $\frac{\dot{x}}{y} - x\dot{y}y^{-2}$; of which,

the fluent of $\frac{\dot{x}}{y}$ is $\frac{x}{y}$ supposing y constant; and

the fluent of $- x\dot{y}y^{-2}$ is also xy^{-1} or $\frac{x}{y}$, when x is constant;

therefore, by that case, $\frac{x}{y}$ is the fluent of the whole $\frac{\dot{x}y - x\dot{y}}{y^2}$.

45. *When the Fluxion of a Quantity is Divided by the Quantity itself:*

Then the fluent is equal to the hyperbolic logarithm of that quantity; or, which is the same thing, the fluent is equal to 2·30258509 multiplied by the common logarithm of the same quantity, by rule 6, for finding fluxions.

So the fluent of $\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$, is the hyp. log. of x.

The fluent of $\frac{2\dot{x}}{x}$ is $2 \times$ hyp. log. of x, or $=$ hyp. log. x^2.

The fluent of $\frac{a\dot{x}}{x}$, is $a \times$ hyp. log. x, or $=$ hyp. log. of x^a.

The fluent of $\frac{\dot{x}}{a+x}$, is

The fluent of $\frac{3x^2\dot{x}}{a+x^3}$, is

46. *Many fluents may be found by the Direct Method thus:*

Take the fluxion again of the given fluxion, or the second fluxion of the fluent sought; into which substitute $\frac{\dot{x}^2}{x}$ for $\ddot{x}$, $\frac{\dot{y}^2}{y}$ for $\ddot{y}$, &c.; that is, make x, $\dot{x}$, $\ddot{x}$, as also y, $\dot{y}$, $\ddot{y}$, &c. to be in continual proportion, or so that $x : \dot{x} :: \dot{x} : \ddot{x}$, and $y : \dot{y} :: \dot{y} : \ddot{y}$, &c.; then divide the square of the given

fluxional expression by the second fluxion, just found, and the quotient will be the fluent required in many cases.

Or the same rule may be otherwise delivered thus:

In the given fluxion $\dot{F}$, write x for $\dot{x}$, y for $\dot{y}$, &c. and call the result G, taking also the fluxion of this quantity, $\dot{G}$; then make $\dot{G} : \dot{F} :: G : F$; so shall the fourth proportional F be the fluent sought in many cases.

It may be proved if this be the true fluent, by taking the fluxion of it again, which if it agree with the proposed fluxion, will show that the fluent is right; otherwise, it is wrong.

EXAMPLES.

Exam. 1. Let it be required to find the fluent of $nx^{n-1}\dot{x}$.

Here $\dot{F} = nx^{n-1}\dot{x}$. Write x for $\dot{x}$, then $nx^{n-1}x$ or $nx^n = G$; the fluxion of this is $\dot{G} = n^2x^{n-1}\dot{x}$; therefore $\dot{G} : \dot{F} :: G : F$, becomes $n^2x^{n-1}\dot{x} : nx^{n-1}\dot{x} :: nx^n : x^n = F$, the fluent sought.

Exam. 2. To find the fluent of $\dot{x}y + x\dot{y}$.

Here $\dot{F} = \dot{x}y + x\dot{y}$; then writing x for $\dot{x}$, and y for $\dot{y}$, it is $xy + xy$ or $2xy = G$; hence $\dot{G} = 2\dot{x}y + 2x\dot{y}$; then $\dot{G} : \dot{F} :: G : F$, becomes $2\dot{x}y + 2x\dot{y} : \dot{x}y + x\dot{y} :: 2xy : xy = F$, the fluent sought.

47. *To find Fluents by means of a Table of Forms of Fluxions and Fluents.*

In the following Table are contained the most usual forms of fluxions that occur in the practical solution of problems, with their corresponding fluents set opposite to them; by means of which, namely, by comparing any proposed fluxion with the corresponding form in the table, the fluent of it will be found.

Forms.	*Fluxions.*	*Fluents.*
1	$x^{n-1}\dot{x}$	$\frac{1}{n}x$
2	$(a \pm x^n)^{m-1}x^{n-1}\dot{x}$	$\pm \frac{1}{mn}(a \pm x^n)^m$
3	$\frac{x^{mn-1}\dot{x}}{(a \pm x^n)^{m+1}}$	$\frac{1}{mna} \times \frac{x^{mn}}{(a \pm x^n)^m}$
4	$\frac{(a \pm x^n)^{m-1}\dot{x}}{x^{mn+1}}$	$\frac{-1}{mna} \times \frac{(a \pm x^n)^m}{x^{mn}}$
5	$(my\dot{x}+nx\dot{y}) \times x^{m-1}y^{n-1}$, or $(\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y})x^m y^n$	$x^m y^n$
6	$mx^{m-1}\dot{x}y^n z^r + nx^m y^{n-1}\dot{y}z^r + rx^m y^n z^{r-1}\dot{z}$, or $(m\dot{x}yz+nx\dot{y}z+rxy\dot{z})x^{m-1}y^{n-1}z^{n-1}z^{r-1}$, or $(\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y} + \frac{r\dot{z}}{z})x^m y^n z^r$,	$x^m y^n z^r$
7	$\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$	log. of x
8	$\frac{x^{n-1}\dot{x}}{a \pm x^n}$	$\pm \frac{1}{n}$ log. of $a \pm x^n$
9	$\frac{x^{-1}\dot{x}}{a \pm x^n}$	$\frac{1}{na}$ log. of $\frac{x^n}{a \pm x^n}$
10	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a - x^n}$	$\frac{1}{n\sqrt{a}}$ log. of $\frac{\sqrt{a}+\sqrt{x^n}}{\sqrt{a}-\sqrt{x^n}}$
11	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a + x^n}$	$\frac{2}{n\sqrt{a}} \times$ arc. to tan. $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n\sqrt{a}} \times$ arc. to cosine $\frac{a-x^n}{a+x^n}$
12	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{(\pm a + x^n)}}$	$\frac{2}{n}$ log. of $\sqrt{x^n} + \sqrt{(\pm a + x^n)}$

Forms.	Fluxions.	Fluents.
13	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{(a-x^n)}}$	$\frac{2}{n}\times$ arc to sin. $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n}\times$ arc to vers. $\frac{2x^n}{a}$
14	$\frac{x^{-1}\dot{x}}{\sqrt{(a\pm x^n)}}$	$\frac{1}{n\sqrt{a}}$ log. of $\frac{\sqrt{a}-\sqrt{(a\pm x^n)}}{\sqrt{a}+\sqrt{(a\pm x^n)}}$
15	$\frac{x^{-1}\dot{x}}{\sqrt{-a+x^n}}$	$\frac{2}{n\sqrt{a}}\times$ arc to secant $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n\sqrt{a}}\times$ arc to cosin. $\frac{2a-x^n}{x^n}$
16	$\dot{x}\sqrt{dx-x^2}$	$\frac{1}{2}$ circ. seg. to diam. d and vers. x
17	$2\dot{x}\sqrt{(a^2-x^2)}$	circ. zone, rad. a, and height from centre x.
18	$c^{nx}\dot{x}$	$\frac{c^{nx}}{n\ \log.\ c}$
19	$\dot{x}y^x \log.\ y+xy^{x-1}\dot{y}$	y^x
20	$x^{\frac{1}{2}}\dot{x}\sqrt{(bx\pm a)}$	$+\frac{x^{\frac{1}{2}}(2bx\pm a)\sqrt{(bx\pm a)}}{4b}-\frac{a^2}{4b\sqrt{b}}+\log.\left\{\sqrt{bx}+\sqrt{(bx\pm a)}\right\}$
21	$x^{\frac{1}{2}}\dot{x}\sqrt{(a-bx)}$	$+\frac{x^{\frac{1}{2}}(2bx-a)\sqrt{(a-bx)}}{4b}+\frac{a^2}{4b\sqrt{b}}\times$ arc. tang. $\sqrt{\frac{bx}{a-bx}}$
22	$\frac{\dot{x}\sqrt{(bx\pm a)}}{x^{\frac{1}{2}}}$	$+x^{\frac{1}{2}}\sqrt{(bx\pm a)}\pm\frac{a}{\sqrt{b}}\times\log.\left\{\sqrt{bx}+\sqrt{(bx\pm a)}\right\}$
23	$\frac{\dot{x}\sqrt{(a-bx)}}{x^{\frac{1}{2}}}$	$+x^{\frac{1}{2}}\sqrt{(a-bx)}+\frac{a}{\sqrt{b}}\times$ arc. tang. $\sqrt{\frac{bx}{a-bx}}$
24	$\frac{\dot{x}\sqrt{(a\pm bx)}}{x}$	$2\sqrt{(a\pm bx)}-2a^{\frac{1}{2}}\times\log.\frac{\sqrt{a}+\sqrt{(a\pm bx)}}{\sqrt{x}}$
25	$\frac{\dot{x}\sqrt{(bx-a)}}{x}$	$2\sqrt{(bx-a)}-2a^{\frac{1}{2}}\times$ arc. tang. $\sqrt{\frac{bx-a}{a}}$

Forms.	Fluxions.	Fluents.
26	$\frac{\dot{x}\sqrt{(a\pm bx^2)}}{x}$	$=+\sqrt{(a\pm bx^2)}-a^{\frac{1}{2}}\times\log.\ \frac{\sqrt{a}+\sqrt{(a\pm bx^2)}}{x}.$
27	$\frac{\dot{x}\sqrt{(bx^2-a)}}{x}$	$=+\sqrt{(bx^2-a)}-a^{\frac{1}{2}}\times\text{arc. tang. }\sqrt{\frac{bx^2-a}{a}}.$
28	$\frac{\dot{x}}{a+bx+cx^2}$	$=\frac{2}{\sqrt{(4ac-b^2)}}\times\text{arc. tan. }\frac{b+2cx}{\sqrt{(4ac-b^2)}}.$
29	$\frac{\dot{x}}{a+bx-cx^2}$	$=\frac{2}{\sqrt{(4ac+b^2)}}\times\log.\ \frac{\sqrt{(4ac+b^2)}-(b-2cx)}{\sqrt{(a+bx-cx^2)}}.$
30	$\frac{\dot{x}}{x(a+bx+cx^2)}$	$=\left\{\begin{array}{l}-\frac{1}{a}\times\log.\ \frac{\sqrt{(a+bx+cx^2)}}{x}.\\ -\frac{b}{a\sqrt{(4ac-b^2)}}\times\text{arc. tan. }\frac{b+2cx}{\sqrt{(4ac-b^2)}}.\end{array}\right.$
31	$\frac{\dot{x}}{x(a+bx-cx^2)}$	$=\left\{\begin{array}{l}-\frac{1}{a}\times\log.\ \frac{\sqrt{(a+bx-cx^2)}}{x}.\\ -\frac{b}{a\sqrt{(4ac+b^2)}}\times\log.\ \frac{\sqrt{(4ac+b^2)}-(b-2cx)}{\sqrt{(a+bx-cx^2)}}.\end{array}\right.$
32	$\frac{x\dot{x}}{b+bx+cx^2}$	$=\left\{\begin{array}{l}+\frac{1}{2c}\times\log.\ (a+bx+cx^2)\\ -\frac{b}{c\sqrt{(4ac-b^2)}}\times\text{arc. tan. }\frac{b+2cx}{\sqrt{(4ac-b^2)}}\end{array}\right.$
33	$\frac{x\dot{x}}{a+bx-cx^2}$	$=\left\{\begin{array}{l}-\frac{1}{2c}\times\log.\ (a+bx-cx^2)\\ +\frac{b}{c\sqrt{(4ac+b^2)}}\times\log.\ \frac{\sqrt{(4ac+b^2)}-(b-2cx)}{\sqrt{(a+bx-cx^2)}}.\end{array}\right.$
34	$\dot{x}\sqrt{(a+bx+cx^2)}$	$=\left\{\begin{array}{l}+\frac{(2cx+b)\sqrt{(a+bx+cx^2)}}{4c}+\frac{4ac-b^2}{8c\sqrt{c}}\times\\ \log.\ \left\{2cx+b+2c^{\frac{1}{2}}\sqrt{(a+bx+cx^2)}\right\}.\end{array}\right.$
35	$\dot{x}\sqrt{(a+bx-cx^2)}$	$=\left\{\begin{array}{l}+\frac{(2cx-b)\sqrt{(a+bx-cx^2)}}{4}+\frac{4ac+b^2}{8c\sqrt{c}}\times\\ \text{arc. tan. }\frac{2cx-b}{2c^{\frac{1}{2}}\sqrt{(a+bx-cx^2)}}.\end{array}\right.$
36	$\frac{(A+Bx)\dot{x}}{a+bx+cx^2}$	$=\left\{\begin{array}{l}+\frac{B}{2c}\times\log.\ (a+bx+cx^2)\\ +\frac{2cA-bB}{c\sqrt{(4ac-b^2)}}\times\text{arc. tan. }\frac{b+2cx}{\sqrt{(4ac-b^2)}}.\end{array}\right.$

Forms.	Fluxions.	Fluents.
37	$\frac{(\text{A}+\text{B}x)\dot{x}}{a+bx-cx^2}$	$=\begin{cases}-\frac{\text{B}}{2c}\times\log.(a+bx-cx^2)\\ +\frac{2c\text{A}+b\text{B}}{c\sqrt{(4ac+b^2)}}\times\log.\frac{\sqrt{(4ac+b^2)}-(b-2cx)}{\sqrt{(a+bx-cx^2)}}.\end{cases}$
38	$\frac{\dot{x}}{\sqrt{(a+bx+cx^2)}}$	$=\begin{cases}+\frac{1}{\sqrt{c}}\times\log.\{2cx+b+2c^{\frac{1}{2}}\\ \sqrt{(a+bx+cx^2)}\}.\end{cases}$
39	$\frac{\dot{x}}{\sqrt{(a+bx-cx^2)}}$	$=\frac{1}{\sqrt{c}}\times\text{arc. tan.}\frac{2cx-b}{2c^{\frac{1}{2}}\sqrt{(a+bx-cx^2)}}.$
40	$\frac{\dot{x}}{x\sqrt{(a+bx+cx^2)}}$	$=-\frac{1}{\sqrt{a}}\times\log.\left\{\frac{2a+bx+2a^{\frac{1}{2}}\sqrt{(a+bx+cx^2)}}{x}.\right\}$
41	$\frac{\dot{x}}{x\sqrt{(-a+bx+cx^2)}}$	$=+\frac{1}{\sqrt{a}}\times\text{arc. tan.}\frac{2a^{\frac{1}{2}}\sqrt{(-a+bx+cx^2)}}{2a-bx}.$

Note. The logarithms, in the above forms, are the hyperbolic ones, which are found by multiplying the common logarithms by 2·302585092994. And the arcs, whose sine, or tangent, &c. are mentioned, have the radius 1, and are those in the common tables of sines, tangents, and secants. Also the numbers m, n, &c., are to be some real quantities, as the forms fail when $m = 0$, or $n = 0$, &c.

The Use of the Foregoing Table of Forms of Fluxions and Fluents.

48. In using the foregoing table, it is to be observed, that the first column serves only to show the number of the form; in the second column are the several forms of fluxions, which are of different kinds or classes; and in the third or last column, are the corresponding fluents.

The method of using the table, is this. Having any fluxion given, to find its fluent: First, Compare the given fluxion with the several forms of fluxions in the second column of the table, till one of the forms be found that agrees with it; which is done by comparing the terms of the given fluxion with the like parts of the tabular fluxion, namely, the radical quantity of the one, with that of the other; and

the exponents of the variable quantities of each, both within and without the vinculum; all which, being found to agree or correspond, will give the particular values of the general quantities in the tabular form; then substitute these particular values in the general or tabular form of the fluent, and the result will be the particular fluent of the given fluxion; after it is multiplied by any co-efficient the proposed fluxion may have.

EXAMPLES.

Exam. 1. To find the fluent of the fluxion $3x^{\frac{5}{3}}\dot{x}$.

This is found to agree with the first form. And, by comparing the fluxions, it appears that $x = x$, and $n - 1 = \frac{5}{3}$, or $n = \frac{8}{3}$; which being substituted in the tabular fluent, or $\frac{1}{n}x^n$, gives, after multiplying by 3 the co-efficient, $3 \times \frac{3}{8}x^{\frac{8}{3}}$,

or $\frac{9}{8}x^{\frac{8}{3}}$, for the fluent sought.

Exam. 2. To find the fluent of $5x^2\dot{x}\sqrt{c^3-x^3}$, or $5x^2\dot{x}(c^3-x^3)^{\frac{1}{2}}$.

This fluxion, it appears, belongs to the 2d tabular form: for $a = c^3$, and $- x^n = - x^3$, and $n = 3$ under the vinculum, also $m - 1 = \frac{1}{2}$, or $m = \frac{3}{2}$, and the exponent $^{n-1}$ of x^{n-1} without the vinculum, by using 3 for n, is $n - 1 = 2$, which agrees with x^2 in the given fluxion: so that all the parts of the form are found to correspond. Then substituting these values into the general fluent, $-\frac{1}{mn}(a-x^n)^m$,

it becomes $- \frac{5}{3} \times \frac{2}{3}(c^3-x^3)^{\frac{3}{2}} = - \frac{10}{9}(c^3-x^3)^{\frac{3}{2}}$.

Exam. 3. To find the fluent of $\frac{x^2\dot{x}}{1+x^3}$.

This is found to agree with the 8th form; where - - - $\pm x^n = + x^3$ in the denominator, or $n = 3$; and the numerator x^{n-1} then becomes x^2, which agrees with the numerator in the given fluxion; also $a = 1$. Hence then, by substituting in the general or tabular fluent, $\frac{1}{n}$ log. of $a+x^n$, it becomes $\frac{1}{3}$ log. $1 + x^3$.

Exam. 4. To find the fluent of $ax^4\dot{x}$.

Exam. 5. To find the fluent of $2(10+x^2)^{\frac{2}{3}}x\dot{x}$.

Exam. 6. To find the fluent of $\frac{a\dot{x}}{(c^2+x^2)^{\frac{3}{2}}}$.

EXAM. 7. To find the fluent of $\frac{3x^2\dot{x}}{(a-x)^4}$.

EXAM. 8. To find the fluent of $\frac{c^2-x^2}{x^5}\dot{x}$.

EXAM. 9. To find the fluent of $\frac{1+3x}{2x^4}\dot{x}$.

EXAM. 10. To find the fluent of $(\frac{3\dot{x}}{x}+\frac{2\dot{y}}{y})x^3y^2$.

EXAM. 11. To find the fluent of $(\frac{\dot{x}}{x}+\frac{\dot{y}}{3y})xy^{\frac{1}{3}}$.

EXAM. 12. To find the fluent of $\frac{3\dot{x}}{ax}$ or $\frac{3}{a}x^{-1}\dot{x}$.

EXAM. 13. To find the fluent of $\frac{a\dot{x}}{3-2x}$.

EXAM. 14. To find the fluent of $\frac{3\dot{x}}{2x-x^2}$ or $\frac{3x^{-1}\dot{x}}{2-x}$.

EXAM. 15. To find the fluent of $\frac{2\dot{x}}{x-3x^3}$ or $\frac{2x^{-1}\dot{x}}{1-3x^2}$.

EXAM. 16. To find the fluent of $\frac{3x\dot{x}}{1-x^4}$.

EXAM. 17. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2-x^5}$.

EXAM. 18. To find the fluent of $\frac{2x\dot{x}}{1+x^4}$.

EXAM. 19. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2+x^5}$.

EXAM. 20. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1+x^4}}$.

EXAM. 21. To find the fluent of $\frac{a\dot{x}}{\sqrt{x^2-4}}$.

EXAM. 22. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1-x^4}}$.

EXAM. 23. To find the fluent of $\frac{a\dot{x}}{\sqrt{4-x^2}}$.

EXAM. 24. To find the fluent of $\frac{2x^{-1}\dot{x}}{\sqrt{1-x^2}}$.

Exam. 25. To find the fluent of $\dfrac{a\dot{x}}{\sqrt{ax^2+x\frac{1}{2}}}$.

Exam. 26. To find the fluent of $\dfrac{2x^{-1}\dot{x}}{\sqrt{x^2-1}}$.

Exam. 27. To find the fluent of $\dfrac{a\dot{x}}{\sqrt{(x^{\frac{1}{2}}-ax)^2}}$

Exam. 28. To find the fluent of $2\dot{x}\sqrt{2x-x^2}$.

Exam. 29. To find the fluent of $a^x\dot{x}$.

Exam. 30. To find the fluent of $3a^{2x}\dot{x}$.

Exam. 31. To find the fluent of $3z^x\dot{x}\log. z+3xz^{x-1}\dot{z}$.

Exam. 32. To find the fluent of $(1+x^3)x\dot{x}$.

Exam. 33. To find the fluent of $(2+x^4)x^{\frac{3}{2}}\dot{x}$.

Exam. 34. To find the fluent of $x^2\dot{x}\sqrt{(a^2+x^2)}$.

To find Fluents by Infinite Series.

49. When a given fluxion, whose fluent is required, is so complex, that it cannot be made to agree with any of the forms in the foregoing table of cases, nor made out from the general rules before given; recourse may then be had to the method of infinite series; which is thus performed:

Expand the radical or fraction, in the given fluxion, into an infinite series of simple terms, by the methods given for that purpose in books of algebra, viz. either by division or extraction of roots, or by the binomial theorem, &c.; and multiply every term by the fluxional letter, and by such simple variable factor as the given fluxional expression may contain. Then take the fluent of each term separately, by the foregoing rules, connecting them all together by their proper signs; and the series will be the fluent sought, after it is multiplied by any constant factor or co-efficient which may be contained in the given fluxion expression.

50. It is to be noted, however, that the quantities must be so arranged, as that the series produced may be a converging one, rather than diverging: and this is effected by placing the greater terms foremost in the given fluxion. When these are known or constant quantities, the infinite series will be an ascending one; that is, the powers of the variable quantity will ascend or increase; but if the variable quantity be set foremost, the infinite series produced will be a descending one, or the powers of that quantity will de-

crease always more and more in the succeeding terms, or increase in the denominators of them, which is the same thing.

For example, to find the fluent of $\frac{1-x}{1+x-x^2}\dot{x}$.

Here, by dividing the numerator by the denominator, the proposed fluxion becomes $\dot{x} - 2x\dot{x} + 3x^2\dot{x} - 5x^3\dot{x} + 8x^4\dot{x} -$ &c.; then the fluents of all the terms being taken, give $x - x^2 + x^3 - \frac{5}{4}x^4 + \frac{8}{5}x^5 -$ &c. for the fluent sought.

Again, to find the fluent of $\dot{x}\sqrt{1-x^2}$.

Here, by extracting the root, or expanding the radical quantity $\sqrt{1-x^2}$, the given fluxion becomes - - - - - - $\dot{x} - \frac{1}{2}x^2\dot{x} - \frac{1}{8}x^4\dot{x} - \frac{1}{16}x^6\dot{x} -$ &c. Then the fluents of all the terms, being taken, give $x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 -$ &c. for the fluent sought.

OTHER EXAMPLES.

EXAM. 1. To find the fluent of $\frac{bx\dot{x}}{a-x}$ both in an ascending and descending series.

EXAM. 2. To find the fluent of $\frac{b\dot{x}}{a+x}$ in both series.

EXAM. 3. To find the fluent of $\frac{3\dot{x}}{(a+x)^2}$.

EXAM. 4. To find the fluent of $\frac{1-x^2+2x^4}{1+x-x^2}\dot{x}$.

EXAM. 5. Given $\dot{z} = \frac{b\dot{x}}{a^2+x^2}$, to find z.

EXAM. 6. Given $\dot{z} = \frac{a^2+x^2}{a+x}\dot{x}$ to find z.

EXAM. 7. Given $\dot{z} = 3\dot{x}\sqrt{a+x}$, to find z.

EXAM. 8. Given $\dot{z} = 2\dot{x}\sqrt{a^2+x^2}$, to find z.

EXAM. 9. Given $\dot{z} = 4\dot{x}\sqrt{a^2-x^2}$, to find z.

EXAM. 10. Given $\dot{z} = \frac{5a\dot{x}}{\sqrt{x^2-a^2}}$, to find z.

EXAM. 11. Given $\dot{z} = 2\dot{x}\sqrt[3]{a^3-x^3}$, to find z.

EXAM. 12. Given $\dot{z} = \frac{3a\dot{x}}{\sqrt{ax-xx}}$, to find z.

EXAM. 13. Given $\dot{z} = 2\dot{x}\sqrt[3]{x^3+x^4+x^5}$, to find z.

EXAM. 14. Given $\dot{z} = 5\dot{x}\sqrt{ax-xx}$, to find z.

To Correct the Fluent of any given Fluxion.

51. The fluxion found from a given fluent is always perfect and complete; but the fluent found from a given fluxion is not always so; as it often wants a correction, to make it contemporaneous with that required by the problem under consideration, &c.: for, the fluent of any given fluxion, as $\dot{x}$, may be either x, which is found by the rule, or it may be $x + c$, or $x - c$, that is x plus or minus some constant quantity c; because both x and $x \pm c$ have the same fluxion $\dot{x}$, and the finding of the constant quantity c, to be added or subtracted with the fluent as found by the foregoing rules, is called *correcting* the fluent.

Now this correction is to be determined from the nature of the problem in hand, by which we come to know the relation which the fluent quantities have to each other at some certain point or time. Reduce, therefore, the general fluential equation, supposed to be found by the foregoing rules, to that point or time; then if the equation be true, it is correct; but if not, it wants a correction; and the quantity of the correction, is the difference between the two general sides of the equation when reduced to that particular point. Hence the general rule for the correction is this:

Connect the constant, but indeterminate, quantity c, with one side of the fluential equation, as determined by the foregoing rules; then, in this equation, substitute for the variable quantites, such values as they are known to have at any particular state, place, or time; and then, from that particular state of the equation, find the value of c, the constant quantity of the correction.

EXAMPLES.

52. Exam. 1. To find the correct fluent of $\dot{z} = ax^3\dot{x}$.

The general fluent is $z = ax^4$, or $z = ax^4 + c$, taking in the correction c.

Now, if it be known that z and x begin together, or that z is $= 0$, when $x = 0$; then writing 0 for both x and z, the general equation becomes $0 = 0 + c$, or $= c$; so that, the value of c being 0, the correct fluents are $z = ax^4$.

But if z be $= 0$, when x is $= b$, any known quantity; then substituting 0 for z, and b for x, in the general equation, it becomes $0 = ab^4 + c$, and hence we find $c = - ab^4$; which being written for c in the general fluential equation, it becomes $z = ax^4 - ab^4$, for the correct fluents.

Or, if it be known that z is $=$ some quantity d, when x is $=$ some other quantity as b; then substituting d for z, and b for x, in the general fluential equation $z = ax^4 + c$, it becomes $d = ab^4 + c$; and hence is deduced the value of the correction, namely, $c = d - ab^4$; consequently, writing this value for c in the general equation, it becomes . . . $z = ax^4 - ab^4 + d$, for the correct equation of the fluents in this case.

53. And hence arises another easy and general way of correcting the fluents, which is this: In the general equation of the fluents, write the particular values of the quantities which they are known to have at any certain time or position; then subtract the sides of the resulting particular equation from the corresponding sides of the general one, and the remainders will give the correct equation of the fluents sought.

So, the general equation being $z = ax^4$;
write d for z, and b for x, then $d = ab^4$;
hence, by subtraction, . $z - d = ax^4 - ab^4$,
or $z = ax^4 - ab^4 + d$, the correct fluents as before.

Exam. 2. To find the correct fluents of $\dot{z} = 5x\dot{x}$; z being $= 0$ when x is $= a$.

Exam. 3. To find the correct fluents of $\dot{z} = 3\dot{x}\sqrt{a + x}$; z and x being $= 0$ at the same time.

Exam. 4. To find the correct fluent of $\dot{z} = \dfrac{2a\dot{x}}{a+x}$; supposing z and x to begin to flow together, or to be each $= 0$ at the same time.

Exam. 5. To find the correct fluent of $\dot{z} = \dfrac{2\dot{x}}{a^2 + x^2}$; supposing z and x to begin together.

OF FLUXIONS AND FLUENTS.

Art. 54. In art. 47, &c. is given a compendious table of various forms of fluxions and fluents, the truth of which it may be proper here in the first place to prove.

55. As to most of those forms indeed they will be easily proved, by only taking the fluxions of the forms of fluents, in the last column, by means of the rules before given in

art. 35 of the direct method; by which they will be found to produce the corresponding fluxions in the 2d column of the table. Thus, the 1st and 2d forms of fluents will be proved by the 1st of the said rules for fluxions: the 3d and 4th forms of the fluents by the 4th rule for fluxions: the 5th and 6th forms, by the 3d rule of fluxions: the 7th, 8th, 9th, 10th, 12th, 14th forms, by the 5th rule of fluxions; the 17th form, by the 7th rule of fluxions: the 18th form, by the 8th rule of fluxions. So that there remains only to prove the 11th, 13th, 15th, and 16th forms.

56. Now, as to the 16th form, that is proved by the 2d example in art. 67, where it appears that $\dot{x}\sqrt{(dx-x^2)}$ is the fluxion of the circular segment, whose diameter is d, and versed sine x. And the remaining three forms, viz. the 11th, 13th, and 15th, will be proved by means of the rectifications of circular arcs, in art. 100.

57. Thus, for the 11th form, it appears by that art. that the fluxion of the circular arc z, whose radius is r, and tangent t, is $\dot{z}=\frac{r^2\dot{t}}{r^2+t^2}$. Now put $t=x^{\frac{1}{2}n}$, or $t^2=x^n$, and $a=r^2$: then is $\dot{t}=\frac{1}{2}nx^{\frac{1}{2}n-1}\dot{x}$, and $r^2+t^2=a+x^n$, and $\dot{z}=\frac{r^2\dot{t}}{r^2+t^2}$ $=\frac{\frac{1}{2}anx^{\frac{1}{2}n-1}\dot{x}}{a+x^n}$; hence $\frac{x^{\frac{1}{2}n-1}\dot{x}}{a+x^n}=\frac{\dot{z}}{\frac{1}{2}an}=\frac{2}{an}\dot{z}$, and the fluent is $\frac{2z}{an}=\frac{2}{nz}\times$ arc to radius $\sqrt{a}$ and tang. $x^{\frac{1}{2}n}$, or $=\frac{2}{n\sqrt{a}}\times$ arc to radius 1 and tang. $\sqrt{\frac{x^n}{a}}$, which is the first form of the fluent in n°. XI.

58. And, for the latter form of the fluent in the same n°; because the coefficient of the former of these, viz. $\frac{2}{n\sqrt{a}}$ is double of $\frac{1}{n\sqrt{a}}$ the coefficient of the latter, therefore the arc in the latter case, must be double the arc in the former. But the cosine of double the arc, to radius 1 and tangent t, is $\frac{1-t^2}{1+t^2}$; and because $t^2=\frac{x^n}{a}$ by the former case, this substituted for t^2 in the cosine $\frac{1-t^2}{1+t^2}$ it becomes $\frac{a-x^n}{a+x^n}$, the cosine as in the latter case of the 11th form.

59. Again, for the first case of the fluent in the 13th form.

By article 100, the fluxion of the circular arc z, to radius r and sine y, is $\dot{z} = \frac{r\dot{y}}{\sqrt{(r^2-y^2)}}$ or $= \frac{\dot{y}}{\sqrt{(1-y^2)}}$ to the radius 1. Now put $y = \sqrt{\frac{x^n}{a}}$, or $y^2 = \frac{x^n}{a}$; hence $\sqrt{(1-y^2)} = \sqrt{1 - \frac{x^n}{a}} = \sqrt{\frac{1}{a}} \times \sqrt{(a-x^n)}$, and $\dot{y} = \sqrt{\frac{1}{a}} \times \frac{1}{2}nx^{\frac{1}{2}n-1}\dot{x}$; then these two being substituted in the value of $\dot{z}$, give $\dot{z}$ or $\frac{\dot{y}}{\sqrt{(1-y^2)}} = \frac{n}{2} \times \frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{(a-x^n)}}$; consequently the given fluxion $\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{(a-x^n)}}$ is $= \frac{2}{n}\dot{z}$, and therefore its fluent is $\frac{2}{n}z$, that is $\frac{2}{n} \times$ arc to sine $\sqrt{\frac{x^n}{a}}$, as in the table of forms, for the first case of form XIII.

60. And, as the coefficient $\frac{1}{n}$, in the latter case of the said form, is the half of $\frac{2}{n}$ the coefficient in the former case, therefore the arc in the latter case must be double of the arc in the former. But, by trigonometry, the versed sine of double an arc, to sine y and radius 1, is $2y^2$, and, by the former case, $2y^2 = \frac{2x^n}{a}$; therefore $\frac{1}{n} \times$ arc to the versed sine $\frac{2x^n}{a}$ is the fluent, as in the 2d case of form XIII.

61. Again, for the first case of fluent in the 15th form. By art. 100, the fluxion of the circular arc z, to radius r and secant s, is $\dot{z} = \frac{r^2\dot{s}}{s\sqrt{(s^2-r^2)}}$, or $= \frac{\dot{s}}{s\sqrt{(s^2-1)}}$ to radius 1. Now, put $s = \sqrt{\frac{x^n}{a}} = \frac{x^{\frac{1}{2}n}}{\sqrt{a}}$, or $s^2 = \frac{x^n}{a}$; hence $s\sqrt{(s^2-1)} = \frac{x^{\frac{1}{2}n}}{\sqrt{a}}\sqrt{(\frac{x^n}{a}-1)} = \frac{x^{\frac{1}{2}n}}{a}\sqrt{(x^n-a)}$, and $\dot{s} = \sqrt{\frac{1}{a}} \times \frac{1}{2}nx^{\frac{1}{2}n-1}\dot{x}$; then these two being substituted in the value of $\dot{z}$, give $\dot{z}$ or $\frac{\dot{s}}{s\sqrt{(s^2-1)}} = \frac{n\sqrt{a}}{3} \times \frac{x^{-1}\dot{x}}{\sqrt{(x^n-a)}}$; consequently the given fluxion $\frac{x^{-1}\dot{x}}{\sqrt{(x^n-a)}} = \frac{2}{n\sqrt{a}}\dot{z}$, and theref. its fluent is $\frac{2}{n\sqrt{a}}z$, that is $\frac{2}{n\sqrt{a}}$

$\times$ arc to secant $\sqrt{\frac{x^n}{a}}$, as in the table of forms, for the first case of form xv.

62. And as the coefficient $\frac{1}{n\sqrt{a}}$, in the latter case of the said form, is the half of $\frac{2}{n\sqrt{a}}$, the coefficient of the former case, therefore the arc in the latter case must be double the arc in the former. But, by trigonometry, the cosine of the double arc, to secant s and radius 1, is $\frac{2}{s^2} - 1$; and, by the former case $\frac{2}{s^2} - 1 = \frac{2a}{x^n} - 1 = \frac{2a - x^n}{x^n}$; therefore $\frac{1}{n\sqrt{a}} \times$ arc to cosine $\frac{2a - x^n}{x^n}$ is the fluent, as in the 2d case of form xv.

Or, the same fluent will be $\frac{2}{n\sqrt{a}} \times$ arc to cosine $\sqrt{\frac{a}{x^n}}$, because the cosine of an arc is the reciprocal of its secant.

63. It has been just above remarked, that several of the tabular forms of fluents are easily shown to be true, by taking the fluxions of those forms, and finding they come out the same as the given fluxions. But they may also be determined in a more direct manner, by the transformation of the given fluxions to another form. Thus, omitting the first form, as too evident to need any explanation, the 2d form is $\dot{z} = (a + x^n)^{m-1} x^{n-1} \dot{x}$, where the exponent $(n-1)$ of the unknown quantity without the vinculum, is 1 less than (n) that under the same. Here, putting $y =$ the compound quantity $a + x^n$: then is $\dot{y} = nx^{n-1}\dot{x}$, and $\dot{z} = \frac{y^{m-1}\dot{y}}{n}$; hence, by art. 88. $z = \frac{y^m}{mn} = \frac{(a+x^n)^m}{mn}$, as in the table.

64. By the above example it appears, that such form of fluxions admits of a fluent in finite terms, when the index $(n-1)$ of the variable quantity (x) without the vinculum, is less by 1 than n, the index of the same quantity under the vinculum. But it will also be found, by a like process, that the same thing takes place in such forms as $(a+x^n)^m x^{cn-1}\dot{x}$, where the exponent $(cn-1)$ without the vinculum, is 1 less than any multiple (c) of that (n) under the vinculum. And further, that the fluent, in each case, will consist of as many terms as are denoted by the integer number c; viz. of one

term when $c = 1$, of two terms when $c = 2$, of three terms when $c=3$, and so on.

65. Thus, in the general form, $\dot{z} = (a + x^n)^m x^{cn-1}\dot{x}$, putting, as before, $a + x^n = y$; then is $x^n = y - a$, and its fluxion $nx^{n-1}\dot{x} = \dot{y}$ or $x^{n-1}\dot{x} = \frac{\dot{y}}{n}$, and $x^{cn-1}\dot{x}$ or x^{cn-n} $x^{n-1}\dot{x} = \frac{1}{n}(y - a)^{c-1}\dot{y}$; also $(a + x^n)^m = y^m$: these values being now substituted in the general form proposed, give $\dot{z} = \frac{1}{n}(y-a)^{c-1}y^m\dot{y}$. Now, if the compound quantity $(y-a)^{c-1}$ be expanded by the binomial theorem, and each term multiplied by $y^m\dot{y}$, that fluxion becomes

$$\dot{z} = \frac{1}{n}(y^{m+c-1}\dot{y} - \frac{c-1}{1}ay^{m+c-2}\dot{y} + \frac{c-1}{1}\cdot\frac{c-2}{2}a^2y^{m+c-3}\dot{y} -$$

&c.); then the fluent of every term being taken by art. 36. it is

$$z = \frac{1}{n}(\frac{y^{m+c}}{m+c} - \frac{c-1}{1}\cdot\frac{ay^{m+c-1}}{m+c-1} + \frac{c-1}{1}\cdot\frac{c-2}{2}\cdot\frac{a^2y^{m+c-2}}{m+c-2} - \text{\&c.}),$$

$$= \frac{y^d}{n}(\frac{1}{n} - \frac{c-1}{d-1}\cdot\frac{a}{y} + \frac{c-1\,.\,c-2}{d-2}\cdot\frac{a^2}{2y^2} - \frac{c-1\,.\,c-2\,.\,c-3}{d-3}\cdot\frac{a^3}{2.3y^3}$$

&c.), putting $d = m + c$, for the general form of the fluent; where, c being a whole number, the multipliers $c-1$, $c-2$, $c-3$, &c., will become equal to nothing, after the first c terms, and therefore the series will then terminate, and exhibit the fluent in that number of terms; viz. there will be only the first term when $c = 1$, but the first two terms when $c = 2$, and the first three terms when $c = 3$, and so on.—Except, however, the cases in which m is some negative number equal to or less than c; in which cases the divisors, $m+c$, $m+c-1$, $m+c-2$, &c. becoming equal to nothing, before the multipliers $c-1$, $c-2$, &c. the corresponding terms of the series, being divided by 0, will be infinite: and then the fluent is said to fail, as in such case nothing can be determined from it.

66. Besides this form of the fluent, there are other methods of proceeding, by which other forms of fluents are derived, of the given fluxion $\dot{z} = (a+x^n)^m x^{cn-1}\dot{x}$, which are of use when the foregoing form fails, or runs into an infinite series; some results of which are given both by Mr. Simpson and Mr. Landen. The two following processes are after the manner of the former author.

67. The given fluxion being $(a+x^n)^m x^{cn-1}\dot{x}$; its fluent

may be assumed equal to $(a+x^n)^{m+1}$ multiplied by a general series, in terms of the powers of x combined with assumed unknown coefficients, which series may be either ascending or descending, that is, having the indices either increasing or decreasing;

viz. $(a+x^n)^{m+1} \times (ax^r + Bx^{r-s} + Cx^{r-2s} + Dx^{r-3s} + \&c.)$,

or $(a+x^n)^{m+1} \times (Ax^r + Bx^{r+s} + Cx^{r+2s} + Dx^{r+3s} + \&c.)$.

And first, for the former of these, take its fluxion in the usual way, which put equal to the given fluxion $(a+x^n)^m x^{cn-1}\dot{x}$, then divide the whole equation by the factors that may be common to all the terms; after which, by comparing the like indices and the coefficients of the like terms, the values of the assumed indices and coefficients will be determined, and consequently the whole fluent. Thus, the former assumed series in fluxions is,

$n(m+1)x^{n-1}\dot{x}\,(a+x^n)^m \times (Ax^r + Bx^{r-s} + Cx^{r-2s}, \&c.) +$
$(a+x^n)^{m+1}\dot{x} \times (rAx^{r-1} + (r-s)\,Bx^{r-s-1} + (r-2s)\,Cx^{r-2s-1}$
&c.); this being put equal to the given fluxion $(a+x^n)^m x^{cn-1}\dot{x}$; and the whole equation divided by $(a+x^n)^m x^{-1}\dot{x}$, there results

$$\left.\begin{array}{l} n(m+1x^n) \times (Ax^r + Bx^{r-s} + Cx^{r-2s} + Dx^{r-3s} + \&c.) \\ +(a+x^n) \times rAx^r + (r-s)Bx^{r-s} + (r-2s)Cx^{r-2s}\&c.) \end{array}\right\} = x^{cn}.$$

Hence, by actually multiplying, and collecting the coefficients of the like powers of x, there results

$$\left.\begin{array}{l} \left.\begin{array}{l} n(m+1) \\ +r \end{array}\right\} Ax^{r+n} \left.\begin{array}{l} +n(m+1) \\ +r-s \end{array}\right\} Bx^{r+n-s} \left.\begin{array}{l} +n(m+1) \\ +r-2s \end{array}\right\} Cx^{r+n-2s}\&c. \\ \quad -x^{cn} \ldots + \ldots\ raAx^r \ \ldots +(r-s)aBx^{r-s}\&c. \end{array}\right\} = 0.$$

Here, by comparing the greatest indices of x, in the first and second terms, it gives $r+n=cn$, and $r+n-s=r$; which give $r=(c-1)n$, and $n=s$. Then these values being substituted in the last series, it becomes

$$\left.\begin{array}{l} (c+m)nAx^{cn} + (c+m-1)nBx^{cn-n} + (c+m-2)ncx^{cn-2n}\&c. \\ \quad -x^{cn} + (c-1)na\ \ Ax^{cn-n} + (c-2)naBx^{cn-2n}\ \&c. \end{array}\right\} = 0.$$

Now, comparing the coefficients of the like terms, and putting $c+m=d$, there result these equalities:

$$A = \frac{1}{dn};\ B = -\frac{c-1.aA}{d-1} = -\frac{c-1\,.\,a}{d-1\,.\,dn};\ C = -\frac{c-2\,.\,aB}{d-2} =$$

$+\dfrac{c-1.c-2.a^2}{d-1.d-2.dn}$, &c.; which values of A, B, C, &c. with those of r and s, being now substituted in the first assumed fluent, it becomes

$$\frac{(a+x^n)^{m+1}x^{cn-n}}{dn} \times \left(\frac{1}{1} - \frac{c-1\,.\,a}{d-1\,.\,x^n} + \frac{c-1.c-2.a^2}{d-1.d-2.x^{2n}} - \right.$$

$\frac{c-1.c-2.c-3.a^3}{d-1.d-2.d-3.x^{3n}}$ + &c.), the true fluent of $(a+x^n)^m x^{cn-1}\dot{x}$, exactly agreeing with the first value of the 19th form in the table of fluents in my Dictionary. Which fluent therefore, when c is a whole positive number, will always terminate in that number of terms; subject to the same exception as in the former case. Thus, if $c = 2$, or the given fluxion be $(a+x^n)^m x^{2n-1}\dot{x}$; then $c+m$ or d being $= m+2$, the fluent becomes

$$\frac{(a+x^n)^{m+1}x^n}{(m+2)n} \times (1 - \frac{ax^{-n}}{m+1}) = \frac{(a+x^n)^{m+1}}{n} \times \frac{(m+1)x^n - a}{m+1.m+2}.$$

And if $c = 3$, or the given fluxion be $(a+x^n)^m x^{3n-1}\dot{x}$; then $m+c$ or d being $= m+3$, the fluent becomes

$$\frac{(a+x^n)^{m+1}x^{2n}}{(m+3)n} \times (1 - \frac{2ax^{-n}}{m+2} + \frac{2a^2x^{-2n}}{m+2.m+1}) = \frac{(a+x^n)^{m+1}}{n} \times$$

$$\frac{x^{2n}}{m+3} - \frac{2ax^n}{m+3\,.\,m+2} + \frac{2a^2}{m+3\,.\,m+2\,.\,m+1}.$$ And so on, when c is = other whole numbers: but, when c denotes either a fraction or a negative number, the series will then be an infinite one, as none of the multipliers $c-1$, $c-2$, $c-3$, can then be equal to nothing.

68. Again, for the latter or ascending form, $(a+x^n)^{m+1} \times (\text{A}x^r + \text{B}x^{r+s} + \text{C}x^{r+2s} + \text{D}x^{r+3s} + \&c.)$, by making its fluxion equal to the proposed one, and dividing, &c., as before, equating the two least indices, &c. the fluent will be obtained in a different form, which will be useful in many cases, when the foregoing one fails, or runs into an infinite series. Thus, if $r + s$, $r + 2s$, &c., be written instead of $r - s$, $r - 2s$, &c., respectively, in the general equation in the last case, and taking the first term of the 2d line into the first line, there results

$$\left.\begin{array}{l} -x^{cn} + \left.\begin{array}{l} n(m+1 \\ +r \end{array}\right\} \text{A}x^{r+n} + \left.\begin{array}{l} n(m+1) \\ +r+s \end{array}\right\} \text{B}x^{r+n+s} \,\&c. \\ +rax^r + (r+s)a\text{B}x^{r+s} + (r+2s)a\text{C}x^{r+2s} \,\&c. \end{array}\right\} = 0.$$

Here, comparing the two least pairs of exponents, and the coefficients, we have $r = cn$, and $s = n$; then $\text{A} = \frac{1}{ra} = \frac{1}{cna}$;

$\text{B} = -\frac{r+n(m+1)}{a(r+s)}$; $\text{A} = -\frac{c+m+1}{c+1} \cdot \frac{\text{A}}{a} = -\frac{c+m+1}{(c+1)cna^2}$;

$\text{C} = -\frac{c+m+2}{(c+2)a}\text{B} = +\frac{c+m+1\,.\,c+m+2}{c\,.\,c+1\,.\,c+2\,.\,na^3}$ &c. Therefore, denoting $c + m$ by d, as before, the fluent of the same fluxion $(a + x^n)^m x^{cn-1}\dot{x}$, will also be truly expressed by

$$\frac{(a+x^n)^{m+1}x^{cn}}{cna} \times \left(\frac{1}{1} - \frac{d+1 \,.\, x^n}{c+1 \,.\, a} + \frac{d+1 \,.\, d+2 \,.\, x^{2n}}{c+1 \,.\, c+2 \,.\, a^2} - \&c.\right);$$

agreeing with the 2d value of the fluent of the 19th form in my Dictionary. Which series will terminate when d or $c+m$ is a negative integer; except when c is also a negative integer less than d; for then the fluent fails, or will be infinite, the divisor in that case first becoming equal to nothing.

To show now the use of the foregoing series, in some example of finding fluents, take first,

69. *Example* 1. To find the fluent of

$$\frac{6x\dot{x}}{\sqrt{(a+x)}} \text{ or } 6x\dot{x}(a+x)^{-\frac{1}{2}}.$$

This example being compared with the general form $x^{cn-1}\dot{x}(a+x^n)^m$, in the several corresponding parts of the first series, gives these following qualities: viz. $a = a$, $n = 1$, $cn - 1 = 1$, or $c - 1 = 1$, or $c = 2$; $m = -\frac{1}{2}$; $y = a + x$, $d = m + c = 2 - \frac{1}{2} = \frac{3}{2}$, $\frac{1}{n}y^d = (a+x)^{\frac{3}{2}}$, $\frac{1}{d} = \frac{2}{3}$, $\frac{c-1}{d-1}\cdot\frac{a}{y} = \frac{2a}{a+x}$; here the series ends, as all terms after this become equal to nothing, because the following terms contain the factor $c - 2 = 0$. These values then being substituted in $\frac{y^d}{n}\left(\frac{1}{d} - \frac{c-1}{d-1}\cdot\frac{a}{y}\right)$, it becomes $(a+x)^{\frac{3}{2}} \times$

$$\left(\frac{2}{3} - \frac{2a}{a+x}\right) = \left(\frac{2a+2x}{3} - 2a\right) \times (a+x)^{\frac{1}{2}} = \frac{2x-4a}{3}\sqrt{(a+x)};$$

which multiplied by 6, the given coefficient in the proposed example, there results $(4x-8a) \,.\, \sqrt{(a+x)}$, for the fluent required.

70. *Exam.* 2. To find the fluent of

$$\frac{3\dot{x}\sqrt{(a^2+x^2)}}{x^6} = (a^2+x^2)^{\frac{1}{2}} \times 3x^{-6}\dot{x}.$$

The several parts of this quantity being compared with the corresponding ones of the general form, give $a = a^2$, $n = 2$, $m = \frac{1}{2}$, $cn - 1$ or $2c - 1 = -6$, whence $c = \frac{1-6}{2} = -\frac{5}{2}$, and $d = m + c = \frac{1}{2} - \frac{5}{2} = -\frac{4}{2} = -2$, which being a negative integer, the fluent will be obtained by the 3d or last form of series; which, on substituting these values of the letters,

gives $\frac{3(a^2+x^2)^{\frac{3}{2}}x^{-5}}{-5a^2} \times \left(\frac{1}{1} - \frac{-1 \,.\, x^2}{-\frac{3}{2}a^2}\right) = \frac{3(a^2+x^2)^{\frac{3}{2}}}{-5a^2x^5}$ . . .

$\times\,(2-\frac{2x^2}{3a^2})=\frac{(a^2+x^2)^{\frac{3}{2}}}{x^5}\times\frac{2x^2-3a^2}{5a^4}$, for the required fluent of the proposed fluxion.

71. *Exam.* 3. Let the fluxion proposed be

$$\frac{5x^{3n-1}\dot{x}}{\sqrt{(b+x^n)}}=5(b+x)^{-\frac{1}{2}}x^{3n-1}\dot{x}.$$

Here, by proceeding as before, we have $a=b$, $n=n$, $m=-\frac{1}{2}$, $c=3$, and $d=c+m=\frac{5}{2}$; whence c being a positive integer, this case belongs to the 2d series; into which therefore the above values being substituted, it becomes

$$\frac{5(b+x^n)^{\frac{1}{2}}x^{2n}}{\frac{5}{2}n}\times(\frac{1}{1}-\frac{2b}{\frac{3}{2}x^n}+\frac{2.1.b^2}{\frac{3}{2}.\frac{1}{2}x^{2n}})=2\sqrt{(b+x^n)}\ \ldots\ \times\frac{3x^{2n}-4bx^n+8b^2}{3n}.$$

72. *Exam.* 4. Let the proposed fluxion be $5(\frac{1}{3}-z^2)^{\frac{1}{2}}z^{-8}\dot{z}$.

Here, proceeding as above, we have $a=\frac{1}{3}$, $n=2$, $m=\frac{1}{2}$, $cn-1$ or $2c-1=-8$, and $c=-\frac{7}{2}$, $x=-z$, $d=c+m=-3$, which being a negative integer, the case belongs to the 3d or last series: which therefore, by substituting these values, becomes $\frac{5(\frac{1}{3}-z^2)^{\frac{3}{2}}}{-7.\frac{1}{2}z^7}\times(\frac{1}{1}+\frac{-2z^2}{-\frac{5}{2}.\frac{1}{3}}+\frac{-2\,.\,-1\,.\,z^4}{-\frac{5}{2}\,.\,-\frac{3}{2}\,.\,\frac{1}{9}})=$

$$\frac{15(\frac{1}{3}-z^2)^{\frac{3}{2}}}{-7z^7}+(1+\frac{12z^2}{5}+\frac{24z^4}{5})=\frac{-3\frac{1}{3}-z^{2\frac{3}{2}}}{7z^7}\ \ldots\ \times(5+12z^2+24z^4),$$

the true fluent of the proposed fluxion. And thus may many other similar fluents be exhibited in finite terms, as in these following examples for practice.

Ex. 5. To find the fluent of $-3x^3\dot{x}\sqrt{(a^2-x^2)}$.

Ex. 6. To find the fluent of $-6x^5\dot{x}\,.\,(a^2-x^2)^{-\frac{3}{2}}$.

Ex. 7. To find the fluent of $\frac{\dot{x}\sqrt{(a-x^n)}}{x^{\frac{7}{2}n-1}}$ or $\ldots$ $(a-x^n)^{\frac{1}{2}}x^{-\frac{7n+1}{2}}\dot{x}$.

73. The case mentioned in art. 42, p. 321, viz. of compound quantities under the vinculum, the fluxion of which is in a given ratio to the fluxion without the vinculum, with only one variable letter, will equally apply when the compound quantities consist of several variables. Thus,

Example 1. The given fluxion being $(4x\dot{x}+8y\dot{y})\times\sqrt{(x^2+2y^2)}$, or $(4x\dot{x}+8y\dot{y})\times(x^2+2y^2)^{\frac{1}{2}}$, the root being x^2+2y^2, the fluxion of which is $2x\dot{x}+4y\dot{y}$. Dividing the former fluxional part by this fluxion, gives the quotient 2:

next, the exponent $\frac{1}{2}$ increased by 1, gives $\frac{3}{2}$: lastly, dividing by this $\frac{3}{2}$, there then results $\frac{4}{3}(x^2 + 2y^2)^{\frac{3}{2}}$, for the required fluent of the proposed fluxion.

Exam. 2. In like manner, the fluent of

$$(x^2 + y^4 + z^6)^{\frac{1}{3}} \times (6x\dot{x} + 12y^3\dot{y} + 18z^5\dot{z}) \text{ is}$$

$$\frac{(x^2 + y^4 + z^6)^{\frac{1}{3}+1} \times (6x\dot{x} + 12y^3\dot{y} + 18z^5\dot{z})}{(2x\dot{x} + 4y^3\dot{y} + 6z^5\dot{z}) \times \frac{4}{3}} = \tfrac{9}{4}(x^2+y^4+z^6)^{\frac{4}{3}}.$$

Exam. 3. In like manner, the fluent of

$2x^2(\dot{x}y^2 + xy\dot{y} + x^2\dot{x})\sqrt{(x^2 + 2y^2)}$, is $\frac{1}{3}(x^4 + 2x^2y^2)^{\frac{3}{2}}$.

74. The fluents of fluxions of the forms

$\dfrac{x^n\dot{x}}{x \pm a}$, $\dfrac{x^n\dot{x}}{x^2 \pm a^2}$, &c. or $\dfrac{x^{cn-1}\dot{x}}{x^n \pm an}$, &c., where c and n are whole numbers, will be found in finite terms, by dividing the numerator by the denominator, using the variable letter x as the first term in the divisor, continuing the division till the powers of x are exhausted; after which, the last remainder will be the fluxion of a logarithm, or of a circular arc, &c.

Example 1. To find the fluent of $\dfrac{x\dot{x}}{a+x}$ or $\dfrac{x\dot{x}}{x+a}$.

By division, $\dfrac{x\dot{x}}{x+a} = \dot{x} - \dfrac{a\dot{x}}{x+a}$, where the remainder $\dfrac{a\dot{x}}{x+a}$ is evidently $= a \times$ the fluxion of the hyperbolic logarithm of $a + x$: therefore the whole fluent of the proposed fluxion is $x - a \times$ hyp. log. of $(a + x)$. In like manner it will be found that,

Ex. 2. The fluent of $\dfrac{x\dot{x}}{x+a}$, is $x + a \times$ hyp. log. of $(x-a)$.

Ex. 3. The flu. of $\dfrac{x\dot{x}}{a-x}$, is $-x - a \times$ hyp. log. of $(a-x)$.

Ex. 4. The flu. of $\dfrac{x^2\dot{x}}{a+x}$, is $\frac{1}{2}x^2 - ax + a^2 \times$ hyp. log. $(a+x)$.

Ex. 5. The flu. of $\dfrac{x^2\dot{x}}{a-x}$, is $-\frac{1}{2}x^2 - ax - a^2 \times$ h. l. of $(a-x)$.

Ex. 6. The flu. of $\dfrac{x^2\dot{x}}{x-a}$, is $\frac{1}{2}x^2 + ax + a^2 \times$ h. l. $(x-a)$.

Ex. 7. The fluent of $\dfrac{x^3\dot{x}}{x+a}$, is

$\frac{1}{3}x^3 - \frac{1}{2}ax^2 + a^2x - a^3 \times$ hyp. log. of $(x + a)$.

Ex. 8. The fluent of $\dfrac{x^3\dot{x}}{x-a}$, is

$\frac{1}{3}x^3 + \frac{1}{2}ax^2 + a^2x + a^3 +$ hyp. log. of $(x - a)$.

Ex. 9. The fluent of $\frac{x^3\dot{x}}{a-x}$, is

$-\frac{1}{3}x^3 - \frac{1}{2}ax^2 - a^2x + a^3 \times$ hyp. log. of $(a - x)$.

Ex. 10. The fluent of $\frac{x^4\dot{x}}{a+x}$, is

$\frac{1}{4}x^4 - \frac{1}{3}ax^3 + \frac{1}{2}a^2x^2 - a^3x + a^4 \times$ hyp. log. $(a + x)$.

Ex. 11. The fluent of $\frac{x^n\dot{x}}{a+x}$, is

$$\frac{x^n}{n} - \frac{ax^{n-1}}{n-1} + \frac{a^2x^{n-2}}{n-2} - \frac{a^3x^{n-3}}{n-3} + \text{\&c.} \pm a^n \times \text{h. l.} (a + x).$$

Ex. 12. The fluent of $\frac{x^n\dot{x}}{a-x}$, is

$$-\frac{x^n}{n} - \frac{ax^{n-1}}{n-1} - \frac{a^2x^{n-2}}{n-2} - \frac{a^3x^{n-3}}{n-3} \text{ \&c.} - a^n \times \text{h. l.} (a - x).$$

Ex. 13. The fluent of $\frac{x^n\dot{x}}{x-a}$, is

$$\frac{x^n}{n} + \frac{ax^{n-1}}{n-1} + \frac{a^2x^{n-2}}{n-2} + \frac{a^3x^{n-3}}{n-3} \text{ \&c.} + a^n \times \text{h. l.} (x - a).$$

Ex. 14. The fluent of $\frac{x^2\dot{x}}{x^2+a^2}$ = (by division) $\dot{x} - \frac{a^2\dot{x}}{x^2 + a^2}$, is, (by form 11) $x -$ cir. arc of radius a and tang. x, or $x - \frac{1}{2}a \times$ cir. arc of rad. 1 and cosine $\frac{a^2-x^2}{a^2+x^2}$. In like manner,

Ex. 15. The fluent of $\frac{x^2\dot{x}}{a^2-x^2}$, or of $-\dot{x} + \frac{a^2\dot{x}}{a^2-x^2}$, is $-x + \frac{1}{2}a \times$ h. l. $\frac{a+x}{a-x}$, by form 10. And

Ex. 16. The fluent of $\frac{x^2\dot{x}}{x^2-a^2} = \dot{x} + \frac{a^2\dot{x}}{x^2-a^2}$, is $x + \frac{1}{2}a \times$ hyp. log. $\frac{x-a}{x+a}$, by the same form.

75. In like manner for the fluents of $\frac{x^4\dot{x}}{a^2 \pm x^2}$. Thus,

Ex. 17. The fluent of $\frac{x^4\dot{x}}{a^2 + x^2} = x^2\dot{x} - a^2\dot{x} + \frac{a^4\dot{x}}{a^2+x^2}$, is (by form 11), $\frac{1}{3}x^3 - a^2x + a^2 \times$ cir. arc to rad. a and tang. x,

or $\frac{1}{3}x^3 - a^2x + \frac{1}{2}a^3 \times$ cir. arc to rad. 1 and cosine $\frac{a^2 - x^2}{a^2 + x^2}$. And

Ex. 18. The fluent of $\frac{x^4\dot{x}}{a^2 - x^2} = -x^2\dot{x} - a^2\dot{x} + \frac{a^4\dot{x}}{a^2 - x^2}$,

is $-\frac{1}{3}x^3 - a^2x + \frac{1}{2}a^3 \times$ hyp. log. $\frac{a+x}{a-x}$, by form 10. Also

Ex. 19. The fluent of $\frac{x^4\dot{x}}{x^2 - a^2} = x^2\dot{x} + a^2\dot{x} + \frac{a^4\dot{x}}{x^2 - a^2}$,

is $\frac{1}{3}x^3 + a^2x + \frac{1}{2}a^3 \times$ hyp. log. $\frac{x - a}{x + a}$, by form 10.

76. And in general for the fluent of $\frac{x^n\dot{x}}{x^2 \pm a^2}$, where n is any even positive number, by dividing till the powers of x in the numerator are exhausted, the fluents will be found as before. And first for the denominator $x^2 + a^2$, as in

Ex. 20. For the fluent of $\frac{x^n\dot{x}}{x^2 + a^2} =$ (by actual division) $x^{n-2}\dot{x} - a^2x^{n-4}\dot{x} + a^4x^{n-6} -$ &c. $\pm a^{n-2}\dot{x} \mp \frac{a^n\dot{x}}{x^2 + a^2}$; the number of terms in the quotient being $\frac{1}{2}n$, and the remainder $\mp \frac{a^4\dot{x}}{x^2 + a^2}$, viz. $-$ or $+$ according as that number of terms is odd or even. Hence, as before, the fluent

is $\frac{x^{n-1}}{n-1} - \frac{a^2x^{n-3}}{n-3} +$ &c. . . . $\pm a^{n-2}x \mp a^{n-2} \times$ arc to rad. a and tan. x, or $\frac{x^{n-1}}{n-1} - \frac{a^2x^{n-3}}{n-3} +$ &c. . . . $\pm a^{n-2}x \mp \frac{1}{2}a^{n-1}$ $\times$ arc to rad. 1 and cos. $\frac{a^2 - x^2}{a^2 + x^2}$.

Ex. 21. In like manner, the fluent of $\frac{x^n\dot{x}}{a^2 - x^2}$, is

$$-\frac{x^{n-1}}{n-1} - \frac{a^2x^{n-3}}{n-3} - \frac{a^4x^{n-5}}{n-5} - \&c. + \tfrac{1}{2}a^{n-1} \times \text{hyp. log.} \frac{a + x}{a - x}.$$

Ex. 22. And of $\frac{x^n\dot{x}}{x^2 - a^2}$, is

$$\frac{x^{n-1}}{n-1} + \frac{a^2x^{n-3}}{n-3} + \&c. + \tfrac{1}{2}a^{n-1} \times \text{hyp. log.} \frac{x - a}{x + a}.$$

77. In a similar manner we are to proceed for the fluents of

$\frac{x^n\dot{x}}{a^2 \mp x^2}$, when n is any odd number, by dividing by the denominator inverted, till the first power of x only be found in the remainder, and when of course there will be one term less in the quotient than in the foregoing case, when n was an even number; but in the present case the log. fluent of the remainder will be found by the 8th form in the table of fluents.

Ex. 23. Thus, for the fluent of $\frac{x^n\dot{x}}{x^2+a^2}$, where n is an odd number, the quotient by division as before, is $x^{n-2}\dot{x} - a^2x^{n-4}\dot{x} + a^4x^{n-6}\dot{x}$ — &c. $\pm\ a^{n-3}x\dot{x}$, the number of terms being $\frac{m-1}{2}$, and the remainder $\mp \frac{a^{n-1}x\dot{x}}{x^2+a^2}$. Therefore the fluent is

$$\frac{x^{n-1}}{n-1} - \frac{a^2x^{n-3}}{n-3} + \&c. \ldots \pm \frac{a^{n-3}x^2}{2} \mp \tfrac{1}{2}a^{n-1} \times \text{h. l. } x^2 + a^2.$$

Ex. 24. The fluent of $\frac{x^n\dot{x}}{x^2-a^2}$ is obtained in the same manner, and has the same terms, but the signs are all positive, and the remainder is $+\ \tfrac{1}{2}a^{n-1} \times$ hyp. log. $x^2 - a^2$.

Ex. 25. Also the fluent of $\frac{x^n\dot{x}}{a^2-x^2}$ is still the same, but the signs are all negative, and the remainder is $-\ \tfrac{1}{2}a^{n-1} \times$ hyp. log. a^2-x^2. Hence also,

Ex. 26. The fluent of $\frac{x^3\dot{x}}{x^2+a^2}$,

is $\tfrac{1}{2}x^2 - \tfrac{1}{2}a^2 \times$hyp. log. of $x^2 + a^2$.

Ex. 27. The fluent of $\frac{x^3\dot{x}}{x^2-a^2}$,

is $\tfrac{1}{2}x^2 + \tfrac{1}{2}a^2 \times$ hyp. log. of $x^2 - a^2$.

Ex. 28. The fluent of $\frac{x^3\dot{x}}{a^2-x^2}$,

is $-\ \tfrac{1}{2}x^2 - \tfrac{1}{2}a^2 \times$ hyp. log. of $a^2 - x^2$.

Ex. 29. The fluent of $\frac{x^5\dot{x}}{x^2+a^2}$,

is $\tfrac{1}{4}x^4 - \tfrac{1}{2}a^2x^2 + \tfrac{1}{2}a^4 \times$ hyp. $x^2 + a^2$.

Ex. 30. The fluent of $\frac{x^5\dot{x}}{x^2-a^2}$,

is $\tfrac{1}{4}x^4 + \tfrac{1}{2}a^2x^2 + \tfrac{1}{2}a^4 \times$ hyp. log. $x^2 - a^2$.

Ex. 31. The fluent of $\frac{x^5\dot{x}}{a^2-x^2}$,

is $-\frac{1}{4}x^4-\frac{1}{2}a^2x^2-\frac{1}{2}a^4\times$ hyp. log. a^2-x^2.

78. *Ex.* 32. In a similar manner may be found the fluents of $\frac{x^{cn-1}\dot{x}}{x^n\mp a^n}$, where c is any whole positive number, by dividing till the remainder be $\frac{a^{(c-1)}x^{n-1}\dot{x}}{x^n\mp a^n}$, which can always be done, and the fluent of that remainder will be had by the 8th form. Thus, by dividing first by x^n+a^n, the terms are, $x^{cn-n-1}\dot{x}-a^nx^{cn-2n-1}\dot{x}+a^{2n}x^{cn-3n-1}\dot{x}-+$ &c. till the last term be $a^{(d-1)n}x^{(c-d)n-1}$, and the remainder - - - - - $\frac{a^dx^{n(c-d)n-1}\dot{x}}{x^n+a^n}=\frac{a^{(c-1)n}x^{n-1}\dot{x}}{x^n+a^n}$ when d is $=c-1$, or 1 less than c, which is also the number of the terms in the quotient; and therefore the fluent is

$$\frac{x^{cn-n}}{cn-n}-\frac{a^nx^{cn-2n}}{cn-2n}+\frac{a2^nx^{cn-3n}}{cn-3n}\cdots\cdots\pm\frac{a^{(c-2)n}x^n}{n}\mp\frac{1}{n}a^{(c-1)n}\times$$

hyp. log. of x^n+a^n. In like manner,

Ex. 33. The fluent of $\frac{x^{cn-1}\dot{x}}{x^n-a^n}$ has all the same terms as the former, but their signs all $+$ or positive, and the remainder $\frac{1}{n}a^{(c-1)n}\times$hyp. log. of x^n-a^n. Also in like manner

Ex. 34. The fluent of $\frac{x^{cn-1}\dot{x}}{a^n-x^n}$ has all the very same terms, but all negative, and the remainder $-\frac{1}{n}a^{(c-1)n}\times$ hyp. log. of a^n-x^n.

Ex. 35. The fluent of $\frac{x^{cn-1}\dot{x}}{b\mp ex^n}=\frac{1}{e}\times\frac{x^{cn-1}\dot{x}}{\frac{b}{e}\mp x^n}$ is also the same with the preceding, by substituting $\frac{b}{e}$ for a^n, and multiplying the whole series by the fraction $\frac{1}{e}$.

79. When the numerator is compound, as well as the denominator, the expression may, in a similar manner by division, be reduced to like terms admitting of finite fluents. Thus, for

Ex. 36. To find the fluent of $\frac{a-bx^2}{c+dx^2} \times x\dot{x} = \frac{ax\dot{x}-bx^3\dot{x}}{c+dx^2}$.

By division this becomes $-\frac{b}{d}x\dot{x} + \frac{ad+bc}{dd} \times \frac{x\dot{x}}{\frac{c}{d}+x^2}$; and its

fluent $-\frac{b}{2d} + \frac{ad+bc}{2d^2} \times$ hyp. log. of $\frac{c}{d} + x^2$.

80. There are certain methods of finding fluents one from another, or of deducing the fluent of a proposed fluxion from another fluent previously known or found. There are hardly any general rules however that will suit all cases; but they mostly consist in assuming some quantity y in the form of a rectangle or product of two factors, which are such, that the one of them drawn into the fluxion of the other may be of the form of the proposed fluxion; then taking the fluxion of the assumed rectangle, there will thence be deduced a value of the proposed fluxion in terms that will often admit of finite fluents. The manner in such cases will better appear from the following examples.

Ex. 1. To find the fluent of $\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$.

Here it is obvious that if y be assumed $= x\sqrt{(x^2+a^2)}$, then one part of the fluxion of this product, viz. $x \times$ flux. of $\sqrt{(x^2+a^2)}$, will be of the same form as the fluxion proposed. Putting theref. the assumed rectangle $y=x\sqrt{(x^2+a^2)}$ into fluxions, it is $\dot{y} = \dot{x}\sqrt{(x^2+a^2)} + \frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$. But as the former part, viz. $\dot{x}\sqrt{(x^2+a^2)}$, does not agree with any of our preceding forms, which have been integrated, multiply it by $\sqrt{(x^2+a^2)}$, and subscribe the same as a denominator to the product, by which that part becomes

$\frac{x^2+a^2}{\sqrt{(x^2+a^2)}}\dot{x} = \frac{x^2\dot{x}+a^2\dot{x}}{\sqrt{(x^2+a^2)}}$; this united with the former part,

makes the whole $\dot{y} = \frac{2x^2\dot{x}}{\sqrt{(x^2+a^2)}} + \frac{a^2\dot{x}}{\sqrt{(x^2+a^2)}}$, hence the given

fluxion $\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{2}\dot{y} - \frac{1}{2}a^2 \times \frac{\dot{x}}{\sqrt{(x^2+a^2)}}$, and its fluent is

therefore $\frac{1}{2}y - \frac{1}{2}a^2 \times f\,\frac{\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{2}x\sqrt{(x^2+a^2)} - \frac{1}{2}a^2 \times$ hyp. log. of $x + \sqrt{(x^2+a^2)}$, by the 12th form of fluents.

Ex. 2. In like manner the fluent of $\frac{x^2\dot{x}}{\sqrt{(x^2-a^2)}}$ will be

found from that of $\frac{\dot{x}}{\sqrt{(x^2-a^2)}}$ by the same 12th form, and

is $= \frac{1}{2}x\sqrt{(x^2-a^2)} + \frac{1}{2}a^2 \times$ hyp. log. $x + \sqrt{(x^2-a^2)}$.

Ex. 3. Also in a similar manner, by the 13th form, the fluent of $\frac{x^2\dot{x}}{\sqrt{(a^2-x^2)}}$ will be found from that of $\frac{\dot{x}}{\sqrt{(a^2-x^2)}}$, and comes out $-\frac{1}{2}x\sqrt{(a^2-x^2)} + \frac{1}{2}a \times$ cir. arc to radius a and sine x.

Ex. 4. In like manner, the fluent of $\frac{x^4\dot{x}}{\sqrt{(x^2+a^2)}}$ will be found from that of $\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$. Here it is manifest that y must be assumed $= x^3\sqrt{(x^2+a^2)}$, in order that one part of its fluxion, viz, $\dot{x} \times$ flux. of $\sqrt{(x^2+a^2)}$ may agree with the proposed fluxion. Thus, by taking the fluxion, and reducing as before, the fluent of $\frac{x^4\dot{x}}{\sqrt{(x^2+a^2)}}$ will be found $=$ $\frac{1}{4}x^3\sqrt{(x^2+a^2)} - \frac{3}{4}a^2 \times f\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$.

Ex. 5. Thus also the fluent of $\frac{x^4\dot{x}}{\sqrt{(x^2-a^2)}}$, is $\frac{1}{4}x^3\sqrt{(x^2-a^2)} + \frac{3}{4}a^2 \times f\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$.

Ex. 6. And the $f\frac{x^4\dot{x}}{\sqrt{(a^2-x^2)}}$, is $-\frac{1}{4}x^3\sqrt{(a^2-x^2)} + \frac{3}{4}a^2 \times f\frac{x^2\dot{x}}{\sqrt{(a^2-x^2)}}$.

In like manner the student may find the fluents of $\frac{x^6\dot{x}}{\sqrt{(x^2\pm a^2)}}$, $\frac{x^8\dot{x}}{\sqrt{(x^2\pm a^2)}}$, &c. to $\frac{x^n\dot{x}}{\sqrt{(x^2\pm a^2)}}$, where n is any even number, each from the fluent of that which immediately precedes it in the series, by substituting for y as before. Thus the fluent of

$$\frac{x^n\dot{x}}{\sqrt{(x^2+a^2)}} \text{ is } \frac{1}{n}x^n - \sqrt{(x+a^2)} - \frac{n-1}{n}a^2 \times f\frac{a^{n-2}\dot{x}}{\sqrt{(x^2+a^2)}}.$$

81. In like manner we may proceed for the series of similar expressions where the index of the power of x in the numerator is some odd number.

Ex. 1. To find the fluent of $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$. Here assuming $y = x^2\sqrt{(x^2+a^2)}$, and taking the fluxion, one part of it will be similar to the fluxion proposed. Thus, $\dot{y} = 2x\dot{x}$

$\sqrt{(x^2+a^2)} + \frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$; hence at once the given fluxion $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}} = \dot{y} - 2x\dot{x}\sqrt{(x^2+a^2)}$; theref. the required fluent is $y - f \cdot 2x\dot{x}\sqrt{(x^2+a^2)} = x^2\sqrt{(x^2+a^2)} - \frac{2}{3}(x^2+a^2)^{\frac{3}{2}}$, by the 2d form of fluents.

Ex. 2. In like manner the fluent of $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$, is $x^2\sqrt{(x^2-a^2)} - \frac{2}{3}(x^2-a^2)^{\frac{3}{2}}$.

Ex. 3. And the fluent of $\frac{x^3\dot{x}}{\sqrt{(a^2-x^2)}}$, is $-x^2\sqrt{(a^2-x^2)} - \frac{2}{3}(a^2-x^2)^{\frac{3}{2}}$.

Ex. 4. To find the flu. of $\frac{x^5\dot{x}}{\sqrt{(x^2+a^2)}}$, from that of $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$. Here it is manifest we must assume $y = x^4\sqrt{(x^2+a^2)}$. This in fluxions and reduced gives $\dot{y} = \frac{5x^5\dot{x}}{\sqrt{(x^2+a^2)}} + \frac{4a^2x^3\dot{x}}{\sqrt{(x^2+a^2)}}$, and hence $\frac{x^5\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{5}\dot{y} - \frac{4a^2}{5} - \frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$; and the flu. is $\frac{1}{5}y - \frac{4}{5}a^2 \times f\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{5}x^4\sqrt{(x^2+a^2)} - \frac{4}{5}a^3 \times f\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$, the fluent of the latter part being as in ex. 1, above.

In like manner the student may find the fluents of $\frac{x^5\dot{x}}{\sqrt{(x^2-a^2)}}$ and $\frac{x^5\dot{x}}{\sqrt{(a^2-x^2)}}$. He may then proceed in a similar way for the fluents of $\frac{x^7\dot{x}}{\sqrt{(x^2\pm a^2)}}$, $\frac{x^9\dot{x}}{\sqrt{(x^2\pm a^2)}}$, &c. $\frac{x^n\dot{x}}{\sqrt{(x^2\pm a^2)}}$, where n is any odd number, viz. always by means of the fluent of each preceding term in the series.

82. In a similar manner may the process be for the fluents of the series of fluxions,

$$\frac{\dot{x}}{\sqrt{(a\pm x)}},\ \frac{x\dot{x}}{\sqrt{(a\pm x)}},\ \frac{x^2\dot{x}}{\sqrt{(a\pm x)}},\ \&c. \ldots \frac{x^n\dot{x}}{\sqrt{(a\pm x)}},$$

using the fluent of each preceding term in the series as a part of the next term, and knowing that the fluent of the first

term $\frac{\dot{x}}{\sqrt{(a \pm x)}}$ is given, by the 2d form of fluents, $= 2\sqrt{(a \pm x)}$, of the same sign as x.

Ex. 1. To find the fluent of $\frac{x\dot{x}}{\sqrt{(x+a)}}$, having given that of $\frac{\dot{x}}{\sqrt{(x+a)}} = 2\sqrt{(x+a)} = \text{A}$ suppose. Here it is evident we must assume $y = x\sqrt{(x+a)}$, for then its flux. $\dot{y} = \frac{\frac{1}{2}x\dot{x}}{\sqrt{(x+a)}} + \dot{x}\sqrt{(x+a)} = \frac{\frac{1}{2}x\dot{x}}{\sqrt{(x+a)}} + \frac{x\dot{x}}{\sqrt{(x+a)}} + \frac{a\dot{x}}{\sqrt{(x+a)}} = \frac{\frac{3}{2}x\dot{x}}{\sqrt{(x+a)}} + a\dot{\text{A}}$; hence $\frac{x\dot{x}}{\sqrt{(x+a)}} = \frac{2}{3}\dot{y} - \frac{2}{3}a\dot{\text{A}}$; and the required fluent is $\frac{2}{3}y - \frac{2}{3}a\text{A} = \frac{2}{3}x\sqrt{(x+a)} - \frac{4}{3}a\sqrt{(x+a)} = (x-2a) \times \frac{2}{3}\sqrt{(x+a)}$.

In like manner the student will find the fluents of

$\frac{x\dot{x}}{\sqrt{(x-a)}}$ and $\frac{x\dot{x}}{\sqrt{(a-x)}}$.

Ex. 2. To find the fluent of $\frac{x^2\dot{x}}{\sqrt{(x+a)}}$, having given that of $\frac{x\dot{x}}{\sqrt{(x+a)}} + \dot{\text{B}}$. Here y must be assumed $= x^2\sqrt{(x+a)}$; for then taking the flu. and reducing, there is found $\frac{x^2\dot{x}}{\sqrt{(x+a)}} = \frac{2}{5}\dot{y} - \frac{4}{5}a\dot{\text{B}}$; theref. $f\frac{x^2\dot{x}}{\sqrt{(x+a)}} = \frac{2}{5}y - \frac{4}{5}a\text{B} = \frac{2}{5}x^2\sqrt{(x+a)} - \frac{4}{5}a\text{B} = \frac{2}{5}x^2\sqrt{(x+a)} - \frac{4}{5}a(x-2a) \times \frac{2}{3}\sqrt{(x+a)} = (9x^2 - 4ax + 8a^2) \times \frac{2}{15}\sqrt{(x+a)}$.

In the same manner the student will find the fluents of $\frac{x^2\dot{x}}{\sqrt{(x-a)}}$ and of $\frac{x^2\dot{x}}{\sqrt{(a-x)}}$. And in general, the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{(x+a)}}$ being given $= \text{C}$, he will find the fluent of $\frac{x^n\dot{x}}{\sqrt{(x+a)}} = \frac{2}{2n+1}x\sqrt{(x+a)} - \frac{2n}{2n+1}a\text{C}$.

83. In a similar way we might proceed to find the fluents of other classes of fluxions by means of other fluents in the table of forms; as, for instance, such as $x\dot{x}\sqrt{(dx-x^2)}$, $x^2\dot{x}\sqrt{(dx-x^2)}$, $x^3\dot{x}\sqrt{(dx-x^2)}$, &c. depending on the fluent of $\dot{x}\sqrt{(dx-x^2)}$, the fluent of which, by the 16th tabular form, is the circular semisegment to diameter d and vers-

ed sine x, or the half or trilineal segment contained by an arc with its right sine and versed sine, the diameter being d.

Ex. 1. Putting then the said semiseg. or flu. of $\dot{x}\sqrt{(dx-x^2)} = \text{A}$, to find the fluent of $x\dot{x}\sqrt{(dx-x^2)}$. Here assuming $y = (dx-x^2)^{\frac{3}{2}}$, and taking the fluxions, they are, $\dot{y} = \frac{3}{2}(d\dot{x} - 2x\dot{x})\sqrt{(dx-x^2)}$; hence $x\dot{x}\sqrt{(dx-x^2)} = \frac{1}{2}d\dot{x}\sqrt{(dx-x^2)} - \frac{1}{3}\dot{y} = \frac{1}{2}d\dot{\text{A}} - \frac{1}{3}\dot{y}$; theref. the required flu. $fx\dot{x}\sqrt{(dx-x^2)}$, is $\frac{1}{2}d\text{A} - \frac{1}{3}y = \frac{1}{2}d\text{A} - \frac{1}{3}(dx-x^2)^{\frac{3}{2}} = \text{B}$ suppose.

Ex. 2. To find the fluent of $x^2\dot{x}\sqrt{(dx-x^2)}$, having that of $x\dot{x}\sqrt{(dx-x^2)}$ given $=\text{B}$. Here assuming $y = x(dx-x^2)$, then taking the fluxions, and reducing, there results $\dot{y} = (\frac{5}{2}dx\dot{x} - 4x^2\dot{x})\sqrt{(dx-x^2)}$; hence $x^2\dot{x}\sqrt{(dx-x^2)} = \frac{5}{8}dx\dot{x}\sqrt{(dx-x^2)} - \frac{1}{4}\dot{y} = \frac{5}{8}d\dot{\text{B}} - \frac{1}{4}\dot{y}$, the flu. theref. of $x^2\dot{x}\sqrt{(dx-x^2)}$ is $\frac{5}{8}d\text{B} - \frac{1}{4}y = \frac{5}{8}d\text{B} - \frac{1}{4}x(dx-x^2)^{\frac{3}{2}}$.

Ex. 3. In the same manner the series may be continued to any extent; so that in general, the flu. of $x^{n-1}\sqrt{(dx-x^2)}$ being given $=$ C, then the next, or the flu. of $x^n\dot{x}\sqrt{(dx-x^2)}$ will be $\frac{2n+1}{n+2} - \frac{1}{2}d\text{C} - \frac{1}{n+2}x^{n-1}(dx-x^2)^{\frac{3}{2}}$.

84. To find the fluent of such expressions as $\frac{\dot{x}}{\sqrt{(x^2 \pm 2ax)}}$, a case not included in the table of forms.

Put the proposed radical $\sqrt{(x^2 \pm 2ax)} = z$, or $x^2 \pm 2ax = z^2$; then, completing the square, $x^2 \pm 2ax + a^2 = z^2 + a^2$, and the root is $x \pm a = \sqrt{(z^2+a^2)}$. The fluxion of this is $\dot{x} = \frac{z\dot{z}}{\sqrt{(z^2+a^2)}}$; theref. $\frac{\dot{x}}{\sqrt{(x^2 \pm 2ax)}} = \frac{\dot{z}}{\sqrt{(z^2+a^2)}}$; the fluent of which, by the 12th form, is the hyp. log. of $z + \sqrt{(z^2+a^2)}$ = hyp. log. of $x \pm a + \sqrt{(x^2 \pm 2ax)}$, the fluent required.

Ex. 2. To find now the fluent of $\frac{x\dot{x}}{\sqrt{(x^2+2ax)}}$, having given, by the above example, the fluent of $\frac{\dot{x}}{\sqrt{(x^2+2ax)}} = \text{A}$ suppose. Assume $\sqrt{(x^2+2ax)} = y$; then its fluxion is $\frac{x\dot{x}+a\dot{x}}{\sqrt{(x^2+2ax)}} = \dot{y}$; theref. $\frac{x\dot{x}}{\sqrt{(x^2+2ax)}} = \dot{y} - \frac{\dot{x}}{\sqrt{(x^2+2ax)}} = \dot{y} - a\dot{\text{A}}$; the fluent of which is $y - a\text{A} = \sqrt{(x^2+2ax)} - a\text{A}$, the fluent sought.

Ex. 3. Thus also, this fluent of $\frac{x\dot{x}}{\sqrt{(x^2+2ax)}}$ being given, the flu. of the next in the series, or $\frac{x^2\dot{x}}{\sqrt{(x^2+2ax)}}$ will be found, by assuming $x\sqrt{(x^2+2ax)}=y$; and so on for any other of the same form. As, if the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{(x^2+2ax)}}$ be given $=$ c; then, by assuming $x^{n-1}\sqrt{(x^2+2ax)}=y$, the fluent of $\frac{x^n\dot{x}}{\sqrt{(x^2+2ax)}}=\frac{1}{n}x^{n-1}\sqrt{(x^2+2ax)}-\frac{2n-1}{n}ac.$

Ex. 4. In like manner, the fluent of $\frac{\dot{x}}{\sqrt{(x^2-2ax)}}$ being given, as in the first example, that of $\frac{x\dot{x}}{\sqrt{(x^2-2ax)}}$ may be found; and thus the series may be continued exactly as in the 3d ex. only taking $-2ax$ for $+2ax$.

85. Again, having given the fluent of $\frac{\dot{x}}{\sqrt{(2ax-x^2)}}$, which, by p. 326, is $\frac{1}{a}\times$ circular arc to radius a and versed sine x, the fluents of $\frac{x\dot{x}}{\sqrt{(2ax-x^2)}}$, $\frac{x^2\dot{x}}{\sqrt{(2ax-x^2)}}$, &c. . . $\frac{x^n\dot{x}}{\sqrt{(2ax-x^2)}}$, may be assigned by the same method of continuation. Thus,

Ex. 1. For the fluent of $\frac{x\dot{x}}{\sqrt{(2ax-x^2)}}$, assume $\sqrt{(2ax-x^2)}=y$; the required fluent will be found $=-\sqrt{(2ax-x^2)}+\text{A}$ or arc to radius a and vers. x.

Ex. 2. In like manner the fluent of $\frac{x^2\dot{x}}{\sqrt{(2ax-x^2)}}$ is fl. $\frac{\frac{3}{2}ax\dot{x}}{\sqrt{(2ax-x^2)}}-\frac{1}{2}x\sqrt{(2ax-x^2)}=\frac{3}{2}a\text{A}-\frac{3a+x}{2}\sqrt{(2ax-x^2)}$, where A denotes the arc mentioned in the last example.

Ex. 3. And in general the fluent of $\frac{x^n\dot{x}}{\sqrt{(2ax-x^2)}}$ is $\frac{2n-1}{n}ac-\frac{1}{n}x^{n-1}\sqrt{(2ax-x^2)}$, where c is the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{(2ax-x^2)}}$, the next preceding term in the series.

86. Thus also, the fluent of $\dot{x}\sqrt{(x-a)}$ being given, $= \frac{2}{3}(x-a)^{\frac{3}{2}}$, by the 2d form, the fluents of $x\dot{x}\sqrt{(x-a)}$, $x^2\dot{x}\sqrt{(x-a)}$, &c. . . $x^n\dot{x}\sqrt{(x-a)}$, may be found. And in general, if the fluent of $x^{n-1}\dot{x}\sqrt{(x-a)} = c$ be given; then by assuming $x^n(x-a)^{\frac{3}{2}} = y$, the fluent of $x^n\dot{x}\sqrt{(x-a)}$ is found $= \frac{2}{2n+3}x^n(x-a)^{\frac{3}{2}} + \frac{2na}{2n+3}c$.

87. Also, given the fluent of $(x-a)^m\dot{x}$, which is $\frac{1}{m+1}(x-a)^{m+1}$ by the 2d form, the fluents of the series $(x-a)^m x\dot{x}$, $(x-a)^m x^2\dot{x}$, &c. . . $(x-a)^m x^n\dot{x}$ can be found. And in general, the fluent of $(x-a)^m x^{n-1}\dot{x}$ being given $= c$; then by assuming $(x-a)^{m+1}x^n = y$, the fluent of $(x-a)^m x^n\dot{x}$ is found $= \frac{x^n(x-a)^{m+1}+nac}{m+n+1}$.

Also, by the same way of continuation, the fluents of $x^n\dot{x}\sqrt{(a \mp x)}$ and of $x^n\dot{x}(a \mp x)^m$ may be found.

88. When the fluxional expression contains a trinomial quantity, as $\sqrt{(b+cx+x^2)}$, this may be reduced to a binomial, by substituting another letter for the unknown one x, connected with half the coefficient of the middle term with its sign. Thus, put $z = x+\frac{1}{2}c$: then $z^2 = x^2+cx+\frac{1}{4}c^2$; theref. $z^2 - \frac{1}{4}c^2 = x^2 + cx$, and $z^2+b-\frac{1}{4}c^2 = x^2+cx+b$ the given trinomial; which is $= z^2+a^2$, by putting $a^2 = b-\frac{1}{4}c^2$.

Ex. 1. To find the fluent of $\frac{3\dot{x}}{\sqrt{(5+4x+x^2)}}$.

Here $z = x+2$; then $z^2 = x^2+4x+4$, and $z^2+1 = 5+4x+x^2$, also $\dot{x} = \dot{z}$; theref. the proposed fluxion reduces to $\frac{3\dot{z}}{\sqrt{(1+z^2)}}$; the fluent of which, by the 12th form is 3 hyp. log. of $z+\sqrt{(1+z)} = 3$ hyp. log. $x+2+\sqrt{(5+4x+x^2)}$.

Ex. 2. To find the fluent of $\dot{x}\sqrt{(b+cx+dx^2)} =$

$$\dot{x}\sqrt{d} \times \sqrt{(\frac{b}{d}+\frac{c}{d}x+x^2)}.$$

Here assuming $x+\frac{c}{2d} = z$; then $\dot{x} = \dot{z}$, and the proposed flux. reduces to $\dot{z}\sqrt{d}\times\sqrt{(z^2+\frac{b}{d}-\frac{c2}{4d^2}} = \dot{z}\sqrt{d}\times\sqrt{(z^2+a^2)}$, putting a^2 for $\frac{b}{d}-\frac{c^2}{4d^2}$; and the fluent will be found by a

similar process to that employed in ex. 1, art. 80.

Ex. 3. In like manner, for the flu. of $x^{n-1}\dot{x}\sqrt{(b+cx^n+dx^{2n})}$, assuming $x^n+\frac{c}{2d}=z$, $nx^{n-1}\dot{x}=\dot{z}$, and $x^{n-1}\dot{x}=\frac{1}{n}\dot{z}$; hence $x^{2n}+\frac{c}{d}x^n+\frac{c^2}{4a^2}=z^2$, and $\sqrt{(dx^{2n}+cx^n+b)}=\sqrt{d}\times\sqrt{(x^{2n}+\frac{c}{d}x+\frac{b}{d})}=\sqrt{d}\times\sqrt{(z^2+\frac{b}{d}-\frac{c^2}{4d^2})}=\sqrt{d}\times\sqrt{(z^2\pm a^2)}$, putting $\pm a^2=\frac{b}{d}-\frac{c^2}{4a^2}$; hence the given fluxion becomes $\frac{1}{n}\dot{z}\sqrt{d}\times\sqrt{(z^2\pm a^2)}$, and its fluent as in the last example.

Ex. 4. Also, for the fluent of $\frac{x^{n-1}\dot{x}}{b+cx+dx^2}$; assume $x^n+\frac{c}{2d}=z$, then the fluxion may be reduced to the form $\frac{1}{dn}\times\frac{\dot{x}}{x^2\pm a^2}$, and the fluent found as before.

So far on this subject may suffice on the present occasion. But the student who may wish to see more on this branch, may profitably consult Mr. Dealtry's very methodical and ingenious treatise on Fluxions, lately published, from which several of the foregoing cases and examples have been taken or imitated.

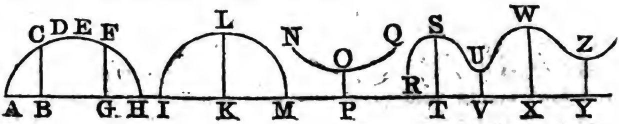
OF MAXIMA AND MINIMA; OR, THE GREATEST AND LEAST MAGNITUDE OF VARIABLE OR FLOWING QUANTITIES.

89. Maximum, denotes the greatest state or quantity attainable in any given case, or the greatest value of a variable quantity: by which it stands opposed to Minimum, which is the least possible quantity in any case.

Thus, the expression or sum a^2+bx, evidently increases as x, or the term bx, increases; therefore the given expression will be the greatest, or a maximum, when x is the greatest, or infinite; and the same expression will be a minimum, or the least, when x is the least, or nothing.

Again, in the algebraic expression $a^2 - bx$, where a and b denote constant or invariable quantities, and x a flowing or variable one, it is evident that the value of this remainder or difference, $a^2 - bx$, will increase, as the term bx, or as x, decreases; therefore the former will be the greatest, when the latter is the smallest; that is, $a^2 - bx$ is a maximum, when x is the least, or nothing at all; and the difference is the least, when x is the greatest.

90. Some variable quantities increase continually; and so have no maximum, but what is infinite. Others again decrease continually; and so have no minimum, but what is of no magnitude, or nothing. But, on the other hand, some variable quantities increase only to a certain finite magnitude, called their Maximum, or greatest state, and after that they decrease again. While others decrease to a certain finite magnitude, called their Minimum, or least state, and afterwards increase again. And lastly, some quantities have several maxima and minima.



Thus, for example, the ordinate BC of the parabola or such-like curve, flowing along the axis AB from the vertex A, continually increases, and has no limit or maximum. And the ordinate GF of the curve EFH, flowing from E towards H, continually decreases to nothing when it arrives at the point H. But in the circle ILM, the ordinate only increases to a certain magnitude, namely, the radius, when it arrives at the middle as at KL, which is its maximum; and after that it decreases again to nothing, at the point M. And in the curve NOQ, the ordinate decreases only to the position OP, where it is least, or a minimum; and after that it continually increases towards Q. But in the curve RSU, &c. the ordinates have several maxima, as ST, WX, and several minima, as VU, YZ, &c.

91. Now, because the fluxion of a variable quantity, is the rate of its increase or decrease; and because the maximum or minimum, of a quantity neither increases nor decreases, at those points or states; therefore such maximum or minimum has no fluxion, or the fluxion is then equal to nothing. From which we have the following rule.

To find the Maximum or Minimum.

92. From the nature of the question or problem, find an algebraical expression for the value, or general state, of the quantity whose maximum or minimum is required; then take the fluxion of that expression, and put it equal to nothing; from which equation, by dividing by, or leaving out, the fluxional letter and other common quantities, and performing other proper reductions, as in common algebra, the value of the unknown quantity will be obtained, determining the point of the maximum or minimum.

So, if it be required to find the maximum state of the compound expression $100x - 5x^2 \pm c$, or the value of x when $100x - 5x^2 \pm c$ is a maximum. The fluxion of this expression is $100\dot{x} - 10x\dot{x}$; which being made $=0$, and divided by $10\dot{x}$, the equation is $10 - x = 0$; and hence $x = 10$. That is, the value of x is 10, when the expression $100x - 5x^2 \pm c$ is the greatest. As is easily tried: for if 10 be substituted for x in that expression, it becomes $\pm c + 500$: but if, for x, there be substituted any other number, whether greater or less than 10, that expression will always be found to be less than $\pm c + 500$, which is therefore its greatest possible value, or its maximum.

93. It is evident, that if a maximum or minimum be any way compounded with, or operated on, by a given constant quantity, the result will still be a maximum or minimum. That is, *if a maximum or minimum be increased, or decreased, or multiplied, or divided, by a given quantity, or any given power or root of it be taken*; the result will still be a maximum or minimum. Thus, if x be a maximum or minimum, then also is $x + a$ or $x - a$, or ax, or $\frac{x}{a}$, or x^a, or $\sqrt[a]{x}$, still a maximum or minimum. Also, *the logarithm of the same will be a maximum or a minimum.* And therefore, *if any proposed maximum or minimum can be made simpler by performing any of these operations, it is better to do so*, before the expression is put into fluxions.

94. When the expression for a maximum or minimum contains several variable letters or quantities; take the fluxion of it as often as there are variable letters; supposing first one of them only to flow, and the rest to be constant; then another only to flow, and the rest constant; and so on for all of them: then putting each of these fluxions $= 0$, there will be as many equations as unknown letters, from which these may be all determined. For the fluxion of the expression must be equal to nothing in each of these cases; otherwise

the expression might become greater or less, without altering the values of the other letters, which are considered as constant.

So, if it be required to find the values of x and y, when $4x^2 - xy + 2y$ is a minimum. Then we have,

First, . $8x\dot{x} - \dot{x}y = 0$, and $8x - y = 0$, or $y = 8x$.
Secondly, $2\dot{y} - x\dot{y} = 0$, and $2 - x = 0$, or $x = 2$.
And hence y or $8x = 16$.

95. *To find whether a proposed quantity admits of a Maximum or a Minimum.*

Every algebraic expression does not admit of a maximum or minimum, properly so called; for it may either increase continually to infinity, or decrease continually to nothing; and in both these cases there is neither a proper maximum nor minimum; for the true maximum is that finite value to which an expression increases, and after which it decreases again: and the minimum is that finite value to which the expression decreases, and after that it increases again. Therefore, when the expression admits of a maximum, its fluxion is positive before the point, and negative after it: but when it admits of a minimum, its fluxion is negative before, and positive after it. Hence then, taking the fluxion of the expression a little before the fluxion is equal to nothing, and again a little after the same; if the former fluxion be positive, and the latter negative, the middle state is a maximum; but if the former fluxion be negative, and the latter positive, the middle state is a minimum.

So, if we would find the quantity $ax - x^2$ a maximum or minimum; make its fluxion equal to nothing, that is, . . $a\dot{x} - 2x\dot{x} = 0$, or $(a - 2x)\,\dot{x} = 0$; dividing by $\dot{x}$, gives $a - 2x = 0$, or $x = \frac{1}{2}a$ at that state. Now, if in the fluxion $(a - 2x)\dot{x}$, the value of x be taken rather less than its true value, $\frac{1}{2}a$, that fluxion will evidently be positive; but if x be taken somewhat greater than $\frac{1}{2}a$ the value of $a - 2x$, and consequently of the fluxion, is as evidently negative. Therefore, the fluxion of $ax - x^2$ being positive before, and negative after the state when its fluxion is $= 0$, it follows that at this state the expression is not a minimum, but a maximum.

Again, taking the expression $x^3 - ax^2$, its fluxion $3x^2\dot{x} - 2ax\dot{x} = (3x - 2a)\,x\dot{x} = 0$; this divided by $x\dot{x}$ gives $3x - 2a = 0$, and $x = \frac{2}{3}a$, its true value when the fluxion of $x^3 - ax^2$ is equal to nothing. But now to know whether the given expression be a maximum or a minimum at that time, take x a little less than $\frac{2}{3}a$ in the value of the fluxion $(3x - 2a)\,x\dot{x}$, and this will evidently be negative; and again, taking x a

little more than $\frac{2}{3}a$, the value of $3x - 2a$, or of the fluxion, is as evidently positive. Therefore the fluxion of $x^3 - ax^2$ being negative before that fluxion is $= 0$, and positive after it, it follows that in this state the quantity $x^3 - ax^2$ admits of a minimum, but not of a maximum.

SOME EXAMPLES FOR PRACTICE.

EXAM. 1. Of all triangles, ACB, constructed on the same base AB, and having the same perimeter, to determine that whose area or surface is the greatest.

Let p denote the semiperimeter, b the base AB, x the side AC, then BC will $= 2p - b - x$. Therefore putting s for the surface, we have by rule 3 for the area of triangles (pa. 408, vol. i.)

$$s^2 = p\,(p-b)\,(p-x)\,(b+x-p).$$

Expressing this equation logarithmically, we have, 2 log. $s =$ log. p + log. $(p-b)$ + log. $(p-x)$ + log. $(b+x-p)$ which (art. 93) is to be a max. or when put into fluxions equal to *zero* or nothing.

Hence $\frac{2\dot{s}}{s} = \frac{-\dot{x}}{p-x} + \frac{\dot{x}}{b+x-p}$;

or dividing by $2\dot{x}$, and multiplying by s,

$$\frac{\dot{s}}{\dot{x}} = \frac{s}{2}\left(\frac{1}{b+x-p} - \frac{1}{p-x}\right) = 0.$$

Now, here it is evident, since s must be a max. that $\frac{s}{2}$ cannot $= 0$; consequently the second factor must: that is,

$\frac{1}{b+x-p} - \frac{1}{p-x} = 0$, or $b + x - p = p - x$.

Therefore, $2p - b - x = x$, or AC = BC; that is, the triangle must be isosceles.

Cor. Hence it follows that of all *isoperimetrical* triangles, the one which has the greatest surface is equilateral. A truth, indeed, which may be readily shown by a direct investigation.

EXAM. 2. Amongst all parallelopipedons of given magnitude, whose planes are respectively perpendicular to one another, to determine that which has the least surface.

Let x, y, and z, be the measures of the three edges of the required parallelopipedon. Then, since the magnitude is given,

we have $xyz = a$, a given magnitude;
and $2xy + 2xz + 2yz =$ a minimum.

Here, substituting for z, and dividing by 2, there results

$$xy + x \cdot \frac{a}{xy} + y \cdot \frac{a}{xy} = \text{a min.}$$

$$\text{or, } u = xy + \frac{a}{y} + \frac{a}{x} = \text{a min.}$$

Therefore, adopting the principle of art. 94,

$$\left.\begin{aligned} \frac{\dot{u}}{\dot{x}} &= y - \frac{a}{x^2} = 0 \\ \text{and } \frac{\dot{u}}{\dot{y}} &= x - \frac{a}{y^2} = 0 \end{aligned}\right\} \text{must both obtain.}$$

$$\text{Hence, } y = \frac{a}{x^2} = a \div \left(\frac{a}{y^2}\right)^2 = a \cdot \frac{y^4}{a^2} = \frac{y^4}{a}.$$

$$\left.\begin{aligned} \text{Consequently } y &= a^{\frac{1}{3}} \\ x &= a^{\frac{1}{3}} \\ \therefore z &= a^{\frac{1}{3}} \end{aligned}\right\} \text{and thus it appears that the required parallelopipedon is a } \textit{cube}.$$

Exam. 3. Divide a given arc A into two such parts, that the mth power of the sine of one part, multiplied into the nth power of the sine of the other part, shall be a maximum.

Let x and y be the parts: then $x + y = A$, and $\sin.^m x \times \sin.^n y =$ a max.

$$\text{In logs.} \quad m \log. \sin x + n \log. \sin. y = \text{a max.}$$

$$\text{Hence, (art. 93)} \frac{m\dot{x} \cos. x}{\sin. x} + \frac{m\dot{y} \cos. y}{\sin. y} = 0.$$

$$\text{But } \dot{y} = -\dot{x} \therefore \frac{m\dot{x} \cos. x}{\sin. x} - \frac{n\dot{x} \cos. y}{\sin. y} = 0.$$

Hence m cot. $x = n$ cot y, or m tan. $y = n$ tan. x.

$$\therefore \frac{m}{n} = \frac{\tan. x}{\tan. y} \ldots \text{and } \frac{m+n}{m-n} = \frac{\tan. x + \tan. y}{\tan. x - \tan. y} = \frac{\sin. (x+y)}{\sin. (x-y)}.$$

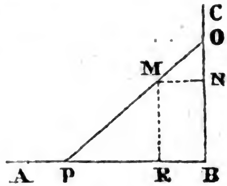
(See equa. 9 and 10, p. 394, vol. i.)

Hence x and y become known: and the same principle is evidently applicable to three or more arcs, making together a given arc.

Exam. 4. To find the longest straight pole that can be put up a chimney, whose height RM $= a$, from the floor to the mantel, and depth MN $= b$, from front to back, are given.

Here the longest pole that can be put up the chimney is, in fact, the *shortest* line PMO, which can be drawn through M, and terminated by BA and BC.

$$\left.\begin{array}{r}\text{Let } x = \text{sine}\\ y = \text{cos.}\end{array}\right\}\text{ of OPB,}$$



$$x : \text{MR}(=a) :: 1 : \frac{a}{x} = \text{PM}$$

$$y : \text{MN}(=b) :: 1 : \frac{b}{y} = \text{MO}$$

$$\frac{a}{x} + \frac{b}{y} = \text{a min.}$$

$$\text{In flux.} - \frac{a\dot{x}}{x^2} - \frac{b\dot{y}}{y^2} = 0$$

$$\text{But } x^2 + y^2 = 1, \therefore 2x\dot{x} = -2y\dot{y},$$

$$\text{and } -\dot{y} = \frac{x\dot{x}}{y}.$$

Substituting this for $\dot{y}$ above, it becomes

$$-\frac{a\dot{x}}{x^2} + \frac{bx\dot{x}}{y^3} = 0,$$

$$\therefore bx^3 = ay^3,$$

$$\frac{x^3}{y^3} = \frac{a}{b},$$

$$\frac{x}{y} = \text{tan. P} = \sqrt[3]{\frac{a}{b}}$$

$$\text{PO} = a \text{ cosec. P} + b \text{ sec. P} = a\sqrt{1 + \frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}} + b\sqrt{1 + \frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}}.$$

EXAM. 5. To divide a line, or any other given quantity a, into two parts, so that their rectangle or product may be the greatest possible.

EXAM. 6. To divide the given quantity a into two parts such, that the product of the m power of one, by the n power of the other, may be a maximum.

EXAM. 7. To divide the given quantity a into three parts such, that the continual product of them all may be a maximum.

EXAM. 8. To divide the given quantity a into three parts such, that the continual product of the 1st, the square of the 2d, and the cube of the 3d, may be a maximum.

EXAM. 9. To determine a fraction such, that the difference between its m power and n power shall be the greatest possible.

EXAM. 10. To divide the number 80 into two such parts, x and y, that $2x^2 + xy + 3y^2$ may be a minimum.

Exam. 11. To find the greatest rectangle that can be inscribed in a given right-angled triangle.

Exam. 12. To find the greatest rectangle that can be inscribed in the quadrant of a given circle.

Exam. 13. To find the least right-angled triangle that can circumscribe the quadrant of a given circle.

Exam. 14. To find the greatest rectangle inscribed in, and the least isosceles triangle circumscribed about, a given semi-ellipse.

Exam. 15. To determine the same for a given parabola.

Exam. 16. To determine the same for a given hyperbola.

Exam. 17. To inscribe the greatest cylinder in a given cone; or to cut the greatest cylinder out of a given cone.

Exam. 18. To determine the dimensions of a rectangular cistern, capable of containing a given quantity a of water, so as to be lined with lead at the least possible expense.

Exam. 19. Required the dimensions of a cylindrical tankard, to hold one quart of ale measure, that can be made of the least possible quantity of silver, of a given thickness.

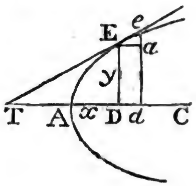
Exam. 20. The cut the greatest parabola from a given cone.

Exam. 21. To cut the greatest ellipse from a given cone.

Exam. 22. To find the value of x when x^x is a minimum.

THE METHOD OF TANGENTS; OR OF DRAWING TANGENTS TO CURVES.

96. The Method of Tangents, is a method of determining the quantity of the tangent and subtangent of any algebraic curve; the equation of the curve being given. Or, *vice versa*, the nature of the curve, from the tangent given.



If AE be any curve, and E be any point in it, to which it is required to draw a tangent TE. Draw the ordinate ED: then if we can determine the subtangent TD, limited between the ordinate and tangent, in the axis produced, by joining the points T, E, the line TE will be the tangent sought.

97. Let dae be another ordinate, indefinitely near to DE, meeting the curve, or tangent produced in e; and let Ea be parallel to the axis AD. Then is the elementary triangle Eea similar to the triangle TDE; and

therefore - $ea : a$E :: ED : DT.

But - - $ea : a$E : : flux. ED : flux. AD.

Therefore - flux. ED : flux. AD : : DE : DT.

That is, - $\dot{y} : \dot{x} : : y : \dfrac{y\dot{x}}{\dot{y}} =$ DT;

which is therefore the general value of the subtangent sought; where x is the absciss AD, and y the ordinate DE.

Hence we have this general rule.

GENERAL RULE.

98. By means of the given equation of the curve, when put into fluxions, find the value of either $\dot{x}$ or $\dot{y}$, or of $\dfrac{\dot{x}}{\dot{y}}$, which value substitute for it in the expression DT $= \dfrac{y\dot{x}}{\dot{y}}$, and, when reduced to its simplest terms, it will be the value of the subtangent sought.

EXAMPLES.

EXAM. 1. Let the proposed curve be that which is defined, or expressed, by the equation $ax^2 + xy^2 - y^3 = 0$.

Here the fluxion of the equation of the curve is $2ax\dot{x} + y^2\dot{x} + 2xy\dot{y} - 3y^2\dot{y} = 0$; then, by transposition, $2ax\dot{x} + y^2\dot{x} = 3y^2\dot{y} - 2xy\dot{y}$; and hence, by division, $\dfrac{\dot{x}}{\dot{y}} = \dfrac{3y^2 - 2xy}{2ax + y^2}$; consequently $\dfrac{y\dot{x}}{\dot{y}} = \dfrac{3y^3 - 2xy^2}{2ax + y^2}$, which is the value of the subtangent TD sought.

EXAM. 2. To draw a tangent to a circle; the equation of which is $ax - x^2 = y^2$; where x is the absciss, y the ordinate, and a the diameter.

EXAM. 3. To draw a tangent to a parabola; its equation being $px = y^2$; where p denotes the parameter of the axis.

EXAM. 4. To draw a tangent to an ellipse; its equation being $c^2(ax - x^2) = a^2y^2$; where a and c are the two axes.

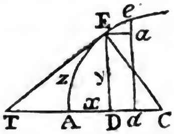
EXAM. 5. To draw a tangent to an hyperbola; its equation being $c^2(ax + x^2) = a^2y^2$; where a and c are the two axes.

Exam. 6. To draw a tangent to the hyperbola referred to the asymptote as an axis; its equation being $xy = a^2$; where a^2 denotes the rectangle of the absciss and ordinate answering to the vertex of the curve.

By slight and obvious extensions of the same principles, tangents may be drawn to spirals, and asymptotes may be drawn to such curves as admit of them.

OF RECTIFICATIONS; OR, TO FIND THE LENGTHS OF CURVE LINES.

99. Rectification, is the finding the length of a curve line, or finding a right line equal to a proposed curve.



By art. 10 it appears, that the elementary triangle Eae, formed by the increments of the absciss, ordinate, and curve, is a right-angled triangle, of which the increment of the curve is the hypothenuse; and therefore the square of the latter is equal to the sum of the squares of the two former; that is, $\text{E}e^2 = \text{E}a^2 + ae^2$. Or, substituting, for the increments, their proportional fluxions, it is $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$, or $\dot{z} = \sqrt{(\dot{x}^2+\dot{y}^2)}$; where z denotes any curve line AE, x its absciss AD, and y its ordinate DE. Hence this rule.

RULE.

100. From the given equation of the curve put into fluxions, find the value of $\dot{x}^2$ or $\dot{y}^2$, which value substitute instead of it in the equation $\dot{z} = \sqrt{(\dot{x}^2+\dot{y}^2)}$; then the fluents, being taken, will give the value of z, or the length of the curve, in terms of the absciss or ordinate.

EXAMPLES.

Exam. 1. To find the length of the arc of a circle, in terms both of the sine, versed sine, tangent, and secant.

The equation of the circle may be expressed in terms of the radius, and either the sine, or the versed sine, or tangent, or secant, &c. of an arc. Let therefore the radius of the circle be CA or CE $= r$, the versed sine AD (of the arc AE)$=x$, the right sine DE $= y$, the tangent TE $= t$, and the secant CT $= s$; then, by the nature of the circle, there arise these equations, viz.

$$y^2 = 2rx - x^2 = \frac{r^2t^2}{r^2+t^2} = \frac{s^2-r^2}{s^2} r^2.$$

Then, by means of the fluxions of these equations, with the general fluxional equation $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$, are obtained the following fluxional forms, for the fluxion of the curve; the fluent of any one of which will be the curve itself; viz.

$$\dot{z} = \frac{r\dot{x}}{\sqrt{2rx-xx}} = \frac{r\dot{y}}{\sqrt{r^2-y^2}} = \frac{r^2\dot{t}}{r^2+t^2} = \frac{r^2\dot{s}}{\sqrt{s^2-r^2}} *.$$

Hence the value of the curve, from the fluent of each of these, expressed in series, gives the four following forms, in series, viz. putting $d = 2r$ the diameter, the curve is

$$z = (1 + \frac{x}{2.3d} + \frac{3x^2}{2.4.5d^2} + \frac{3.5x^3}{2.4.6.7d^3} + \text{\&c.}) \sqrt{dx},$$

$$= (1 + \frac{y^2}{2.3r^2} + \frac{3y^4}{2.4.5r^4} + \frac{3.5y^6}{2.4.6.7r^6} + \text{\&c.})\, y,$$

$$= (1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \text{\&c.})\, t,$$

$$= (\frac{s-r}{s} + \frac{s^3-r^3}{2.3s^3} + \frac{3(s^5-r^5)}{2.4.5s^5} + \text{\&c.})\, r.$$

Now, it is evident, that the simplest of these series, is the third in order, or that which is expressed in terms of the tangent. That form will therefore be the fittest to calculate an example by in numbers. And for this purpose it will be convenient to assume some arc whose tangent, or at least the square of it, is known to be some small simple number. Now, the arc of 45 degrees, it is known, has its tangent equal to the radius; and therefore, taking the radius $r = 1$, and consequently the tangent of $45°$, or $t, = 1$ also, in this case the arc of $45°$ to the radius 1, or the arc of the

---

* These formulæ are, obviously, analogous to those given in art. 30, p. 312, and are so many forms of fluxions whose fluents become known. Thus the fluent of an expression, such as $\frac{r\dot{x}}{\sqrt{(2rx-x^2)}}$, is a circular arc whose radius is $= r$ and versed sine $= x$. The fluent of an expression such as $\frac{r^2\dot{t}}{r^2+t^2}$ is a circular arc whose radius is $= r$ and tangent $= t$: and so of the rest.

Conversely, the same formulæ, or those just referred to, serve to assign the relative magnitudes of the *differences* in any parts of a table of natural sines, of natural tangents, &c. Thus $\dot{t} = \frac{r^2+t^2}{r^2}\dot{z} = \dot{z} \times \text{sec.}^2$ of arc to tan. t, consequently, the tabular differences of the tangents vary as the squares of the secants. Hence, those differences, at $0°$, at $45°$, and at $60°$, are as 1^2, $(\sqrt{2})^2$, and 2^2, or as 1, 2, and 4. This suggests an application of these formulæ which will often be found useful.

quadrant to the diameter 1, will be equal to the infinite series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ &c.

But as this series converges very slowly, it will be proper to take some smaller arc, that the series may converge faster; such as the arc of 30 degrees, the tangent of which is $= \sqrt{\frac{1}{3}}$, or its square $t^2 = \frac{1}{3}$: which being substituted in the series, the length of the arc of 30° comes out - - - $(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \text{\&c.})\sqrt{\frac{1}{3}}$. Hence, to compute these terms in decimal numbers, after the first, the succeeding terms will be found by dividing, always by 3, and these quotients again by the absolute numbers, 3, 5, 7, 9, &c.; and lastly, adding every other term together, into two sums, the one the sum of the positive terms, and the other the sum of the negative ones; then lastly, the one sum taken from the other, leaves the length of the arc of 30 degrees; which being the 12th part of the whole circumference when the radius is 1, or the 6th part when the diameter is 1, consequently 6 times that arc will be the length of the whole circumference to the diameter 1. Therefore, multiplying the first term $\sqrt{\frac{1}{3}}$ by 6, the product is $\sqrt{12} = 3{\cdot}4641016$; and hence the operation, true to 7 places of decimals, will be conveniently made as follows:

		+ Terms.	— Terms.
1)	3·4641016 (	3·4641016	
3)	1·1547005 (		0·3849002
5)	3849002 (	769800	
7)	1283001 (		183286
9)	427667 (	47519	
11)	142556 (		12960
13)	47519 (	3655	
15)	15840 (		1056
17)	5280 (	311	
19)	1760 (		93
21)	587 (	28	
23)	196 (		8
25)	65 (	3	
27)	22 (		1
		+3·5462332	—0·4046406
		—0·4046406	

So that at last 3·1415926 is the whole circumference to the diameter 1*.

* For this value, true to 100 places of decimals; and indeed for many

Exam. 2. To find the length of a parabola.

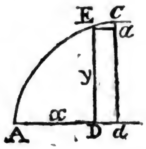
Exam. 3. To find the length of the semicubical parabola, whose equation is $ax^2 = y^3$.

Exam. 4. To find the length of an elliptical curve.

Exam. 5. To find the length of an hyperbolic curve.

OF QUADRATURES; OR, FINDING THE AREAS OF CURVES.

101. The Quadrature of Curves, is the measuring their areas, or finding a square, or other right-lined space, equal to a proposed curvilineal one.



By art. 9, it appears, that any flowing quantity being drawn into the fluxion of the line along which it flows, or in the direction of its motion, there is produced the fluxion of the quantity generated by the flowing. That is, DE $\times$ Dd or $y\dot{x}$ is the fluxion of the area ADE. Hence this rule.

RULE.

102. From the given equation of the curve, find the value either of $\dot{x}$ or of y; which value substitute instead of it in the expression $y\dot{x}$; then the fluent of that expression, being taken, will be the area of the curve sought.

EXAMPLES.

Exam. 1. To find the area of the common parabola.

The equation of the parabola being $ax = y^2$; where a is the parameter, x the absciss AD, or part of the axis, and y the ordinate DE.

From the equation of the curve is found $y = \sqrt{ax}$. This substituted in the general fluxion of the area $y\dot{x}$ gives $\dot{x}\sqrt{ax}$ or $a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$ the fluxion of the parabolic area; and the fluent of this, or $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}x\sqrt{ax} = \frac{2}{3}xy$, is the area of the parabola ADE, which is therefore equal to $\frac{2}{3}$ of its circumscribing rectangle.

curious, and important investigations in reference to rectifications, quadratures, &c. see *Hutton's Mensurations*.

EXAM. 2. To square the circle, or find its area.

The equation of the circle being $y^2 = ax - x^2$, or $y = \sqrt{ax - x^2}$, where a is the diameter; by substitution, the general fluxion of the area $y\dot{x}$, becomes $\dot{x}\sqrt{ax - x^2}$, for the fluxion of the circular area. But as the fluent of this cannot be found in finite terms, the quantity $\sqrt{ax - x^2}$ is thrown into a series, by extracting the root, and then the fluxion of the area becomes

$$\dot{x}\sqrt{ax} \times \left(1 - \frac{x}{2a} - \frac{x^2}{2.4a^2} - \frac{1.3x^3}{2.4.6a^3} - \frac{1.3.5x^4}{2.4.6.8a^4} - \&c.\right);$$

and then the fluent of every term being taken, it gives

$$x\sqrt{ax} \times \left(\frac{2}{3} - \frac{1.x}{5a} - \frac{1.x^2}{4.7a^2} - \frac{1.3x^3}{4.6.9a^3} - \frac{1.3.5x^4}{4.6.8.11a^4} - \&c.\right);$$

for the general expression of the semisegment ADE.

And when the point D arrives at the extremity of the diameter, then the space becomes a semicircle, and $x = a$; and then the series above becomes barely

$$a^2\left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4.7} - \frac{1.3}{4.6.9} - \frac{1.3.5}{4.6.8.11} - \&c.\right)$$

for the area of the semicircle whose diameter is a.

If, instead of taking the equation of the circle having the origin of the co-ordinates at the circumference, the equation $x^2 + y^2 = r^2$ be taken, regarding the origin of the co-ordinates at the centre; and if, still farther, r be taken $= 1$, then $y = \sqrt{1 - x^2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \&c.$ Taking this value of y for it, in the expression $y\dot{x}$, the correct fluent will be $x - \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{112} - \frac{5x^9}{1152} - \frac{7x^{11}}{2816}$, for the area of the portion CFBD (fig. pa. 378, vol. 1.) Now, if arc BD $= 30°$, then CF $= x = \frac{1}{2}$, and the sum of the series $= \cdot 4783057$. From which deducting the area of the triangle CFB $= \frac{1}{4} \cdot \frac{1}{2}\sqrt{3} = \cdot 2165063$, there remains $\cdot 2617994$ for the area of the sector CBD. Twelve times this, or $3 \cdot 1415928$, &c. expresses the area of the circle whose diameter is 2.

EXAM. 3. To find the area of any parabola, whose equation is $a^m x^n = y^{m+n}$.

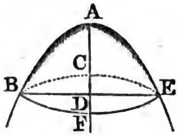
EXAM. 4. To find the area of an ellipse.

EXAM. 5. To find the area of an hyperbola.

EXAM. 6. To find the area between the curve and asymptote of an hyperbola.

EXAM. 7. To find the like area in any other hyperbola whose general equation is $x^m y^n = a^{m+n}$.

TO FIND THE SURFACES OF SOLIDS.



103. In the solid formed by the rotation of any curve about its axis, the surface may be considered as generated by the circumference of an expanding circle, moving perpendicularly along the axis, but the expanding circumference moving along the arc or curve of the solid. Therefore, as the fluxion of any generated quantity, is produced by drawing the generating quantity into the fluxion of the line or direction in which it moves, the fluxion of the surface will be found by drawing the circumference of the generating circle into the fluxion of the curve. That is, the fluxion of the surface whose radius is the ordinate DE.

104. But if π be $=3\cdot141593$, the circumference of a circle whose diameter is 1, $x =$ AD the absciss, $y =$ DE the ordinate, and $z =$ AE the curve; then $2y =$ the diameter BE, and $2\pi y =$ the circumference BCEF; also, $\dot{\text{AE}} = \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$: therefore $2\pi y\dot{z}$ or $2\pi y \sqrt{\dot{x}^2 + \dot{y}^2}$ is the fluxion of the surface. And consequently if, from the given equation of the curve, the value of $\dot{x}$ or $\dot{y}$ be found, and substituted in this expression $2\pi y \sqrt{\dot{x}^2 + \dot{y}^2}$, the fluent of the expression, being then taken, will be the surface of the solid required.

EXAMPLES.

EXAM. 1. To find the surface of a sphere, or of any segment.

In this case, AE is a circular arc, whose equation is $y^2 = ax - x^2$, or $y = \sqrt{(ax - x^2)}$.

The fluxion of this gives $\dot{y} = \dfrac{a-2x}{2\sqrt{(ax-x^2)}}\dot{x} = \dfrac{a-2x}{2y}\dot{x}$;

hence $\dot{y}^2 = \dfrac{a^2 - 4ax + 4x^2}{4y^2}\dot{x}^2 = \dfrac{a^2 - 4y^2}{4y^2}\dot{x}^2$; consequently

$\dot{x}^2 + \dot{y}^2 = \dfrac{a^2\dot{x}^2}{4y^2}$, and $\dot{z} = \sqrt{(\dot{x}^2 + \dot{y}^2)} = \dfrac{a\dot{x}}{2y}$.

This value of $\dot{z}$, the fluxion of a circular arc, may be found more easily thus: In the fig to art. 99, the two triangles EDC, E*ae* are equiangular, being each of them equiangular to the triangle ETC : conseq. ED : EC : : F*a* : E*e*, that is, - - ,

$y : \frac{1}{2}a :: \dot{x} : \dot{z} = \dfrac{a\dot{x}}{2y}$, the same as before.

The value of $\dot{z}$ being found, by substitution is obtained $2\pi y\dot{z} = a\pi\dot{x}$ for the fluxion of the spherical surface, generated by the circular arc in revolving about the diameter AD. And the fluent of this gives $a\pi x$ for the said surface of the spherical segment BAE.

But $a\pi$ is equal to the whole circumference of the generating circle; and therefore it follows, that the surface of any spherical segment, is equal to the same circumference of the generating circle, drawn into x or AD, the height of the segment.

Also when x or AD becomes equal to the whole diameter a, the expression $a\pi x$ becomes $a\pi a$ or πa^2, or 4 times the area of the generating circle, for the surface of the whole sphere.

And these agree with the rules before found in Mensuration of Solids.

Exam. 2. To find the surface of a spheroid.

Exam. 3. To find the surface of a paraboloid.

Exam. 4. To find the surface of an hyperboloid.

TO FIND THE CONTENTS OF SOLIDS.

105. Any solid which is formed by the revolution of a curve about its axis (see last fig.), may also be conceived to be generated by the motion of the plane of an expanding circle, moving perpendicularly along the axis. And therefore the area of that circle being drawn into the fluxion of the axis will produce the fluxion of the solid. That is, $\dot{\text{AD}}$ × area of the circle BCF, whose radius is DE, or diameter BE, is the fluxion of the solid, by art. 9.

106. Hence, if AD $= x$, DE $= y$, $\pi = 3{\cdot}141593$; because πy^2 is equal to the area of the circle BCF; therefore $cy^2\dot{x}$ is the fluxion of the solid. Consequently if, from the given equation of the curve, the value of either y^2 or x be found, and that value substituted for it in the expression $\pi y^2\dot{x}$, the fluent of the resulting quantity, being taken, will be the solidity of the figure proposed.

EXAMPLES.

Exam. 1. To find the solidity of a sphere, or any segment.

The equation to the generating circle being $y^2 = ax - x^2$, where a denotes the diameter, by substitution, the general fluxion of the solid $\pi y^2\dot{x}$, becomes $\pi ax\dot{x} - \pi x^2\dot{x}$, the fluent of

which gives $\frac{1}{2}\pi a x^2 - \frac{1}{3}\pi x^3$, or $\frac{1}{6}\pi x^2(3a-2x)$, for the solid content of the spherical segment BAE, whose height AD is x.

When the segment becomes equal to the whole sphere, then $x = a$, and the above expression for the solidity, becomes $\frac{1}{6}\pi a^3$ for the solid content of the whole sphere.

And these deductions agree with the rules before given and demonstrated in the Mensuration of Solids.

Exam. 2. To find the solidity of a spheroid.

Exam. 3. To find the solidity of a paraboloid.

Exam. 4. To find the solidity of an hyperboloid.

Exam. 5. To find the solidity of a body, or segment, or frustum, produced by the revolution upon its axis of any curve denoted by the general equation

$$y^2 = A + Bx + Cx^2.$$

Where AP $= x$, PM $= y$; and taking the several cases when A, B, or C, become equal to nothing, and those in which they have finite values.

TO FIND LOGARITHMS.

107. It has been proved, art. 76, that the fluxion of the hyperbolic logarithm of a quantity, is equal to the fluxion of the quantity divided by the same quantity. Therefore, when any quantity is proposed, to find its logarithm; take the fluxion of that quantity, and divide it by the same quantity; then take the fluent of the quotient, either in a series or otherwise, and it will be the logarithm sought; when corrected as usual, if need be; that is, the hyperbolic logarithm.

108. But, for any other logarithm, multiply the hyperbolic logarithm, above found, by the modulus of the system, for the logarithm sought.

Note. The modulus of the hyperbolic logarithms, is 1; and the modulus of the common logarithms, is ·43429448190 &c.; and, in general, the modulus of any system, is equal to the logarithm of 10 in that system divided by the number 2·3025850929940, &c. which is the hyp. log. of 10. Also, the hyp. log. of any number, is in proportion to the com. log. of the same number, as unity or 1 is to ·43429, &c. or as the number 2·302585, &c. is to 1; and therefore, if the common log. of any number be multiplied by 2·302585, &c. it will

$$\tfrac{1}{3}\left(\frac{1}{2n^2-1}\right)^3 + \tfrac{1}{5}\left(\frac{1}{(2n^2-1)}\right)^5 + \&c. \Big\}$$

But $\log. \dfrac{n^2}{(n-)\ (n+1)} = 2 \log. n - \log. (n-1) -$ $\log. (n+1)$; therefore, putting N for the series

$$2\text{M} \left\{ \frac{1}{2n^2-1} + \tfrac{1}{3}\left(\frac{1}{2n^2-1}\right)^3 + \tfrac{1}{5}\left(\frac{1}{2n^2-1}\right)^5 + \&c. \right\}$$

we have this formula,

$$2 \log. n - \log. (n-1) - \log. (n+1) = \text{N}:$$

and hence, as often as we have the logarithms of any two of three numbers whose common difference is unity, the logarithm of the remaining number may be found. Example. Having given

the common log. of $9 = 0.95424250943$
the common log. of $10 = 1$;

it is required to find the common logarithm of 11.

Here we have $n = 10$, so that the formula gives in this case $2 \log. 10 - \log. 9 - \log. 11 = \text{N}$, and hence we have

$$\log. 11 = 2 \log. 10 - \log. 9 - \text{N},$$

$$\text{where N} = \frac{2\text{M}}{199} + \frac{2\text{M}}{3 \cdot 199^3} + \&c.$$

M being $\cdot 43429448190$.

Calculation of N.

$$\text{A} = \frac{2\text{M}}{199} = \cdot 00436476866$$

$$\text{B} = \frac{\text{A}}{3 \cdot 199^2} = \cdot 00000003674$$

$$\text{N} = \cdot 00436480540$$

$$2 \log. 10 = 2 \cdot 00000000000$$

$$\log. 9 = 0 \cdot 95424250943$$
$$\text{N} = 0 \cdot 00436480540$$

$\log. 9 + \text{N} = 0 \cdot 95860731483$: taken from

2 log. 10, leaves $\log. 11 = 1 \cdot 04139268517$

110. Here the series expressed by N converges very fast, so that two of its terms are sufficient to give the logarithm true to 10 places of decimals. But the logarithm of 11 may be expressed by the logarithms of smaller numbers and a series which converges still more rapidly, by the following artifice, which will apply also to some other numbers. Because the numbers 98, 99, and 100 are the products of numbers, the greatest of which is 11, for $98 = 2 \times 7^2$, $99 = 9 \times 11$,

and $100 = 10 \times 10$, it follows that if we have an equation composed of terms which are the logarithms of these three numbers, it may be resolved into another, the terms of which shall be the logarithms of the number 11 and other smaller numbers. Now by the preceding formula, if we put 99 for n, we have

$$2 \log. 99 - \log. 98 - \log. 100 = \text{N}.$$

that is, substituting log. 9 + log. 11 for log. 99, log. 2 + 2 log. 7 for log. 98, and 2 log. 10 for log. 100,

$$2 \log. 9 + 2 \log. 11 - \log. 2 - 2 \log. 7 - 2 \log. 10 = \text{N},$$

and hence by transposition, &c.

$$\log. 11 = \tfrac{1}{2}\text{N} + \tfrac{1}{2}\log. 2 + \log. 7 - \log. 9 + \log. 10;$$

and in this equation,

$$\text{N} = \frac{2\text{M}}{19601} + \frac{1}{3}\frac{2\text{M}}{19601^3} + \&c.$$

The first term alone of this series is sufficient to give the logarithm of 11 true to 14 places.

111. When it is required to find the logarithm of a high number, as for example 1231, we may proceed as follows:

$$\log. 1231 = \log. (1230 + 1) = \log. \left\{ 1230\left(1 + \frac{1}{1230}\right) \right\}$$

$$= \log. 1230 + \log.\left(1 + \frac{1}{1230}\right)$$

Again, log. 1230 = log. 2 + log. 5 + log. 123, and

$$\log. 123 = \log. \left\{ 120\left(1 + \frac{1}{40}\right) \right\}$$

$$= \log. 120 + \log.\left(1 + \frac{1}{40}\right)$$

$\log. 120 = \log. (2^3 \times 3 \times 5) = 3 \log. 2 + \log. 3 + \log. 5.$
Therefore

$$\log. 1231 = 4 \log. 2 + \log. 3 + 2 \log. 5 + \log. \left(1 + \frac{1}{40}\right)$$

$$+ \log.\left(1 + \frac{1}{1230}\right).$$

Thus the logarithm of the proposed number is expressed by the logarithms of 2, 3, 5, and the logarithms of

$1 + \frac{1}{40}$, $1 + \frac{1}{1230}$, all of which may be easily found by the formulæ already delivered.

112. When it is required to interpose one logarithm between a series of equidistant terms in a table, it may be effected upon the principle of interpolation by means of the well-known theorem; viz.

$$a - nb + n \cdot \frac{n-1}{2} \cdot c - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} d + \&c. = 0.$$

Thus, suppose there were given the logs. of 101, 102, 104, and 105, and that of 103 were required.

Here the number of equal intervals is 4, and of terms 5; so that the general form becomes

$a - 4b + 6c - 4d + e = 0$; and $c = \frac{1}{6}[4(b+d) - (a+e)]$

$$\begin{aligned} a &= 2\cdot0043214 \\ b &= 2\cdot0086002 \\ d &= 2\cdot0170333 \\ e &= 2\cdot0211893 \end{aligned}$$

$$\begin{aligned} 4(b+d) &= 16\cdot1025340 \\ a+e &= 4\cdot0255107 \end{aligned}$$

$$6)12\cdot0770233$$

$$\text{Log. of } 103 = 2\cdot0128372$$

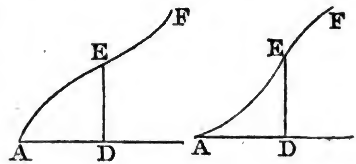
Exam. 1. Given the logs. of 999 and 1000. Required the log. of 1001.

Exam. 2. Given the logs. of 51, 53, 57, and 59; to find the log. of 55.

On this interesting and important subject, consult the Introduction to Dr. Hutton's Mathematical Tables, and Hellins's Mathematical Essays.

TO FIND THE POINTS OF INFLEXION, OR OF CONTRARY FLEXURE IN CURVES.

113. The Point of Inflexion in a curve is that point of it which separates the concave from the convex part, lying between the two: or where the curve changes from concave to convex, or from convex to concave, on the same side of the curve. Such as the point E in the annexed figures, where the former of the two is concave towards the axis AD, from A to E, but convex from E to F; and on the contrary,



the latter figure is convex from A to E, and concave from E to F.

114. From the nature of curvature, as has been remarked before at art. 85, it is evident, that when a curve is concave towards an axis, then the fluxion of the ordinate decreases, or is in a decreasing ratio, with regard to the fluxion of the absciss; but, on the contrary, that it increases, or is in an increasing ratio to the fluxion of the absciss, when the curve is convex towards the axis; and consequently those two fluxions are in a constant ratio at the point of inflexion, where the curve is neither convex nor concave; that is, $\dot{x}$ is to $\dot{y}$ in a constant ratio, or $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$ is a constant quantity. But constant quantities have no fluxion, or their fluxion is equal to nothing: so that, in this case, the fluxion of $\frac{\dot{y}}{\dot{x}}$ or of $\frac{\dot{x}}{\dot{y}}$ is equal to nothing. And hence we have this general rule:

115. Put the given equation of the curve into fluxions; from which find either $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$. Then take the fluxion of this ratio, or fraction, and put it equal to 0 or nothing; and from this last equation find also the value of the same $\frac{\dot{x}}{\dot{y}}$ or $\frac{\dot{y}}{\dot{x}}$. Then put this latter value equal to the former, which will form an equation; from which, and the first given equation of the curve, x and y will be determined, being the absciss and ordinate answering to the point of inflexion in the curve, as required.

EXAMPLES.

Exam. 1. To find the point of inflexion in the curve whose equation is $ax^2 = a^2y + x^2y$.

This equation in fluxions is $2ax\dot{x} = a^2\dot{y} + 2xy\dot{x} + x^2\dot{y}$, which gives $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{2ax - 2xy}$. Then the fluxion of this quantity made $= 0$, gives $2x\dot{x}\,(ax - xy) = (a^2 + x^2) \times (a\dot{x} - \dot{x}y - x\dot{y})$; and this again gives $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{a^2 - x^2} \times \frac{x}{a - y}$.

Lastly, this value of $\frac{\dot{x}}{\dot{y}}$ being put equal to the former, gives

$\frac{a^2+x^2}{a^2-x^2}\cdot\frac{x}{a-y} = \frac{a^2+x^2}{2x}\cdot\frac{1}{a-y}$; and hence $2x^2 = a^2 - x^2$, or $3x^2 = a^2$, and $x = a\sqrt{\frac{1}{3}}$, the absciss.

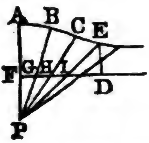
Hence also, from the original equation, ...

$y = \frac{ax^2}{a^2+x^2} = \frac{\frac{1}{3}a^3}{\frac{4}{3}a^2} = \frac{1}{4}a$, the ordinate of the point of inflexion sought.

Exam. 2. To find the point of inflexion in a curve defined by the equation $ay = a\sqrt{ax+x^2}$.

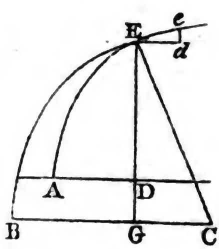
Exam. 3. To find the point of inflexion in a curve defined by the equation $ay^2 = a^2x + x^3$.

Exam. 4. To find the point of inflexion in the Conchoid of Nicomedes, which is generated or constructed in this manner: From a fixed point P, which is called the pole of the conchoid, draw any number of right lines, PA, PB, PC, PE, &c. cutting the given line FD in the points F, G, H, I, &c.; then make the distances FA, GB, HC, IE, &c. equal to each other, and equal to a given line; then the curve line ABCE, &c. will be the conchoid; a curve so called by its inventor Nicomedes.



TO FIND THE RADIUS OF CURVATURE OF CURVES.

116. The Curvature of a Circle is constant, or the same in every point of it, and its radius is the radius of curvature. But the case is different in other curves, every one of which has its curvature continually varying, either increasing or decreasing, and every point having a degree of curvature peculiar to itself; and the radius of a circle which has the same curvature with the curve at any given point, is the radius of curvature at that point; which radius it is the business of this chapter to find.



117. Let AE*e* be any curve, concave towards its axis AD; draw an ordinate DE to the point E, where the curvature is to be found; and

suppose EC perpendicular to the curve, and equal to the radius of curvature sought, or equal to the radius of a circle having the same curvature there, and with that radius describe the said equally curved circle BE*e*; lastly, draw E*d* parallel to AD, and *de* parallel and indefinitely near to DE : thereby making E*d* the fluxion or increment of the absciss AD, also *de* the fluxion of the ordinate DE, and E*e* that of the curve AE. Then put $x =$ AD, $y =$ DE, $z =$ AE, and $r =$ CE the radius of curvature ; then E$d = \dot{x}$, $de = \dot{y}$, and E$e = \dot{z}$.

Now, by sim. triangles, the three lines E*d*, *de*, E*e*,
which vary as $\dot{x}, \dot{y}, \dot{z}$,
are respectively as the three - - GE, GC, CE ;
therefore - - - - - $\mathrm{GC} . \dot{x} = \mathrm{GE} . \dot{y}$;
and the flux. of this eq. is $\mathrm{GC} . \ddot{x} + \mathrm{G\dot{C}} . \dot{x} = \mathrm{GE} . \ddot{y} + \mathrm{G\dot{E}} . \dot{y}$,
or because $\mathrm{G\dot{C}} = -\mathrm{B\dot{G}}$, it is $\mathrm{GC} . \ddot{x} - \mathrm{B\dot{G}} . \dot{x} = \mathrm{GE} . \ddot{y} + \mathrm{G\dot{E}} . \dot{y}$.

But since the two curves AE and BE have the same curvature at the point E, their abscisses and ordinates have the same fluxions at that point, that is, E*d* or $\dot{x}$ is the fluxion both of AD and BG, and *de* or $\ddot{y}$ $\propto$ the fluxion both of DE and GE. In the equation above therefore substitute $\dot{x}$ for $\mathrm{B\dot{G}}$, and $\dot{y}$ for $\mathrm{G\dot{E}}$, and it becomes

$$\mathrm{GC}\ddot{x} - \dot{x}\dot{x} = \mathrm{GF}\ddot{y} + \dot{y}\dot{y},$$
$$\text{or } \mathrm{GC}\ddot{x} - \mathrm{GF}\ddot{y} = \dot{x}^2 + \dot{y}^2 = \dot{z}^2.$$

Now multiply the three terms of this equation respectively by these three quantities, $\frac{\dot{y}}{\mathrm{GC}}, \frac{\dot{x}}{\mathrm{GE}}, \frac{\dot{z}}{\mathrm{CE}}$, which are all equal, and

it becomes . $\dot{y}\ddot{x} - \dot{x}\ddot{y} = \frac{\dot{z}^3}{\mathrm{CE}}$, or $\frac{\dot{z}^3}{r}$;

and hence is found $r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, for the general value of the radius of curvature, for all curves whatever, in terms of the fluxions of the absciss and ordinate.

118. Further, as in any case either x or y may be supposed to flow equably, that is, either $\dot{x}$ or $\dot{y}$ constant quantities, or $\ddot{x}$ or $\ddot{y}$ equal to nothing, it follows that, by this supposition, either of the terms of the denominator, of the value of r, may be made to vanish. Thus, when $\dot{x}$ is supposed constant, $\ddot{x}$ being then $= 0$, the value of r is barely - - - - - - - -
$\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$; or r is $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$ when $\dot{y}$ is constant.

EXAMPLES.

EXAM. 1. To find the radius of curvature to any point of a parabola, whose equation is $ax=y^2$, its vertex being A, and axis AD.

Here, the equation to the curve being $ax = y^2$, the fluxion of it is $a\dot{x} = 2y\dot{y}$; and the fluxion of this again is $a\ddot{x} = 2\dot{y}^2$, supposing $\dot{y}$ constant; hence then r or

$$\frac{\dot{z}^3}{\dot{y}\ddot{x}} \text{ or } \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{y}\ddot{x}} \text{ is } = \frac{(a^2+4y^2)^{\frac{3}{2}}}{2a^2} \text{ or } \frac{(a+4x)^{\frac{3}{2}}}{2\sqrt{a}},$$

for the general value of the radius of curvature at any point E, the ordinate to which cuts off the absciss AD $= x$.

Hence, when the absciss x is nothing, the last expression becomes barely $\frac{1}{2}a = r$, for the radius of curvature at the vertex of the parabola; that is, the diameter of the circle of curvature at the vertex of a parabola, is equal to a, the parameter of the axis. See, also, pa. 535, vol. i.

EXAM. 2. To find the radius of curvature of an ellipse, whose equation is $a^2y^2 = c^2(ax - x^2)$.

$$\text{Ans. } r = \frac{[a^2c^2+4(a^2-c^2)\times(ax-x^2)]^{\frac{3}{2}}}{2a^4c}.$$

EXAM. 3. To find the radius of curvature of an hyperbola, whose equation is $a^2y^2 = c^2(ax + x^2)$.

$$\text{Ans. } r = \frac{[a^2c^2+4(a^2+c^2)\times(ax+x^2)]^{\frac{3}{2}}}{2a^4c}.$$

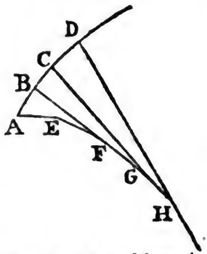
EXAM. 4. To find the radius of curvature of the cycloid.

Ans. $r = 2\sqrt{(aa - ax)}$, where x is the absciss, and a the diameter of the generating circle.

OF INVOLUTE AND EVOLUTE CURVES.

119. AN Evolute is any curve supposed to be evolved or opened, which having a thread wrapped close about it, fastened at one end, and beginning to evolve or unwind the thread from the other end, keeping always tight stretched the part which is evolved or wound off: then this end of the thread will describe another curve, called the Involute. Or, the same involute is described in the contrary way, by wrap-

ping the thread about the curve of the evolute, keeping it at the same time always stretched.

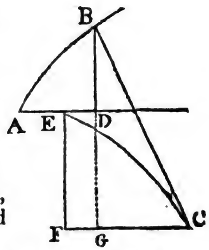


120. Thus, if EFGH be any curve, and AE be either a part of the curve, or a right line: then if a thread be fixed to the curve at H, and be wound or plied close to the curve, &c. from H to A, keeping the thread always stretched tight; the other end of the thread will describe a certain curve ABCD, called an Involute; the first curve EFGH being its evolute. Or, if the thread, fixed at H, be unwound from the curve, beginning at A, and keeping it always tight, it will describe the same involute ABCD.

121. If AE, BF, CG, DH, &c. be any positions of the thread, in evolving or unwinding; it follows, that these parts of the thread are always the radii of curvature, at the corresponding points, A, B, C, D; and also equal to the corresponding lengths AE, AEF, AEFG, AEFGH, of the evolute; that is,

AE = AE is the radius of curvature to the point A,
BF = AEF is the radius of curvature to the point B,
CG = AEG is the radius of curvature to the point C,
DH = AEH is the radius of curvature to the point D.

122. It also follows, from the premises, that any radius of curvature, BF, is perpendicular to the involute at the point B, and is a tangent to the evolute curve at the point F. Also, that the evolute is the locus of the centre of curvature of the involute curve.



123. Hence, and from art. 117, it will be easy to find one of these curves, when the other is given. To this purpose, put

x = AD, the absciss of the involute,
y = DB, an ordinate to the same,
z = AB, the involute curve,
r = BC, the radius of curvature,
v = EF, the absciss of the evolute EC,
u = FC, the ordinate of the same, and
a = AE, a certain given line.

Then by the nature of the radius of curvature, it is

$r = \dfrac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$ = BC = AE + EC; also, by sim. triangles,

$\dot{z} : \dot{x} :: r :$ GB $= \dfrac{r\dot{x}}{\dot{z}} = \dfrac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \dfrac{\dot{z}^2}{-\ddot{y}}$;

$$\dot z : \dot y :: r : \text{GC} = \frac{r\dot y}{\dot z} = \frac{\dot y \dot z^2}{\dot y \ddot x - \dot x \ddot y} = \frac{\dot y \dot z^2}{-\dot x \ddot y}.$$

Hence $\text{EF} = \text{GB} - \text{DB} = \dfrac{\dot z^2}{-\ddot y} - y = v$;

and $\text{FC} = \text{AD} - \text{AE} + \text{GC} = x - a + \dfrac{\dot y \dot z^2}{-\dot x \ddot y} = u$;

which are the values of the absciss and ordinate of the evolute curve EC; from which therefore these may be found, when the involute is given.

On the contrary, if v and u, or the evolute, be given: then, putting the given curve $\text{EC} = s$, since $\text{CB} = \text{AE} + \text{EC}$, or $r = a + s$, this gives r the radius of curvature. Also, by similar triangles, there arise these proportions, viz.

$$\dot s : \dot v :: r : \frac{r\dot v}{\dot s} = \frac{a+s}{\dot s}\dot v = \text{GB},$$

and $\dot s : \dot u :: r : \dfrac{r\dot u}{\dot s} = \dfrac{a+s}{\dot s}\dot u = \text{GC}$;

theref. $\text{AD} = \text{AE} + \text{FC} - \text{GC} = a + u - \dfrac{a+s}{\dot s}\dot u = x$,

and $\text{DB} = \text{GB} - \text{CD} = \text{GB} - \text{EF} = \dfrac{a+s}{\dot s}\dot v - v = y$;

which are the absciss and ordinate of the involute curve, and which may therefore be found, when the evolute is given. Where it may be noted, that $\dot s^2 = \dot v^2 + \dot u^2$, and $\dot z^2 = \dot x^2 + \dot y^2$. Also, either of the quantities x, y, may be supposed to flow equably, in which case the respective second fluxion, $\ddot x$ or $\ddot y$, will be nothing, and the corresponding term in the denominator $\dot y \ddot x - \dot x \ddot y$ will vanish, leaving only the other term in it; which will have the effect of rendering the whole operation simpler.

EXAMPLES.

EXAM. 1. To determine the nature of the curve by whose evolution the common parabola AB is described.

Here the equation of the given evolute AB, is $cx = y^2$ where c is the parameter of the axis AD. Hence then $y = \sqrt{cx}$, and $\dot y = \frac{1}{2}\dot x\sqrt{\dfrac{c}{x}}$, also $\ddot y = \dfrac{-\dot x^2}{4x}\sqrt{\dfrac{c}{x}}$ by making $\dot x$

constant. Consequently the general values of v and u, or of the absciss and ordinate, EF and FC, above given, become, in that case,

$$\text{EF} = v = \frac{\dot{z}^2}{-\ddot{y}} - y = \frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}} - y = 4x\sqrt{\frac{x}{c}}; \text{ and}$$

$$\text{FC} = u = x - a + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}} = 3x + \tfrac{1}{2}c - a.$$

But the value of the quantity a or AE, by exam. 1 to art. 118, was found to be $\frac{1}{2}c$; consequently the last quantity, FC or u, is barely $= 3x$.

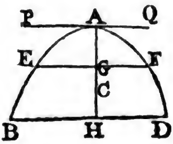
Hence then, comparing the values of v and u, there is found $3v\sqrt{c} = 4u\sqrt{x}$, or $27cv^2 = 16u^3$; which is the equation between the absciss and ordinate of the evolute curve EC, showing it to be the semicubical parabola.

Exam. 2. To determine the evolute of the common cycloid.

Ans. another cycloid, equal to the former.

TO FIND THE CENTRE OF GRAVITY.

124. By referring to art. 108, &c. in the *Statics*, it is seen what are the principles and nature of the Centre of Gravity in any figure, and how it is generally expressed. It there appears, that if PAQ be a line, or a plane, drawn through any point, as suppose the vertex of any body, or figure, ABD, and if s denote any section EF of the figure, d = AG, its distance below PQ, and b = the whole body or figure ABD; then the distance AC, of the centre of gravity below PQ, is universally denoted by $\frac{\text{sum of all the } ds}{b}$ whether ABD be a line, or a plane surface, or a curve superficies, or a solid.



But the sum of all the ds, is the same as the fluent of $d\dot{b}$, and b is the same as the fluent of $\dot{b}$; therefore the general expression for the distance of the centre of gravity, is AC $= \frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent } x\dot{b}}{b}$; putting $x = d$ the variable distance AG. Which will divide into the following four cases.

125. Case 1. When AE is some line, as a curve suppose. In this case $\dot b$ is $= \dot z$ or $\sqrt{\dot x^2 + \dot y^2}$, the fluxion of the curve; and $b = z$: theref. AC $= \frac{\text{fluent of } x\dot z}{z} = \frac{\text{fluent of } x\sqrt{\dot x^2 + \dot y^2}}{z}$ $= \frac{fx\dot z}{z}$ is the distance of the centre of gravity in a curve.

126. Case 2. When the figure ABD is a plane; then $\dot b = y\dot x$; therefore the general expression becomes AC $=$ $\frac{fyx\dot x}{fy\dot x}$ for the distance of the centre of gravity in a plane.

127. Case 3. When the figure is the superficies of a body generated by the rotation of a line AEB, about the axis AH. Then putting $\pi = 3{\cdot}14159$, &c. $2\pi y$ will denote the circumference of the generating circle, and $2\pi y\dot z$ the fluxion of the surface: therefore AC $= \frac{\text{fluent of } 2\pi yx\dot z}{\text{fluent of } 2\pi y\dot z} = \frac{fyx\dot z}{fy\dot z}$ will be the distance of the centre of gravity for a surface generated by the rotation of a curve line z.

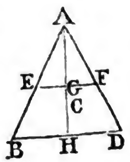
128. Case 4. When the figure is a solid generated by the rotation of a plane ABH, about the axis AH.

Then is $\pi y^2 =$ the area of the circle whose radius is y, and $\pi y^2\dot x = \dot b$, the fluxion of the solid; therefore AC $= \frac{\text{fluent of } x\dot b}{\text{fluent of } \dot b} = \frac{\text{fluent of } \pi y^2 x\dot x}{\text{fluent of } \pi y^2\dot x} = \frac{fy^2x\dot x}{fy^2\dot x}$ is the distance of the centre of gravity below the vertex in a solid.

EXAMPLES.

Exam. 1. Let the figure proposed be the isosceles triangle ABD.

It is evident that the centre of gravity C, will be somewhere in the perpendicular AH. Now, if a denote AH, $c =$ BD, $x =$ AG, and $y =$ EF any line parallel to the base BD: then as $a : c :: x : y = \frac{cx}{a}$; therefore, by the 2d Case, AC $= \frac{\text{fluent } yx\dot x}{\text{fluent } y\dot x} = \frac{\text{fluent } x^2\dot x}{\text{fluent } x\dot x} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2}$ $= \frac{2}{3}x = \frac{2}{3}$AH, when x becomes $=$ AH: consequently CH $= \frac{1}{3}$AH.



In like manner, the centre of gravity of any other plane triangle, will be found to be at $\frac{1}{3}$ of the altitude of the triangle; the same as it was found in art. 111, *Statics*.

Exam. 2. In a parabola; the distance from the vertex is $\frac{3}{5}x$, or $\frac{3}{5}$ of the axis.

Exam. 3. In a circular arc; the distance from the centre of the circle, is $\frac{cr}{a}$; where a denotes the arc, c its chord, and r the radius.

Exam. 4. In a circular sector; the distance from the centre of the circle, is $\frac{2cr}{3a}$: where a, c, r, are the same as in exam. 3.

Exam. 5. In a circular segment; the distance from the centre of the circle is $\frac{c^3}{12a}$; where c is the chord, and a the area, of the segment.

Exam. 6. In a cone, or any other pyramid; the distance from the vertex is $\frac{3}{4}x$, or $\frac{3}{4}$ of the altitude.

Exam. 7. In the semisphere, or semispheroid; the distance from the centre is $\frac{3}{8}r$, or $\frac{3}{8}$ of the radius: and the distance from the vertex $\frac{5}{8}$ of the radius.

Exam. 8. In the parabolic conoid; the distance from the base is $\frac{1}{3}x$, or $\frac{1}{3}$ of the axis. And the distance from the vertex $\frac{2}{3}$ of the axis.

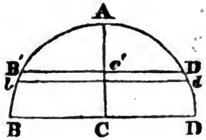
Exam. 9. In the segment of a sphere, or of a spheroid; the distance from the base is $\frac{2a-x}{6a-4x}x$; where x is the height of the segment, and a the whole axis, or diameter of the sphere.

Exam. 10. In the hyperbolic conoid; the distance from the base is $\frac{2a+x}{6a+4x}x$; where x is the height of the conoid, and a the whole axis or diameter.

129. Among the preceding examples, those which relate to circles and spheres, furnish pleasing applications of the fluxional formulæ for the trigonometrical quantities. Thus,

1. To find the centre of gravity of a circular arc.

Let AB′ $= z$, B′$b = \dot{z}$
rad. $= 1$.



Then $\dfrac{f x\dot{z}}{z} = \dfrac{f 2 \cos. z\dot{z}}{2z} = \dfrac{2 \sin. z}{2z}$.

Hence, BAD : BD : : rad. : dist. C. G. from A.

2. To find the centre of gravity of a circular segment.

Here $\dfrac{f xy\dot{x}}{f y\dot{x}} = \dfrac{f 2 \sin. z \cos. z \varphi \cos. z}{f 2 \sin. z \varphi \cos. z} = \dfrac{f 2 \sin.^2 z \cos. z\dot{z}}{f 2 \sin.^2 z\dot{z}}$

$$= \frac{f 2 \sin.^2 z \varphi \sin. z}{f(1-\cos. 2z)\dot{z}} = \frac{\frac{2}{3} \sin.^3 z}{z - \frac{1}{2} \sin. 2z}.$$

3. To find the distance of C. G. of a spheric surface from its centre.

$$\frac{f 2\pi \sin. z \cos. z\dot{z}}{f 2\pi \sin. z\dot{z}} = \frac{\pi \sin.^2 z}{2\pi(1-\cos. z)} = \tfrac{1}{2}(1 + \cos. z):$$

that is, the middle point of the versed sine.

4. For the C. G. of a spherical segment.

$$\frac{f y^2 x\dot{x}}{f y^2 \dot{x}} = \frac{f \sin.^2 z \cos. \ z(-\varphi \cos. z)}{f \sin.^2 z \ (-\varphi \cos. z)} = \frac{f \sin.^2 z \cos. z \varphi \cos. z}{f \sin.^2 z \varphi \cos. z}$$

$$= \frac{f \sin.^2 z \cos. z \sin. z\dot{z}}{f \sin.^2 z(-\varphi \cos. z)} = \frac{f \sin.^3 z \cos. z\dot{z}}{f \sin.^2 z \ (-\varphi \cos. z)}$$

$$= \frac{f \sin.^3 z \varphi \sin. z}{f \sin.^2 z(-\varphi \cos. z)} = \frac{\frac{1}{4} \sin.^4 z}{f(1-\cos.^2 z)(-\varphi \cos. z)}$$

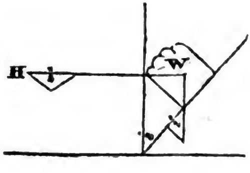
$$= \frac{\frac{1}{4} \sin.^4 z}{f(\cos.^2 z - 1) \varphi \cos. z} = \frac{\frac{1}{4} \sin.^4 z}{\frac{1}{3}(\cos.^3 z - 3 \cos. z)},$$

which corrected becomes $= \dfrac{\frac{1}{4} \sin.^4 z}{\frac{1}{3}(\cos.^3 z - 3 \cos. z + 2)}$.

When the segment becomes a hemisphere, this becomes $= \frac{3}{8}$ of radius, as it ought to be.

130. *Pressure of Earth against Walls.*

Lemma. A weight, W, being placed on a plane, inclined to the vertical in an angle i, to find a horizontal force, H, sufficient to sustain it, so that it shall not run down the plane, taking friction into the account.



Each of the forces, W, H, being resolved into two, the one parallel, the other perpendicular, to the plane; there will result,

parallel to the plane, a force $= \text{W} \cos. i - \text{H} \sin. i$,
perp. to the plane, a force $= \text{W} \sin. i + \text{H} \cos. i$.

In order to an equilibrium, the first of these forces ought to be precisely equal to the friction down the plane.

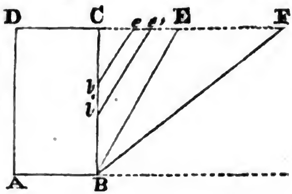
That is, $\text{W} \cos. i - \text{H} \sin. i = f\text{W} \sin. i + f\text{H} \cos. i$,
whence $f\text{H} \cos. i + \text{H} \sin. i = -f\text{W} \sin. i + \text{W} \cos. i$,

$$\text{and } \text{H} = \text{W} \frac{\cos. i - f \sin. i}{\sin. i + f \cos. i} = \text{W} \frac{1 - f \tan. i,}{\tan. i + f}.$$

Corol. Hence, if instead of a horizontal force, the weight W were sustained by a wall, or by any obstacle whatever, the horizontal effort exerted by the weight against the obstacle would be $\text{W} . \dfrac{1 - f \tan. i}{\tan. i + f}$.

131. PROP. To determine the horizontal stress of the terrace whose vertical section is BCEF, against the wall whose section is ABCD, and the momentum of the pressure to overturn the wall about the angle A.

Considering, first, the stress of a triangle CBE, whose sloping side BE makes the angle i with the vertical: let be, $b'e'$, be each parallel to BE, limiting the elementary trapezoid $bb'e'e$. Let $\text{BC} = a$, $cb = x$, $bb' = \dot{x}$; then area of $bb'e'e = x\dot{x} \tan. i$; and if s be the specific gravity of the earth, the weight of the portion $bb'e'e$ will be $= sx\dot{x} \tan. i$. Therefore the horizontal effort, against the line bb', will be



$$= sx\dot{x} \tan. i . \frac{1 - f \tan. i}{\tan. i + f} = sx\dot{x} \frac{1 - f \tan. i}{1 + f \cot. i}$$

$$= sx\dot{x}\text{M} \text{ ; putting } \frac{1 - f \tan. i}{1 + f \tan. i} = \text{M}.$$

The fluent of $sx\dot{x}\text{M}$, when $x = a$, gives $\frac{1}{2}a^2 s\text{M}$, for the whole horizontal thrust of the triangle CBE.

Referring the momentum of the thrust of the elementary portion $bb'e'e$, to the length of lever $b\text{B} = a - x$, we have for that momentum $\text{M}s\,(a - x)x\dot{x}$. The fluent of this when $x = a$, is $= \frac{1}{6}a^3 s\text{M}$.

132. It remains to determine the angle i.

Now, it is evident that $\frac{1-f\tan. i}{1+f\cot. i} = \text{M}$, vanishes, and consequently, both the horizontal thrust and its momentum vanish, whether $\tan. i = 0$, or $= \frac{1}{f}$. Between these two values, therefore, there is one which gives both the greatest thrust and the greatest momentum. This value is found by making

$\varphi\text{M} = 0$, that is, $\varphi\,\frac{1-f\tan. i}{1+f\cot. i} = 0$. Put $\tan. i = z$,

$$\text{then } -f\dot{z}\left(1+\frac{f}{z}\right)+\frac{f\dot{z}}{z^2}(1-fz)=0\,;$$

$$\text{or } \dot{z}+\frac{f\dot{z}}{z}=\frac{\dot{z}-fz\dot{z}}{z^2},$$

$$1+\frac{f}{z}=\frac{1-fz}{z^2}\quad .\ .\quad z^2+fz=1-fz,$$

$$z^2+2fz=1\ .\ .\ z^2+2fz+f^2=1+f^2,$$

$$z+f=\sqrt{(1+f)^2}\quad .\ .\ .\quad z=-f+\sqrt{1+f^2},$$

that is,

$$\tan. i=-f+\sqrt{1+f^2}.$$

Substituting this value of $\tan. i$ for it in the above expression for M, we have for the horizontal thrust

$$\tfrac{1}{2}a^2s\left\{-f+\sqrt{(1+f^2)}\right\}^2=\tfrac{1}{2}\,a^2s\tan.^2 i,$$

while the momentum of the stress is found to be

$$\tfrac{1}{6}a^3s\left\{-f+\sqrt{(1+f^2)^2}\right\}=\tfrac{1}{6}a^3s\tan.^2 i,$$

which was to be found.

133. The angle which has for its tangent $\frac{1}{f}$ is the angle of the slope, which the earth would, of itself, naturally take, if it were not sustained by any wall.

For a body has a tendency to descend along a plane (inclination to vertical $=i$) with a force $=g\cos. i$, and it presses the plane with a force $=g\sin. i$. Wherefore the friction $=fg\sin. i$; and since it counterbalances the force with which the body endeavours to descend, we have

$$fg\sin. i=g\cos. i\ \therefore\ \frac{\sin. i}{\cos. i}=\tan. i=\frac{1}{f}\,;$$

$$\text{also } f=\cot. i.$$

Farther, the angle whose tangent is $-f+\sqrt{1+f^2}$ is half the angle whose tangent is $\frac{1}{f}$.

For $\tan. i = \frac{2\tan.\frac{1}{2}i}{1-\tan.^2\frac{1}{2}i}$. (Equa. 17, pa. 395, vol. i.)

$$\text{Or, } \frac{2[-f+\sqrt{1+f^2}]}{1-[-f+\sqrt{1+f^2}]^2} = \frac{-2f+2\sqrt{1+f^2}}{-2f^2+2f\sqrt{1+f^2}} = \frac{1}{f}.$$

Let, therefore, BF be the slope which loose earth would, of itself, naturally assume: then, the line BE which determines the triangle of earth that exerts the greatest horizontal stress against the vertical wall bisects the angle CBF.

134. SCHOLIUM.

Sandy and loose earth takes a natural declivity of 60° from the vertical; stronger earth will take a declivity of 53°. Therefore, for a terrace of loose earth we have $i = 30°$; for another of strong and close earth $i = 26\frac{1}{2}°$.

Hence, for the former kind, where $\tan. 30° = \frac{1}{3}\sqrt{3}$, the value of the stress is $\frac{1}{6}a^2s$, and that of the momentum of the stress $\frac{1}{18}a^3s$.

For the latter kind, where $\tan. 26\frac{1}{2}° = \frac{1}{2}$ nearly, the stress $= \frac{1}{8}a^2s$, its momentum $= \frac{1}{24}a^3s$.

135. The horizontal stress and momentum being thus known, it is easy to proportion to them the resistance of the wall ABCD.

Let $b =$ AB, while BC $= a$, and let s be the spec. grav. of the wall. For brick, $s = 2000$, for strong earth, $s = 1428$. Then the momentum of the resistance referred to the point AB, being $\frac{1}{2}ab^2$s; we shall have

$$\tfrac{1}{2}ab^2s = \tfrac{1}{24}a^3s \text{ (for strong earth)}$$

$$\therefore b = a\sqrt{\frac{s}{12s}} = \cdot 28034 \times a\sqrt{\frac{s}{\text{s}}}.$$

Thus, if $a = 39{\cdot}37$ feet, s and s as above, we shall find $b = 9{\cdot}326$ feet.

EXAM. 2. Supposing the earth of the same kind as in the above example, s to s, as 4 to 5, and the height of the wall and bank each 12 feet; required the thickness of the wall, being rectangular. Ans. 2·986 feet.

Note. The preceding investigation proceeds upon the principles assumed by *Coulomb* and *Prony*. They who wish to go thoroughly into this subject, and have not opportunity to make experiments, may advantageously consult *Traité Expérimental, Analytique et Pratique de la Poussée des Terres, &c. par M. Mayniel.*

ON THE FLEXIBILITY, STRENGTH, AND RUPTURE OF TIMBER, &c.

A piece of solid matter may be exposed to, at least, four distinct kinds of strains: viz.

1st. It may be pulled, or torn, asunder, as in the case of ropes, stretchers, king-posts, tie-beams, &c.

2dly. It may be crushed, as in the case of pillars, posts, and truss-beams.

3dly. It may be broken across, as in the case of a *joint* or rafter.

4thly. It may be wrenched, or twisted, as in the case of the axle of a wheel, the nail of a press, &c.

The complete investigation of these particulars, only in their principal varieties, would require a volume. The student who wishes to go into the inquiry with scientific precision, may consult M. Girard's Treatise on the Resistance of Solids, an interesting essay on the Flexibility of Wood, by M. Dupin, in *Journal de l'Ecole Polytechnique*, tome 10, Tredgold's Principles of Carpentry, and Mr. Barlow's valuable *Essay on the Strength and Stress of Timber*. Having attended many of the experiments recorded in the latter-mentioned work, I can with confidence recommend its principal results as accurate and useful; and shall, therefore, refer to the work itself for the experiments and investigations from which the following formulæ and rules are deduced.

Let l denote the length, a the breadth, d the depth of a rectangular beam, all in inches, w the weight with which it is loaded in the middle (being supported at both ends), δ the deflection occasioned by that weight, and E the measure of the elasticity: then it is found that $\frac{wl^3}{ad^3\delta} = E$ is a constant quantity, for the same timber; or, which amounts to the same, that $\frac{wl^3}{Ead^3} = \delta$.

This formula is equally applicable to beams fixed at one end, and loaded at the other, and those which are supported at both ends and loaded in the middle; only the value of E in the one case will be to that in the other, as 32 to 1.

For the ultimate deflection of beams before their rupture, the theorem is $\frac{l^2}{d\Delta} = U$, where Δ is the last deflection.

If the resistance of a rod an inch square be s, then ad^2s

will be the resistance of a beam the same length, whose breadth is a and depth d : also, if the angle of deflection be Δ, and the breaking weight be w ; then

1. *When the beam is fixed at one end, and loaded at the other.*

$$l\text{w} \cos. \Delta = ad^2\text{s}, \text{ or } \frac{l\text{w} \cos. \Delta}{ad^2} = \text{s, a constant quantity.}$$

2. *When the beam is supported at each end, and loaded in the middle.*

$$\tfrac{1}{4}l\text{w}sec^2 \Delta = ad^2\text{s}, \text{ or } \frac{l\text{w}sec^2 \Delta}{4ad^2} = \text{s, } constant.$$

3. *When the beam is fixed at each end, and loaded in the middle.*

$$\tfrac{1}{6}l\text{w}sec^2 \Delta = ad^2\text{s}, \text{ or } \frac{l\text{w}sec^2 \Delta}{6ad^2} = \text{s, } constant.$$

4. *When the beam in either of the two last cases is loaded at any other point than the centre.*

We shall have, in the former case, by denoting the two unequal lengths by m and n,

$$\frac{mn\text{w}}{l} sec^2 \Delta = ad^2\text{s}, \text{ or } \frac{mn\text{w}sec^2 \Delta}{lad^2} = \text{s} :$$

and in the second,

$$\frac{2mn\text{w}}{3l} sec^2 \Delta = ad^2\text{s}, \text{ or } \frac{2mn\text{w}sec^2 \Delta}{3lad^2} = \text{s},$$

still the same constant quantity.

And the first formula will also apply to a beam fixed at any given angle of inclination; observing only, that the angle Δ, in this case, will represent the angle of the beam's inclination, increased or diminished by the angle of its deflection, according as its first position is ascending or descending; or rather, it will denote the angle of the beam's inclination at the moment of fracture.

In all these cases, when it is only intended to apply the results to the common application of timber to architectural and other purposes, the angle of deflection may be omitted, and the equations then become simply,

1. $\frac{l\text{w}}{ad^2} = \text{s}$, 2. $\frac{l\text{w}}{4ad^2} = \text{s}$,

3. $\frac{l\text{w}}{6ad^2} = \text{s}$, 4. $\frac{mn\text{w}}{lad^2} = \text{s}$,

$$5. \quad \frac{2mnw}{3lad^2} = s.$$

The absolute value of direct cohesion on a square inch is $c = \frac{s'd^2}{(d-D)^2}$; where D is the depth of the natural axis, or of the line which separates the compressed from the stretched portion of the wood.

The subjoined portion of data for different kinds of wood, results from the union of these formulæ with experiments.

Name of the kind of Wood.	Spec. Grav.	Value of v.	Value of E	Value of s.	Value of s'.	Value of c.
Teak	745	818	9657802	2462	2488	15555
Poon	579	596	6759200	2221	2266	14787
Eng. Oak . . .	969	598	3494730	1181	1205	9836
Do. Spec. 2. .	934	435	5806200	1672	1736	10853
Canadian Oak	872	588	8595864	1766	1803	11428
Dantzic Oak .	756	724	4765750	1457	1477	7386
Adriatic Oak .	993	610	3885700	1583	1409	8808
Ash	760	395	6580750	2026	2124	17337
Beech	696	615	5417266	1556	1586	9912
Elm	553	509	2799347	1013	1042	5767
Pitch Pine . . .	660	588	4900466	1632	1666	10415
Red Pine	657	605	7359700	1341	1368	10000
New Eng. Fir.	553	757	5967400	1102	1116	9947
Riga Fir	753	588	5314570	1108	1131	10707
Do. Spec. 2. .	738	. .	3962800	1051	1081	. . . .
Mar Forest Fir	696	588	2581400	1144	1168	9539
Do. Spec. 2 . .	693	403	3478328	1262	1310	10691
Larch	531	411	2465433	658	890	. . . .
Do. Spec. 2 . .	522	518	3591133	832	850	. . . .
Do. Spec. 3 . .	556	518	4210830	1127	1149	7655
Do. Spec. 4 . .	560	518	4210830	1149	1172	7352
Norway Spar .	577	648	5832000	1474	1492	12180

Other tables and observations on the cohesive strength of metals, &c. are given in a subsequent part of this volume.

Solution of Practical Problems, from the preceding Data.

PROB. I. *To find the Strength of Direct Cohesion of a Piece of Timber of any given Dimensions.*

Rule.—Multiply the area of the transverse section, in inches, by the value of c, in the preceding table of data, and the product will be the strength required.

Note. If the specific gravity be not the same as the mean tabular specific gravity; say, as the latter is to the former, so is the above product to the correct result.

Exam. 1. What weight will it require to tear asunder a piece of teak 3 inches square, the specific gravity being 745?

Ans. 139950lbs.

Exam. 2. What weight will break vertically a cylinder of ash, 2 inches in diameter, and specific gravity 700?

Ans. 50166lbs.

Prob. II. *To compute the Deflection of Beams fixed at one End and loaded at the other with any given Weight.*

Rule 1. Multiply the tabular value of E by the breadth and cube of the depth of the given beam, both in inches.

2. Multiply also the cube of the length in inches by the given weight, and that product again by 32.

3. Divide the latter product by the former, for the deflection sought.

Exam. 1. An ash batten, 3 inches square, is fixed in a wall, and projects from it 4 feet. If a weight of 200lbs. be hung on its extremity, how much will it be deflected?

Ans. $1\frac{1}{3}$ inches.

Exam. 2. What would the same beam be deflected if a prop or shore, proceeding from the wall, met it at half its length?

Here, without repeating the operation, as we know that the deflections are as the cubes of the lengths; and as by means of the shore the length is reduced to one half the former, viz. to 2 feet, we have

$$4^3 : 2^3 :: 1\tfrac{1}{3} \text{ inches (former deflec.)} :$$

$$\frac{1\frac{1}{3} \times 2^3}{4^3} = \frac{1\frac{1}{3}}{8} = \frac{4}{24} = \tfrac{1}{6} \text{ of an inch, } \textit{answer}.$$

Exam. 3. A batten of New England fir, 6 feet long and 4 inches deep, by $2\frac{1}{2}$ inches in breadth, is fixed at one end, and loaded, uniformly throughout its length, with 200lbs., how much will its extremity be deflected?

Note. The same rule will apply, when the weight is distributed throughout the length, by multiplying the second product by 12 instead of 32.

Prob. III. *To compute the Deflection of Beams, supported at each End, and loaded in the Middle with any given Weight.*

Rule. 1. Multiply the tabular value of E by the breadth and cube of the depth, both in inches.

2. Multiply also the cube of the length, in inches, by the given weight in lbs.; then divide the latter product by the former for the deflection sought.

Exam. 1. A square beam of English oak, whose side is 6 inches, is supported on two walls, 20 feet distant, and is to be loaded at its middle point with 1000lbs., what will *it* bo deflected? Ans. 1·8 inch.

Exam. 2. A beam of red pine, 8 inches in breadth, and 1 foot deep, is supported on two walls, distant 33 feet 4 inches: how much will it be deflected with 2000lbs. suspended at its centre? Ans. $1\frac{1}{4}$ inches.

Note. If the beam be *fixed* at each end, the deflexion will, with equal weights, be two-thirds of that found by the above rule.

Prob. iv. *To compute the Deflection of Beams supported at each end, and loaded uniformly throughout their Length with a given Weight.*

Rule. Compute the deflection the same as in the last problem. Multiply that result by 5, and divide the product by 8, and the quotient will be the answer.

Exam. 1. A uniform bar of Adriatic oak, 2 inches square, is rested upon two props, distant 24 feet, how much will it be deflected by its own weight, its specific gravity being 960, or 60lbs. to the cubic foot? Ans. $9\frac{1}{2}$ inches.

Exam. 2. A beam of Riga fir, 12 inches square, is to support the brick work over a gateway, 12 feet wide; the computed weight of the brick work is 30000lbs., what deflection may be expected? Ans. ·58 inch.

Prob. v. *To compute the ultmate Deflection of Beams, or Rods, before their Rupture.*

Note. The beams are supposed to be supported at each end.

Rule. Multiply the tabular value of υ, in the preceding table of data, by the depth of the beam in inches, and divide the square of the length, also in inches, by that product, for the ultimate deflection sought.

Exam. A square inch rod of ash, 6 feet long, is broken by a weight applied to its centre: how much will it be deflected before it breaks? Ans. 13·1 inches.

Prob. vi. *To find the ultimate transverse Strength of any rectangular Beam of Timber, fixed at one End and loaded at the other.*

Rule i. Multiply the value of s, in the preceding table of data, by the breadth and square of the depth, both in inches, and divide that product by the length, also in inches, and the quotient will be the weight in lbs. This is approximative.

Rule. ii. 1. Take the ultimate deflection 8 times that of the last problem, and divide the deflection by the length, which will give the sine of the angle of deflection; whence, by a table, find the secant.

2. Multiply this secant by the breadth and square of the depth in inches, and the product again by the value of s′ in the table of data.

3. Divide this last product by the length in inches, and the quotient will be the answer, in lbs.

Exam. 1. What weight will it require to break a piece of Mar forest fir, fixed by one end in a wall, and loaded at the other; the breadth being 2 inches, depth 3 inches, and length 4 feet? Ans. 518lbs.

Exam. 2. A square oaken balk, 12 inches square, projects 8 feet 4 inches from a solid wall, in which it is fixed; what weight will be sufficient to break it? Ans. 50345lbs.

Exam. 3. A piece of ash, 2 inches square, projects 6 feet from a wall in which it is fixed; what weight, uniformly distributed through its length, will be required to break it?

Prob. vii. *To compute the ultimate transverse Strength of any rectangular Beam, when supported at both Ends and loaded in the Centre.*

Rule i. Multiply the tabular value of s by 4 times the breadth and square of the depth in inches, and divide that product by the length, also in inches, for the weight.

Rule ii. 1. Compute the ultimate deflection by Prob. v.; square that deflection, and divide it by the square of half the length of the beam, and add the quotient to 1, for the square of the secant of deflection; which multiply by the length in inches.

2. Multiply the tabular value of s′ by 4 times the breadth, and the square of the depth; and divide that product by the former, for the answer in lbs.

EXAM. What weight will be necessary to break a piece of larch similar to the 3rd specimen, the length being 8 feet 4 inches, the breadth 8 inches, and depth 10 inches; being supported at each end, and loaded in the middle?

Ans. 36676lbs.

Note 1. When the beam is loaded uniformly throughout its length, the same rule will apply, but the result must be doubled.

2. If the beam be *fixed* at each end and loaded in the middle, then the result obtained in the problem must be increased by its half.

3. If the beam be fixed at both ends and loaded uniformly throughout its length, the same result must be multiplied by 3. That is, the strength under these several circumstances:

Supported and loaded in the centre...	are as	1	:	2
Do. and loaded throughout its length		2	:	4
Fixed and loaded in the centre........		$1\frac{1}{2}$	:	3
Do. loaded throughout its length......		3	:	6

EXAM. A piece of New England fir, 10 feet long and 6 inches square, being fixed at each end, and loaded uniformly through its entire length: it is required to find the weight necessary to break it. Ans. 24036lbs.

PROB. VIII. *To find the Weight under which a Column of Timber of given Dimensions and Elasticity will begin to bend, when placed, vertically, on a horizontal Plane.*

Rule. Multiply into one sum the value of E for the proposed wood, the cube of the least thickness, and the greatest thickness, the two latter both in inches; and that product again by the constant number ·2056. Then divide the last product by the square of the length, in inches, for the answer, or weight in lbs.*

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* This rule is founded upon the formulæ which have been given for this particular case, by *Euler*, *Poisson*, &c.

viz. $\text{E}kk = \dfrac{\text{P}f^3}{3b} =$ *absolute elasticity.*

$\text{Q} = \dfrac{\pi^2 \text{E}kk}{4f^2} = \dfrac{\pi^2 \text{P}f}{12b} =$ *the weight*, under which a column begins to bend. Where P is half the weight, f half the length, and b the deflection, when the beam or column is loaded in the middle, and supported at its two ends: also, $\pi = 3{\cdot}14159$, &c. or the semicircumference of a circle to radius 1; that is, according to our notation, $\text{E}kk = \dfrac{\frac{1}{2}\text{W}(\frac{1}{2}l)^3}{3\delta}$, or

EXAM. 1. What weight will be requisite to bend a rod of red pine, 10 inches in length and 1 inch square, when placed vertically on a plane, the weight being applied at its upper extremity? Ans. 15131lbs.

EXAM. 2. Assuming the elasticity of English oak at 5806200, what weight will it require to bend a column, 8 feet 4 inches in length and 10 inches square?

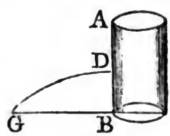
Ans. 1193754lbs.

EXAM. 3. What weight will it require to bend a column of the same wood, and the same lateral dimensions, but of double the length? Ans. 298438lbs.

PRACTICAL QUESTIONS.

QUESTION I.

A LARGE vessel, of 10 feet, or any other given depth, and of any shape, being kept constantly full of water, by means of a supplying cock, at the top; it is proposed to assign the place where a small hole must be made in the side of it, so that the water may spout through it to the greatest distance on the plane of the base.



Let AB denote the height or side of the vessel; D the required hole in the side, from which the water spouts, in the parabolic curve DG, to the greatest distance BG, on the horizontal plane.

By the scholium art. 268, Hydraulics, the distance BG is always equal to 2 $\sqrt{(\text{AD} \cdot \text{DB})}$, which is equal to 2 $\sqrt{[x(a-x)]}$ or 2 $\sqrt{(ax-x^2)}$, if a be put to denote the whole height AB of the vessel, and $x = \text{AD}$ the depth of the hole. Hence $2\sqrt{(ax - x_2)}$, or $ax - x^2$, must be a maximum. In fluxions, $a\dot{x} - 2x\dot{x} = 0$, or $a - 2x = 0$, and $2x = a$, or

---

$\text{E}kk = \frac{wl^3}{48\delta}$, but we have $\text{E} = \frac{wl^3}{ad^3\delta}$; whence $\text{E}kk = \frac{\text{E}ad^3}{48}$. And substituting this in our second formula for $\text{E}kk$, and l for $2f$, we have

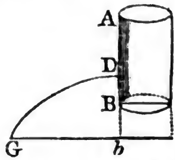
$$\text{Q} = \frac{\pi^2\text{E}ad^3}{48l^2} = \frac{\cdot 2056\text{E}ad^3}{l^2};$$

which is the same as the rule in words.

$x = \frac{1}{2}a$. So that the hole D must be in the middle between the top and bottom; the same as before found at the end of the scholium above quoted.

QUESTION II.

If the same vessel as in QUEST. 1, stand on high, with its bottom a given height above a horizontal plane below; it is proposed to determine where the small hole must be made, so as to spout farthest on the said plane.

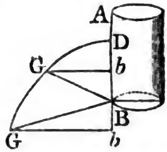


Let the annexed figure represent the vessel as before, and bG the greatest distance spouted by the fluid, DG, on the plane bG.

Here, as before, $bG = 2\sqrt{(AD \cdot Db)}$ $= 2\sqrt{[x(c-x)]} = 2\sqrt{(cx - x^2)}$, by putting $Ab = c$, and $AD = x$. So that $2\sqrt{(cx - x^2)}$ or $cx - x^2$ must be a maximum. And hence, like as in the former question, $x = \frac{1}{2}c = \frac{1}{2}Ab$. So that the hole D must be made in the middle between the top of the vessel, and the given plane, that the water may spout farthest.

QUESTION III.

But if the same vessel, as before, stand on the top of an inclined plane, making a given angle, as suppose of 30 degrees, with the horizon; it is proposed to determine the place of the small hole, so as the water may spout the farthest on the said inclined plane.



Here again (D being the place of the hole, and BG the given inclined plane), $bG = 2\sqrt{(AD \cdot Db)} = 2\sqrt{[x(a - x \pm z)]}$, putting $z = Bb$, and, as before, $a = AB$, and $x = AD$. Then bG must still be a maximum, as also Bb, being in a given ratio to the maximum BG, on account of the given angle B. Therefore $ax - x^2 \pm xz$, as well as z, is a maximum. Hence, by art. 94 of the Fluxions, $a\dot{x} - 2x\dot{x} \pm z\dot{x} = 0$, or $a - 2x \pm z = 0$; conseq. $\pm z = 2x - a$; and hence $bG = 2\sqrt{x(a - x \pm z)}$ becomes barely $2x$. But as the given angle BGb is $= 30°$, the sine of which is $\frac{1}{2}$; therefore $BG = 2Bb$ or $2z$, and $bG^2 = BG^2 - Bb^2 = 3z^2 = 3(2x - a)^2$, or $bG = \pm(2x - a)\sqrt{3}$.

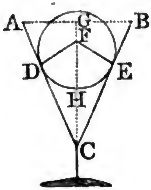
Putting, now, these two values of bG equal to each other, gives the equation $2x = \pm(2x - a)\sqrt{3}$, from which is found

$x = \frac{\frac{1}{2}a\sqrt{3}}{\sqrt{3} \pm 1} = \frac{3 \pm \sqrt{3}}{4} a$, the value of AD required.

A neat solution may also be deduced from a trigonemetrical analysis.

QUESTION IV.

It is required to determine the size of a ball, which, being let fall into a conical glass full of water, shall expel the most water possible from the glass; its depth being 6, and diameer 5 inches.



Let ABC represent the cone of the glass, and DHE the ball, touching the sides in the points D and E, the centre of the ball being at some point F in the axis GC of the cone.

Put AG = GB = $2\frac{1}{2} = a$,
CG = $6 = b$,
AC = $\sqrt{(AG^2 + GC^2)} = 6\frac{1}{2} = c$,
FD = FE = FH = x the radius of the ball.

The two triangles ACG and DCF are equiangular; theref. AG : AC : : DF : FC, that is, $a : c : : x : \frac{cx}{a}$ = FC; hence GF = GC — FC = $b - \frac{cx}{a}$, and GH = GF + FH = $b + x - \frac{cx}{a}$, the height of the segment immersed in the water. Then (by rule 1 for the spherical segment, p. 428, vol. i.), the content of the said immersed segment will be (6DF — 2GH) × GH² × ·5236 = $(2x - b + \frac{cx}{a}) \times (x + b - \frac{cx}{a})^2 \times 1{\cdot}0472$, which must be a maximum, by the question; the fluxion of this made = 0, and divided by $2\dot{x}$ and the common factors, gives $\frac{2a+c}{a} \times (b - \frac{c-a}{a}x) - (\frac{2a+c}{a}x - b) \times \frac{c-a}{a} \times 2 = 0$; this reduced gives $x = \frac{abc}{(c-a) \times (c+2a)} = 2\frac{11}{62}$, the radius of the ball. Consequently its diameter is $4\frac{11}{31}$ inches, as required.

PRACTICAL EXERCISES CONCERNING FORCES; WITH THE RELATION BETWEEN THEM AND THE TIME, VELOCITY, AND SPACE DESCRIBED.

Before entering on the following problems, it will be convenient here to lay down a synopsis of the theorems which express the several relations between any forces, and their corresponding times, velocities, and spaces described; which are all comprehended in the following 12 theorems, as collected from the principles in the foregoing parts of this work.

Let f, F, be any two constant accelerative forces, acting on any body, during the respective times t, T, at the end of which are generated the velocities v, V, and described the spaces s, S. Then, because the spaces are as the times and velocities conjointly, and the velocities as the forces and times; we shall have,

1. *In Constant Forces.*

$$1.\quad \frac{s}{S} = \frac{tv}{TV} = \frac{t^2 f}{T^2 F} = \frac{v^2 F}{V^2 f}.$$

$$2.\quad \frac{v}{V} = \frac{ft}{FT} = \frac{sT}{St} = \sqrt{\frac{fs}{FS}}.$$

$$3.\quad \frac{t}{T} = \frac{Fv}{fV} = \frac{sV}{Sv} = \sqrt{\frac{Fs}{fS}}.$$

$$4.\quad \frac{f}{F} = \frac{Tv}{tV} = \frac{T^2 s}{t^2 S} = \frac{v^2 S}{V^2 s}.$$

And if one of the forces, as F, be the force of gravity at the surface of the earth, and be called 1, and its time T be $= 1''$; then it is known by experiment that the corresponding space S is $= 16\frac{1}{12}$ feet, and consequently its velocity $V = 2S = 32\frac{1}{6}$, which call g. Then the above four theorems, in this case, become as here below:

$$5.\quad s = \tfrac{1}{2}tv = \tfrac{1}{2}gft^2 = \frac{v^2}{2gf}.$$

$$6.\quad v = \frac{2s}{t} = gft = \sqrt{2gfs}.$$

$$7. \quad t = \frac{2s}{v} = \frac{v}{gf} = \sqrt{\frac{s}{\frac{1}{2}gf}}.$$

$$8. \quad f = \frac{v}{gt} = \frac{2s}{gt^2} = \frac{v^2}{2gs}.$$

And from these are deduced the following four theorems, for variable forces, viz.

II. *In Variable Forces.*

$$9. \quad \dot{s} = v\dot{t} = \frac{v\dot{v}}{gf}.$$

$$10. \quad \dot{v} = gf\dot{t} = \frac{gf\dot{s}}{v}.$$

$$11. \quad \dot{t} = \frac{\dot{s}}{v} = \frac{\dot{v}}{gf}.$$

$$12. \quad f = \frac{v\dot{v}}{g\dot{s}} = \frac{\dot{v}}{g\dot{t}}.$$

In these last four theorems, the force f, though variable, is supposed to be constant for the indefinitely small time $\dot{t}$, and they are to be used in all cases of variable forces, as the former ones in constant forces; namely, from the circumstances of the problem under consideration, an expression is deduced for the value of the force f, which being substituted in one of these theorems, that may be proper to the case in hand; the equation thence resulting will determine the corresponding values of the other quantities, required in the problem.

When a motive force happens to be concerned in the question, it may be proper to observe, that the motive force m, of a body, is equal to fq, the product of the accelerative force, and the quantity of matter in it q; and the relation between these three quantities being universally expressed by this equation $m = qf$, it follows that, by means of it, any one of the three may be expelled out of the calculation, or else brought into it.

Also, the momentum, or quantity of motion in a moving body, is qv, the product of the velocity and matter.

It is also to be observed, that the theorems equally hold good for the destruction of motion and velocity, by means of retarding forces, as for the generation of the same, by means of accelerating forces.

To the following problems, which are all resolved by the application of these theorems, it has been thought proper to

subjoin their solutions, for the better information and convenience of the student.

PROBLEM I.

To determine the time and velocity of a body descending, by the force of gravity, down an inclined plane; the length of the plane being 20 feet, and its height 1 foot.

Here, by Mechanics, the force of gravity being to the force down the plane, as the length of the plane is to its height, therefore as 20 : 1 :: 1 (the force of gravity) : $\frac{1}{20} = f$, the force on the plane.

Therefore, by theor. 6, v or $\sqrt{2gfs}$ is $\sqrt{(4 \times 16\frac{1}{12} \times \frac{1}{20} \times 20)} = \sqrt{(4 \times 16\frac{1}{12})} = 2 \times 4\frac{1}{96}$ or $8\frac{1}{48}$ feet nearly, the last velocity per second. And,

By theor. 7, t or $\sqrt{\frac{s}{\frac{1}{2}gf}}$ is $\sqrt{\frac{20}{16\frac{1}{12} \times \frac{1}{20}}} = \sqrt{\frac{400}{16\frac{1}{12}}} = \frac{20}{4\frac{1}{96}} =$ $4\frac{76}{77}$ seconds, the time of descending.

PROBLEM II.

If a cannon ball be fired with a velocity of 1000 feet per second, up a smooth inclined plane, which rises 1 foot in 20: it is proposed to assign the length which it will ascend up the plane, before it stops and begins to return down again, and the time of its ascent.

Here $f = \frac{1}{20}$ as before.

Then, by theor. 5, $s = \frac{v^2}{2gf} = \frac{1000^2}{4 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{60000000}{193}$ $= 310880\frac{160}{193}$ feet, or nearly 59 miles, the distance moved.

And, by theor. 7, $t = \frac{v}{gf} = \frac{1000}{2 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{120000}{193}$ $= 621''\frac{147}{193} = 10'21''\frac{147}{193}$, the time of ascent.

PROBLEM III.

If a ball be projected up a smooth inclined plane, which rises 1 foot in 10, and ascend 100 feet before it stop: required the time of ascent, and the velocity of projection.

First, by theor. 6, $v = \sqrt{2gfs} = \sqrt{(4 \times 16\frac{1}{12} \times \frac{1}{10} \times 100)}$ $= 8\frac{1}{48}\sqrt{10} = 25{\cdot}36408$ feet per second, the velocity.

And, by theor. 7, $t = \sqrt{\frac{s}{\frac{1}{2}gf}} = \sqrt{\frac{100}{16\frac{1}{12} \times \frac{1}{10}}} = \frac{10}{4\frac{1}{96}}\sqrt{10} =$ $\frac{192}{77}\sqrt{10} = 7{\cdot}88516$ seconds, the time in motion.

PROBLEM IV.

If a ball be observed to ascend up a smooth inclined plane, 100 feet in 10 seconds, before it stop, to return back again: required the velocity of projection, and the angle of the plane's inclination.

First, by theor. 6, $v = \frac{2s}{t} = \frac{200}{10} = 20$ feet per second, the velocity.

And, by theor. 8, $f = \frac{2s}{gt^2} = \frac{2 \cdot 100}{2 \cdot 16\frac{1}{12} \times 100} = \frac{12}{193}$. That is, the length of the plane is to its height, as 193 to 12.

Therefore 193 : 12 :: 100 : 6·2176 the height of the plane, or the sine of elevation to radius 100, which answers to 3° 34′, the angle of elevation of the plane.

PROBLEM V.

By a mean of several experiments, I have found, that a cast-iron ball, of 2 inches diameter, fired perpendicularly into the face or end of a block of elm wood, or in the direction of the fibres, with a velocity of 1500 feet per second, penetrated 13 inches deep into its substance. It is proposed thence to determine the time of the penetration, and the resisting force of the wood, as compared to the force of gravity, supposing that force to be a constant quantity.

First, by theor. 7, $t = \frac{2s}{v} = \frac{2 \times 13}{1500 \times 12} = \frac{1}{692}$ part of a second, the time in penetrating.

And, by theor. 8, $f = \frac{v^2}{2gs} = \frac{1500^2}{4 \times 16\frac{1}{12} \times \frac{13}{12}} = \frac{81000000}{13 \times 193}$ $= 32284$. That is, the resisting force of the wood, is to the force of gravity, as 32284 to 1.

But this number will be different, according to the diameter of the ball, and its density or specific gravity. For since f is as $\frac{v^2}{s}$ by theor. 4, the density and size of the ball remaining the same; if the density, or specific gravity, n, vary, and all the rest be constant, it is evident that f will be as n; and therefore f as $\frac{nv^2}{s}$ when the size of the ball

only is constant. But when only the diameter d varies, all the rest being constant, the force of the blow will vary as d^3, or as the magnitude of the ball; and the resisting surface, or force of resistance, varies as d^2; therefore f is as $\frac{d^3}{d^2}$, or as d only, when all the rest are constant. Consequently f is as $\frac{dnv^2}{s}$ when they are all variable.

And so $\frac{f}{F} = \frac{dnv^2 S}{DNV^2 s}$, and $\frac{s}{S} = \frac{dnv^2 F}{DNV^2 f}$; where f denotes the strength or firmness of the substance penetrated, and is here supposed to be the same, for all balls and velocities, in the same substance, which is either accurately or nearly so. See page 214, vol. iii. of my Tracts.

Hence, taking the numbers in the problem, it is - - -

$$f = \frac{dnv^2}{s} = \frac{\frac{2}{12} \times 7\frac{1}{3} \times 1500^2}{\frac{13}{12}} = \frac{44 \times 1500^2}{39} = 2538462$$ the value of f for elm wood. Where the specific gravity of the ball is taken $7\frac{1}{3}$, which is a little less than that of solid cast iron, as it ought, on account of the air-bubble which is found in all cast balls.

PROBLEM VI.

To find how far a 24lb. ball of cast iron will penetrate into a block of sound elm, when fired with a velocity of 1600 *feet per second.*

Here, because the substance is the same as in the last problem, both of the balls and wood, $N = n$, and $F = f$; therefore $\frac{S}{s} = \frac{DV^2}{dv^2}$, or $S = \frac{DV^2 s}{dv^2} = \frac{5\cdot55 \times 1600^2 \times 13}{2 \times 1500^2} = 41\frac{2}{43}$ inches nearly, the penetration required.

PROBLEM VII.

It was found by Mr. Robins (vol. i. p. 273, *of his works*), *that an* 18-*pounder ball, fired with a velocity of* 1200 *feet per second, penetrated* 34 *inches into sound dry oak. It is required thence to ascertain the comparative strength or firmness of oak and elm.*

The diameter of an 18lb. ball is 5·04 inches = D. Then, by the numbers given in this problem for oak, and in prob. 5,

for elm, we have - - - - - - - -

$$\frac{f}{\text{F}} = \frac{dv^2\text{s}}{\text{DV}^2s} = \frac{2 \times 1500^2 \times 34}{5 \cdot 04 \times 1200^2 \times 13} = \frac{100 \times 17}{5 \cdot 04 \times 16 \times 13} = \frac{1700}{1048} \text{ or}$$

$= \frac{8}{5}$ nearly.

From which it would seem, that elm timber resists more than oak, in the ratio of about 8 to 5; which is not probable, as oak is a much firmer and harder wood. But it is to be suspected that the great penetration in Mr. R.'s experiment was owing to the splitting of the timber in some degree.

PROBLEM VIII.

A 24-pounder ball being fired into a bank of firm earth, with a velocity of 1300 *feet per second, penetrated* 15 *feet. It is required thence to ascertain the comparative resistances of elm and earth.*

Comparing the numbers here with those in prob. 5, it is

$$\frac{f}{\text{F}} = \frac{dv^2\text{s}}{\text{DV}^2s} = \frac{2 \times 1500^2 \times 15 \times 12}{5 \cdot 55 \times 1300^2 \times 13} = \frac{15^2 \times 24}{13^3 \times 0 \cdot 37} = \frac{1300}{271} = \frac{20}{3}$$

nearly $= 6\frac{2}{3}$ nearly. That is, elm timber resists about $6\frac{2}{3}$ times more than earth.

PROBLEM XI.

To determine how far a leaden bullet, of $\frac{3}{4}$ of an inch diameter, will penetrate dry elm: supposing it fired with a velocity of 1700 *feet per second, and that the lead does not change its figure by the stroke against the wood.*

Here $\text{D} = \frac{3}{4}$, $\text{N} = 11\frac{1}{3}$, $n = 7\frac{1}{3}$. Then, by the numbers and theorem in prob. 5, it is $\text{s} =$ - - - - -

$$\frac{\text{DNV}^2s}{dnv^2} = \frac{\frac{3}{4} \times 11\frac{1}{3} \times 1700^2 \times 13}{2 \times 7\frac{1}{3} \times 1500^2} = \frac{17^3 \times 13}{200 \times 33} = \frac{63869}{6600} = 9\frac{2}{3}$$

inches nearly, the depth of penetration.

But as Mr. Robins found this penetration, by experiment, to be only 5 inches; it follows, either that his timber must have resisted about twice as much; or else, which is much more probable, that the defect in the penetration arose from the change of figure in the leaden ball he used, from the blow against the wood.

PROBLEM X.

A one pound ball, projected with a velocity of 1500 *feet per second, having been found to penetrate* 13 *inches deep into dry elm: It is required to ascertain the time of passing through every single inch of the* 13, *and the velocity lost at*

each of them; supposing the resistance of the wood constant or uniform.

The velocity v being 1500 feet, or $1500 \times 12 = 18000$ inches, and velocities and times being as the roots of the spaces, in constant retarding forces, as well as in accelerating ones, and t being $= \frac{2s}{v} = \frac{26}{12 \times 1500} = \frac{13}{9000} = \frac{1}{692}$ part of a second, the whole time of passing through the 13 inches; therefore as

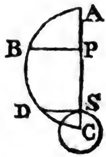
$$\sqrt{13} : \sqrt{13} - \sqrt{12} :: v :$$

$$\begin{array}{lll}
\text{veloc. lost} & & \text{Time in the} \\
\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}}v = 58{\cdot}9 :: t : & \frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}}t = {\cdot}00005 & \text{1st inch.} \\
\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}}v = 61{\cdot}4 :: t : & \frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}}t = {\cdot}00006 & \text{2d} \\
\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}}v = 64{\cdot}2, \text{ \&c.} & \frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}}t = {\cdot}00006 & \text{3d} \\
\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}}v = 67{\cdot}5 & \frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}}t = {\cdot}00007 & \text{4th} \\
\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}}v = 71{\cdot}4 & \frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}}t = {\cdot}00007 & \text{5th} \\
\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}}v = 76{\cdot}0 & \frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}}t = {\cdot}00007 & \text{6th} \\
\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}}v = 81{\cdot}7 & \frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}}t = {\cdot}00008 & \text{7th} \\
\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}}v = 88{\cdot}8 & \frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}}t = {\cdot}00008 & \text{8th} \\
\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}}v = 98{\cdot}2 & \frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}}t = {\cdot}00009 & \text{9th} \\
\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}}v = 111{\cdot}4 & \frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}}t = {\cdot}00011 & \text{10th} \\
\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}}v = 132{\cdot}2 & \frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}}t = {\cdot}00013 & \text{11th} \\
\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}}v = 172{\cdot}3 & \frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}}t = {\cdot}00017 & \text{12th} \\
\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}}v = 417.0 & \frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}}t = {\cdot}00040 & \text{13th} \\
\hline
\text{Sum } 1500{\cdot}0 & \text{Sum } \tfrac{1}{692} \text{ or } {\cdot}00144 &
\end{array}$$

Hence, as the motion lost at the beginning is very small; and consequently the motion communicated to any body, as an inch plank, in passing through it, is very small also; we can conceive how such a plank may be shot through, when standing upright, without oversetting it.

PROBLEM XI.

The force of attraction, above the earth, being inversely as the square of the distance from the centre; it is proposed to determine the time, velocity, and other circumstances, attending a heavy body falling from any given height; the descent at the earth's surface being $16\frac{1}{12}$ *feet, or* 193 *inches, in the first second of time.*



Put
r = CS the radius of the earth,
a = CA the dist. fallen from,
x = CP any variable distance,
v = the velocity at P,
t = time of falling there, and
$\frac{1}{2}g = 16\frac{1}{12}$, half the veloc. or force at S,
f = the force at the point P.

Then we have the three following equations, viz.

$x^2 : r^2 :: 1 : f = \frac{r^2}{x^2}$ the force at P, when the force of gravity at the surface is considered as 1;
$\dot{t}v = -\dot{x}$, because x decreases; and
$v\dot{v} = -gf\dot{x} = -\frac{gr^2\dot{x}}{x^2}$.

The fluents of the last equation give $v^2 = \frac{2gr^2}{x}$. But when $x = a$, the velocity $v = 0$; therefore, by correction,
$v^2 = \frac{2gr^2}{x} - \frac{2gr^2}{a} = 2gr^2 \times \frac{a-x}{ax}$; or $v = \sqrt{(\frac{2gr^2}{a} \times \frac{a-x}{x})}$,
a general expression for the velocity at any point P.

When $x = r$, this gives $v = \sqrt{(2gr \times \frac{a-r}{a})}$ for the greatest velocity, or the velocity when the body strikes the earth.

When a is very great in respect of r, the last velocity becomes $(1 - \frac{r}{2a}) \times \sqrt{2gr}$ very nearly, or nearly $\sqrt{2gr}$ only,

which is accurately the greatest velocity by falling from an infinite height. And this, when $r = 3965$ miles, is 6·9506 miles per second. Also, the velocity acquired in falling from the distance of the sun, or 12000 diameters of the earth, is 6·9505 miles per second. And the velocity acquired in falling from the distance of the moon, or 30 diameters, is 6·8927 miles per second.

Again, to find the time; since $\dot{t}v = -\dot{x}$ therefore $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{2gr^2}} \times \frac{-x\dot{x}}{\sqrt{(ax - xx)}}$; the correct fluent of which gives $t = \sqrt{\frac{a}{2gr^2}} \times (\sqrt{ax - xx} +$ arc to diameter a and vers. $a - x)$; or the time of falling to any point P $= \frac{1}{2r}\sqrt{\frac{a}{\frac{1}{2}g}} \times (\text{AB} + \text{BP})$. And when $x = r$, this becomes $t = \frac{1}{2}\sqrt{\frac{a}{\frac{1}{2}g}} \times \frac{\text{AD} + \text{DS}}{\text{SC}}$ for the whole time of falling to the surface at S; which is evidently infinite when a or AC is infinite, though the velocity is then only the finite quantity $\sqrt{2gr}$.

When the height above the earth's surface is given $= g$; because r is then nearly $= a$, and AD nearly $=$ DS, the time t for the distance g will be nearly - - - - - - - - - -
$\sqrt{\frac{1}{2gr^2}} \times 2\text{DS} = \sqrt{\frac{1}{2gr}} \times \sqrt{2gr} = 1''$, as it ought to be.

If a body, at the distance of the moon at A, fall to the earth's surface at S. Then $r = 3965$ miles, $a = 60r$, and $t = 416806'' = 4$ da. 19 h. 46 m. 46 s. which is the time of falling from the moon to the earth.

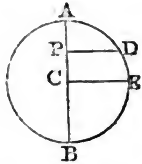
In like manner the time of falling from the distance of the sun would be 64 da. 13 h. 45 m. 46 s.

When the attracting body is considered as a point C; the whole time of descending to C will be - - - - - - - -
$$\frac{1}{2r}\sqrt{\frac{a}{\frac{1}{2}g}} \times \text{ABCD} = \frac{\cdot 7854a}{r}\sqrt{\frac{a}{\frac{1}{2}g}} = \frac{10a}{51r}\sqrt{a} = \frac{\cdot 7854}{r}\sqrt{\frac{a^3}{\frac{1}{2}g}}.$$
Hence, the times employed by bodies, in falling from quiescence to the centre of attraction, are as the square roots of the cubes of the heights from which they respectively fall.

PROBLEM XII.

The force of attraction below the earth's surface being directly as the distance from the centre: it is proposed to determine the circumstances of velocity, time, and space fallen by a

heavy body from the surface, through a perforation made straight to the centre of the earth: abstracting from the effect of the earth's rotation, and supposing it to be a homogeneous sphere 3965 *miles radius.*



Put r = AC the radius of the earth,
x = CP the dist. from the centre,
v = the velocity at P,
t = the time there,
$\frac{1}{2}g = 16\frac{1}{12}$, half the force at A,
f = the force at P.
Then CA : CP :: 1 : f; and the three equations are $rf = x$, and $v\dot{v} = -gf\dot{x}$, and $\dot{t}v = -\dot{x}$.

Hence $f = \frac{x}{r}$, and $v\dot{v} = \frac{-gx\dot{x}}{r}$; the correct fluent of which gives $v = \sqrt{(g \times \frac{r^2 - x^2}{r})} = \text{PD}\sqrt{\frac{g}{r}} = \text{PD}\sqrt{\frac{g}{\text{CE}}}$, the velocity at the point P; where PD and CE are perpendicular to CA. So that the velocity at any point P, is as the perpendicular or sine PD at that point.

When the body arrives at C, then $v = \sqrt{gr} = \sqrt{(g \cdot \text{AC})}$ = 25950 feet or 4·9148 miles per second, which is the greatest velocity, or that at the centre C.

Again, for the time; $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{r}{g}} \times \frac{-\dot{x}}{\sqrt{(r^2 - x^2)}}$; and the fluents give $t = \sqrt{\frac{r}{g}} \times$ arc to cosine $\frac{x}{r} = \sqrt{\frac{1}{gr}} \times$ arc AD. So that the time of descent to any point P, is as the corresponding arc AD.

When P arrives at C, the above becomes t = ... $\sqrt{\frac{1}{gr}} \times$ quadrant AE $= \frac{\text{AE}}{\text{AC}}\sqrt{\frac{r}{g}} = 1\cdot5708\sqrt{\frac{r}{g}} = 1267\frac{1}{4}$ seconds = 21 m. $7\frac{1}{4}$ s. for the time of falling to the centre C.

The time of falling to the centre is the same quantity $1\cdot5708\sqrt{\frac{r}{g}}$, from whatever point in the radius AC the body begins to move. For, let n be any given distance from C at which the motion commences: then by correction, $v = \sqrt{(\frac{g}{r} . \overline{n^2 - x^2})}$, and hence $\dot{t} = \sqrt{\frac{r}{g}} \times \frac{-\dot{x}}{\sqrt{(n^2 - x^2)}}$, the fluents of which give $t = \sqrt{\frac{r}{g}} \times$ arc to cosine $\frac{x}{n}$; which, when x

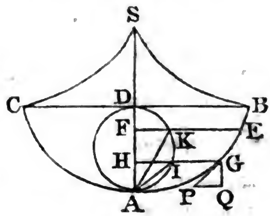
$= 0$, gives $t = \sqrt{\frac{r}{g}} \times$ quadrant $= 1\cdot5708 \sqrt{\frac{r}{g}}$, for the time of descent to the centre C, the same as before.

As an equal force, acting in contrary directions, generates or destroys an equal quantity of motion in the same time; it follows that, after passing the centre, the body will just ascend to the opposite surface at B, in the same time in which it fell to the centre from A. Then from B it will return again in the same manner, through C to A; and so oscillate continually between A and B, the velocity being always equal at equal distances from C on both sides; and the whole time of a double oscillation, or of passing from A and arriving at A again, will be quadruple the time of passing over the radius AC, or $= 2 \times 3\cdot1416 \sqrt{\frac{r}{g}} =$ 1h. 24m. 29s.

PROBLEM XIII.

To find the Time of a Pendulum vibrating in the Arc of a Cycloid.

Let S be the point of suspension;
SA the length of pendulum;
CAB, the whole cycloidal arc;
AIKD, the generating circle, to which FKE, HIG are perpendiculars.
SC, SB two other equal semicycloids, on which the thread wrapping, the end A is made to describe the cycloid BAC.



By the nature of the cycloid, AD = DS; and SA = 2AD = SC = SB = CA = AB. Also, if at any point G be drawn the tangent GP; GQ parallel and PQ perpendicular to AD: then PG is parallel to the chord AI, by the nature of the curve. And, by the nature of forces, the force of gravity : force in direction GP : : GP : GQ : : AI : AH : : AD : AI; in like manner, the force of gravity : force in the curve at E : : AD : AK; that is, the accelerative force in the curve, is every where as the corresponding chord AI or AK of the circle, or as the arc AG or AE of the cycloid, since AG is always = 2AI, by the nature of the curve. So that the process and conclusions,

for the velocity and time of describing any arc in this case, will be the very same as in the last problem, the nature of the forces being the same, viz. as the distance to be passed over to the lowest point A.

From which it follows, that the time of a semi-vibration, in all arcs, AG, AE, &c. is the same constant quantity $1{\cdot}5708\sqrt{\frac{r}{g}} = 1{\cdot}5708\sqrt{\frac{AS}{g}} = 1{\cdot}5708\sqrt{\frac{l}{g}}$; and the time of a whole vibration from B to C, or from C to B, is $3{\cdot}1416\sqrt{\frac{l}{g}}$; where l = AS = AB is the length of the pendulum, and 3·1416 the circumference of a circle whose diameter is 1,

Since the time of a body's falling by gravity through $\frac{1}{2}l$, or half the length of the pendulum, by the nature of descents, is $\sqrt{\frac{l}{g}}$, which being in proportion to $3{\cdot}1416\sqrt{\frac{l}{g}}$, as 1 is to 3·1416; therefore the diameter of a circle is to its circumference, as the time of falling through half the length of a pendulum, is to the time of one vibration.

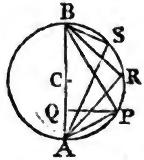
If the time of the whole vibration be 1 second, this equation arises, viz. $1'' = 3{\cdot}1416\sqrt{\frac{l}{g}}$; hence $l = \frac{g}{3{\cdot}1416^2} = \frac{\frac{1}{2}g}{4{\cdot}9348}$, and $\frac{1}{2}g = 3{\cdot}1416^2 \times \frac{1}{2}l = 4{\cdot}9348l$. So that if one of these, g or l, be given by experiment, these equations will give the other. See pages 206 and 231.

Hence the times of vibration of pendulums, are as the square roots of their lengths; and the number of vibrations made in a given time, is reciprocally as the square roots of the lengths. And hence also, the length of a pendulum vibrating n times in a minute, or 60″, is $l = 39\frac{1}{8} \times \frac{60^2}{n^2} = \frac{140850}{n^2}$; as at page 241.

When a pendulum vibrates in a circular arc: as the length of the string is constantly the same, the time of vibration will be longer than in a cycloid; but the two times will approach nearer together as the circular arc is smaller; so that when it is very small, the times of vibration will be nearly equal. And hence it happens that $39\frac{1}{8}$ inches is the length of a pendulum vibrating seconds, in the very small arc of a circle.

PROBLEM XIV.

To determine the time of a Body descending down the Chord of a Circle.



Let C be the centre; AB the vertical diameter; AP any chord, down which a body is to descend from P to A; and PQ perpendicular to AB.

Now, as the natural force of gravity in the vertical direction BA, is to the force urging the body down the plane PA, as the length of the plane AP, is to its height AQ; therefore the velocity in PA and QA, will be equal at all equal perpendicular distances below PQ; and consequently the time in PA : time in QA : : PA : QA : : BA : PA; but time in BA : time in QA : : $\sqrt{BA} : \sqrt{QA} : : \sqrt{(BA \cdot BA)} : \sqrt{(QA \cdot BA)}$: : BA : PA; hence, as three of the terms in each proportion are the same, the fourth terms must be equal, namely, the time in BA = the time PA.

And, in like manner, the time in BP = the time in BA. So that, in general, the times of descending down all the chords, BA, BP, BR, BS, &c. or PA, RA, SA, &c. are all equal, and each equal to the time of falling freely through the diameter; as before found at art. 194, Dynamics. Which time is $\sqrt{\frac{2r}{\frac{1}{2}g}}$, where $\frac{1}{2}g = 16\frac{1}{12}$ feet, and $r =$ the radius AC, for $\sqrt{\frac{1}{2}g} : \sqrt{2r} : : 1'' : \sqrt{\frac{2r}{\frac{1}{2}g}}$.

PROBLEM XV.

To determine the Time of filling the Ditches of a Work with Water, at the Top, by a Sluice of 2 *Feet square; the Head of Water above the Sluice being* 10 *Feet, and the Dimensions of the Ditch being* 20 *Feet wide at Bottom,* 22 *at Top,* 9 *deep, and* 1000 *Feet long.*

The capacity of the ditch is 189000 cubic feet.

But $\sqrt{\frac{1}{2}g} : \sqrt{10} : : g : 2\sqrt{5g}$ the velocity of the water through the sluice, the area of which is 4 square feet; therefore $8\sqrt{5g}$ is the quantity per second running through it; and consequently $8\sqrt{5g} : 189000 : : 1'' : \frac{23625}{\sqrt{5g}} = 1863''$ or 31 m. 3 s. nearly, which is the time of filling the ditch.

PROBLEM XVI.

To determine the Time of emptying a Vessel of Water by a Sluice in the Bottom of it, or in the Side near the Bottom: the Height of the Aperture being very small in respect of the Altitude of the Fluid.

Put a = the area of the aperture or sluice;
$g = 32\frac{1}{6}$ feet, the force of gravity;
d = the whole depth of water;
x = the variable altitude of the surface above the aperture;
A = the area of the surface of the water.

Then $\sqrt{\frac{1}{2}g} : \sqrt{x} :: 2g : 2\sqrt{\frac{1}{2}gx}$ the velocity with which the fluid will issue at the sluice; and hence $A : a :: 2\sqrt{\frac{1}{2}gx} : \frac{2a\sqrt{\frac{1}{2}gx}}{A}$, the velocity with which the surface of the water will descend at the altitude x, or the space it would descend in 1 second with the velocity there. Now, in descending the space $\dot{x}$, the velocity may be considered as uniform; and uniform descents are as their times; therefore $\frac{2a\sqrt{\frac{1}{2}gx}}{A} : -\dot{x} :: 1'' : \frac{-A\dot{x}}{2a\sqrt{\frac{1}{2}gx}}$ the time of descending $\dot{x}$ space, or the fluxion of the time of exhausting. That is, $\dot{t} = \frac{-A\dot{x}}{24\sqrt{\frac{1}{2}gx}}$; which is made negative, because x is a decreasing quantity, or its fluxion negative.

Now, when the nature or figure of the vessel is given, the area A will be given in terms of x; which value of A being substituted into this fluxion of the time, the fluent of the result will be the time of exhausting sought.

So if, for example, the vessel be any prism, or everywhere of the same breadth; then A is a constant quantity, and therefore the fluent is $-\frac{A}{a}\sqrt{\frac{x}{\frac{1}{2}g}}$. But when $x = d$, this becomes $-\frac{A}{a}\sqrt{\frac{d}{\frac{1}{2}g}}$, and should be 0; therefore the correct fluent is $t = \frac{A}{a} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{\frac{1}{2}g}}$ for the time of the surface descending till the depth of the water be x. And when $x = 0$,

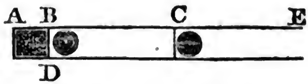
the whole time of exhausting is barely $\frac{A}{a}\sqrt{\frac{d}{\frac{1}{2}g}}$.*

Hence, if A be $= 10000$ square feet, $a = 1$ square foot, and $d=10$ feet; the time is $7885\frac{1}{3}$ seconds, or 2h. 11m. $25\frac{1}{3}$s.

Again, if the vessel be a ditch, or canal of 20 feet broad at the bottom, 22 at the top, 9 deep, and 1000 feet long; then is $90 : 90 + x :: 20 : \frac{90+x}{9} \times 2$ the breadth of the surface of the water when its depth in the canal is x; and therefore $A = \frac{90+x}{9} \times 2000$ is the surface at that time. Consequently $\dot{t}$ or $\frac{-A\dot{x}}{2a\sqrt{\frac{1}{2}gx}} = 1000 \times \frac{90+x}{9} \times \frac{-\dot{x}}{a\sqrt{\frac{1}{2}gx}}$ is the fluxion of the time; the correct fluent of which, when $x = 0$, is $1000 \times \frac{180+\frac{2}{3}d}{9a} \times \sqrt{\frac{d}{\frac{1}{2}g}} = \frac{1000 \times 186 \times 3}{9 \times 4\frac{1}{96}} =$ $15459''\frac{2}{3}$ nearly, or 4h. 17m. $39\frac{2}{3}$s., being the whole time of exhausting by a sluice of 1 foot square.

PROBLEM XVII.

To determine the Velocity with which a Ball is discharged from a given Piece of Ordnance, with a given Charge of Gunpowder.



Let the annexed figure represent the bore of the gun; AD being the part filled with gunpowder. And put

$a =$ AB, the part at first filled with powder and the bag;
$b =$ AE, the whole length of the gunbore;
$c = \cdot 7854$, the area of a circle whose diameter is 1;
$d =$ BD, the diameter of the ball;
$e =$ the specific gravity of the ball, or weight of 1 cubic foot;
$\frac{1}{2}g = 16\frac{1}{12}$ feet, descended by a body in 1 second;
$m =$ 240oz, 15lb, the pressure of the atmosphere on a sq. inch;
n to 1 the ratio of the first force of the fired powder, to the pressure of the atmosphere;
$w =$ the weight of the ball. Also, let

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* This last result is obviously inconsistent with the result in Prob. 39, Promiscuous Exercises, near the end of this volume. The reason is, that the supposition of $x = 0$, is incompatible with the hypothesis that the height of the aperture is very small compared with x. The rest of the solution is correct.

$x = $ AC, be any variable distance of the ball from A, in moving along the gunbarrel.

First, cd^2 is = the area of the circle BD of the ball; theref. mcd^2 is the pressure of the atmosphere on BD; conseq. $mncd^2$ is the first force of the powder on BD.

But the force of the inflamed powder is proportional to its density, and the density is inversely as the space it fills; therefore the force of the powder on the ball at B, is to the force on the same at C, as AC is to AB; that is, ...

$x : a :: mncd^2 : \frac{mnacd^2}{x} = $ F, the motive force at C:

conseq. $\frac{F}{w} = \frac{mnacd^2}{wx} = f$, the accelerating force there.

Hence, theor. 10 of forces gives $v\dot{v} = gf\dot{x} = \frac{gmnacd^2}{w} \times \frac{\dot{x}}{x}$;

the fluent of which is $v^2 = \frac{2gmnacd^2}{w} \times$ hyp. log. of x.

But when $v = 0$, then $x = a$; therefore by correction, $v^2 = \frac{2gmnacd^2}{w} \times$ hyp. log $\frac{x}{a}$ is the correct fluent; conseq.

$v = \sqrt{(\frac{2gmnacd^2}{w} \times \text{hyp. log. } \frac{x}{a})}$ is the veloc. of the ball at C,

and $v = \sqrt{(\frac{2gmnhcd^2}{w} \times \text{hyp. log. } \frac{b}{a})}$ the velocity with which the ball issues from the muzzle at E; where h denotes the length of the cylinder filled with powder; and a the length to the hinder part of the ball, which will be more than h when the ball does not touch the powder.

Or, by substituting the numbers for g, m, c, and changing the hyperbolic logarithms for the common ones, then

$v = \sqrt{(\frac{2306nhd^2}{w} \times \text{com. log. } \frac{b}{a})}$, the velocity at E, in feet.

But, the content of the ball being $\frac{2}{3}cd^3$, its weight is . .

$w = \frac{\frac{2}{3}cd^3e}{12^3} = \frac{ced^3}{2592} = \frac{ed^3}{3300}$; which being substituted for w, in the value of v, it becomes

$v = 2758\sqrt{(\frac{nh}{de} \times \text{com. log. } \frac{b}{a})}$, the velocity at E.

When the ball is of cast iron; taking $e = 7368 = 10(27{\cdot}14)^2$,

the rule becomes $v = 32\sqrt{(\frac{nh}{d} \times \log. \frac{b}{a})}$ for the veloc. of the cast iron ball.

Or, when the ball is of lead; then $e = 11325 = 10(33{\cdot}652)^2$,

and $v = 26\surd(\frac{nh}{d} \times \log.\ \frac{b}{a})$ for the veloc. of the leaden ball*.

Corol. From the general expression for the velocity v, above given, may be derived what must be the length of the charge of powder a, in the gun-barrel, so as to produce the greatest possible velocity in the ball: namely, by making the

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* Some practical artillerists having expressed a wish to the Editor, to see a solution to this problem upon the supposition *that the gun-powder explodes gradually, so as to be all ignited when the ball quits the mouth of a gun*, he avails himself of the opportunity of giving it in this place.

Here we must first investigate the relation between the time and the space described. By the hypothesis we shall have the force as $\frac{t}{x}$; and we have x varying as t^n to find n. We have, from the primitive formulæ, pa. 400. $v\dot{v} = gf\dot{x} \propto f\dot{x}$, and $f \propto \frac{t}{x} \propto \frac{x^{\frac{1}{n}}}{x} \propto \frac{1}{x^{\frac{n-1}{n}}}$. Therefore $v\dot{v} \propto \frac{\dot{x}}{x^{\frac{n-1}{n}}}$; and the fluent $v^2 \propto nx^{\frac{1}{n}}$.

Farther,

$\dot{x} \propto v\dot{t} \propto v \cdot \frac{1}{n} x^{\frac{1}{n}-1}\dot{x} \propto x^{\frac{1}{2n}} \times x^{\frac{1}{n}-1}\dot{x} \propto x^{\frac{3-2n}{2n}}\dot{x}$; so that the fluent $x \propto x^{\frac{3}{2n}}$. Conseq. $1 = \frac{3}{2n}$, and $n = \frac{3}{2}$. That is, in this case, $x \propto t^{\frac{3}{2}}$.

Now, taking the notation of the problem in the text, we have $f' = \frac{mncd^2}{w}$, the accelerative force at the first instant of explosion; and, by hyp. at E, where the whole is exploded, the force will be $\frac{af'}{b}$. To find the force at C, if T be put for the whole time of explosion, we shall have

$$\frac{T}{b}\left(\frac{\text{time}}{\text{space}}\right) : \frac{af'}{b} \text{ (force at E)} :: \frac{t}{x}\left(\frac{\text{time}}{\text{space}}\right) : \frac{af't}{Tx},$$

force at C, or the value of f there, in the general formulæ.

Hence we have $v\dot{v} = gf\dot{x} = \frac{gaf't\dot{x}}{Tx}$.

But $b : T^{\frac{3}{2}} :: x : t^{\frac{3}{2}} = \frac{T^{\frac{3}{2}}x}{b}$ $\therefore t = \frac{Tx^{\frac{2}{3}}}{b^{\frac{2}{3}}}$; and by substituting this value of t for it, we have

value of v a maximum, or, by squaring and omitting the constant quantities, the expression $a \times$ hyp. log. of $\frac{b}{a}$ a maximum, or its fluxion equal to nothing; that is $\dot a \times$ hyp. log. $\frac{b}{a} - \dot a = 0$, or hyp. log. of $\frac{b}{a} = 1$; hence $\frac{b}{a} = 2{\cdot}71828$, the number whose hyp. log. is 1. So that $a : b :: 1 : 2{\cdot}71828$, or as 4 to 11 nearly, or nearer as 7 to 19; that is the length of the charge, to produce the greatest velocity, is the $\frac{4}{11}$th part of the length of the bore, or nearer $\frac{7}{19}$ of it.

By actual experiment it is found, that the charge for the greatest velocity, is but little less than that which is here

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$$v\dot v = \frac{gaf'}{\text{T}x} \cdot \frac{\text{T}x^{\frac{2}{3}}\dot x}{b^{\frac{2}{3}}} = \frac{gaf'x^{-\frac{1}{3}}\dot x}{b^{\frac{2}{3}}}.$$

Taking the fluents, we have

$$\tfrac{1}{2}v^2 = \frac{gaf'x^{\frac{2}{3}}}{\frac{2}{3}b^{\frac{2}{3}}}, \text{ whence } v = \sqrt{\frac{3gaf'x^{\frac{2}{3}}}{b^{\frac{2}{3}}}}.$$

When x becomes $= b$, this becomes

$$v = \sqrt{(3gaf')} = \sqrt{\left(3gmne \cdot \frac{ad^2}{w}\right)}.$$

Cor. 1. Hence, so long as a and d remain the same, the velocity at the muzzle E will be the same whatever be the length of the gun; for b does not appear in the ultimate value of v.

This is contrary to all experience, and proves that the hypothesis is untenable.

Cor. 2. The powder being the same, the velocity at the muzzle (d remaining the same) will be as the square root of the charge.

Cor. 3. In guns of different bores, the velocity at the muzzle will be as $\sqrt{\frac{a}{d}}$. For $v \propto \sqrt{\frac{ad^2}{w}} \propto \sqrt{\frac{ad^2}{d^3}} \propto \sqrt{\frac{a}{d}}$.

Cor. 4. If the charge be given, a will be inversely as d^2, and $v \propto \sqrt{\frac{1}{d^3}}$.

Cor. 5. If b be the length of a gun in which the charge of powder will be all fired when the ball reaches the muzzle, then in a shorter gun AC, the same powder, and an equal charge will give an ultimate velocity varying as $\sqrt{\frac{x^{\frac{2}{3}}}{b^{\frac{2}{3}}}}$ or as $\sqrt[3]{\frac{x}{b}}$.

computed from theory; as may be seen by turning to page 213, vol. 3, of the Tracts, where the corresponding parts are found to be, for four different lengths of gun, thus, $\frac{3}{10}$, $\frac{3}{13}$, $\frac{3}{16}$, $\frac{3}{20}$; the parts here varying, as the gun is longer, which allows time for the greater quantity of powder to be fired, before the ball is out of the bore.

SCHOLIUM.

In the calculation of the foregoing problem, the value of the constant quantity n remains to be determined. It denotes the first strength or force of the fired gunpowder, just before the ball is moved out of its place. This value is assumed, by Mr. Robins, equal to 1000, that is, 1000 times the pressure of the atmosphere, on any equal spaces.

But the value of the quantity n may be derived much more accurately, from the experiments related in my Tracts, by comparing the velocities there found by experiment, with the rule for the value of v, or the velocity, as computed by theory, viz. . . . .

$$v = 100\sqrt{(\frac{na}{10d} \times \text{log. of } \frac{b}{a})}, \text{ or} = 100\sqrt{(\frac{nh}{10d} \times \text{log of } \frac{b}{a})}.$$

Now, supposing that v is a given quantity, as well as all the other quantities, excepting only the number n, then by reducing this equation, the value of the letter n is found to be as follows, viz. . . . .

$$n = \frac{dvv}{1000a} \div \text{com. log. of } \frac{b}{a}, \text{ or} = \frac{dvv}{1000h} \div \text{log. of } \frac{b}{a},$$

when h is different from a.

Now, to apply this to the experiments. By pa. 69. vol. 3, of the Tracts, the velocity of the ball, of 1·96 inches diameter, with 4 ounces of powder, in the gun No. 1, was 1100 feet per second; and, by pa. 316, vol. 2, the length of the gun, when corrected for the spheroidal hollow in the bottom of the bore, was 28·53; also, by pa. 48, vol. 3, the length of the charge, when corrected in like manner, was 3·45 inches of powder and bag together, but 2·54 of powder only: so that the values of the quantities in the rule, are thus: $a = 3{\cdot}45$; $b = 28{\cdot}53$; $d = 1{\cdot}96$; $h = 2{\cdot}54$; and $v = 1100$: then, by substituting these values instead of the letters, in the theorem $n = \frac{dvv}{1000a} \div \text{com. log of } \frac{b}{a}$, it comes out $n = 750$, when h is considered as the same as a. And so on, for the other experiments there treated of.

It is here to be noted, however, that there is a circumstance in the experiments delivered in the Tracts, just mentioned, which will alter the value of the letter a in this theorem, which is this, viz that a denotes the distance of the shot from the bottom of the bore; and the length of the charge of powder alone ought to be the same thing; but, in the experiments, that length included, besides the length of real powder, the substance of the thin flannel bag in which it was always contained, of which the neck at least extended a considerable length, being the part where the open end was wrapped and tied close round with a thread. This circumstance causes the value of n, as found by the theorem above, to come out less than it ought to be, for it shows the strength of the inflamed powder when just fired, and when the flame fills the whole space a before occupied both by the real powder and the bag, whereas it ought to show the first strength of the flame when it is supposed to be contained in the space only occupied by the powder alone, without the bag. The formula will therefore bring out the value of n too little, in proportion as the real space filled by the powder is less than the space filled both by the powder and its bag. In the same proportion therefore must we increase the formula, that is, in the proportion of h, the length of real powder, to a the length of powder and bag together. When the theorem is so corrected, it becomes $\frac{dvv}{1000h} \div$ com. log. of $\frac{b}{a}$.

Now, by pa. 48 and 49, vol. 3. Tracts, there are given both the lengths of all the charges, or values of a, including the bag, and also the length of the neck and bottom of the bag, which is 0·91 of an inch, which therefore must be subtracted from all the values of a, to give the corresponding values of h. This in the example above reduces 3·45 to 2·54.

Hence, by increasing the above result 750, in proportion of 2·54 to 3·45, it becomes 1018. And so on for the other experiments.

But it will be best to arrange the results in a table, with the several dimensions, when corrected, from which they are computed, as follows.

Table of Velocities of Balls and First Force of Powder, &c:

Gun.		Charge of Powder.			Velocity or value of v.	First force, or value of n.
No.	Length, or value of b.	Weight in ounces.	Length or value of a.	Length or value of h.		
1	inches. 28·53	4	3·45	2·54	1100	1018
		8	5·99	5·08	1340	1191
		16	11·07	10·16	1430	967
2	38·43	4	3·45	2·54	1180	1077
		8	5·99	5·08	1580	1193
		16	11·07	10·16	1660	984
3	57·70	4	3·45	2·54	1300	1067
		8	5·99	5·08	1790	1256
		16	11·07	10·16	2000	1076
4	80·23	4	3·45	2·54	1370	1060
		8	5·99	5·08	1940	1289
		16	11.07	10·16	2200	1085

Where it may be observed, that the numbers in the column of velocities, 1430 and 2200, are a little increased, as, from a view of the table of experiments, they evidently required to be. Also the value of the letter d is constantly 1·96 inch.

Hence it appears, that the value of the letter n, used in the theorem, though not yet greatly different from the number 1000, assumed by Mr. Robins, is rather various, both for the different lengths of the gun, and for the different charges with the same gun.

But this diversity in the value of the quantity n, or the first force of the inflamed gunpowder, is probably owing in some measure to the omission of a material datum in the calculation of the problem, namely, the weight of the charge of powder, which has not at all been brought into the computation. For it is manifest, that the elastic fluid has not only the ball to move and impel before it, but its own weight of matter also. The computation may therefore be renewed, in the ensuing problem, to take that datum into the account.

PROBLEM XVIII.

To determine the same as in the last Problem; taking both the Weight of Powder and the Ball into the Calculation.

Besides the notation used in the last problem, let $2p$ denote the weight of the powder in the charge, with the flannel bag in which it was inclosed.

Now, because the inflamed powder occupies at all times the part of the gun bore which is behind the ball, its centre of gravity, or the middle part of the same, will move with only half the velocity that the ball moves with; and this will require the same force as half the weight of the powder, &c. moved with the whole velocity of the ball. Therefore, in the conclusion derived in the last problem, we are now, instead of w, to substitute the quantity $p+w$; and when that is done, the last velocity will come out, $v = \sqrt{(\frac{2230nhd^2}{p+w} \times \text{com. log.} \frac{b}{a})}$.

And from this equation is found the value of n, which is $n = \frac{p+w}{2230hd^2}v^2 \div \text{log. of } \frac{b}{a}, = \frac{p+w}{8567h}v^2 \div \text{log. of } \frac{b}{a}$, by substituting for d its value 1·96, the diameter of the ball.

Now as to the ball, its medium weight was 16 oz. 13 dr. = 16·81 oz. And the weights of the bags containing the several charges of powder, viz. 4 oz., 8 oz., 16 oz., were 8 dr., 12 dr., and 1 oz., 5 dr.; then, adding these to the respective contained weights of powder, the sums, 4·5 oz., 8·75 oz., 17·31 oz., are the values of $2p$, or the weights of the powder and bags; the halves of which, or 2·25, and 4·38, and 8·66, are the values of the quantity p for those three charges; and these being added to 16·81, the constant weight of the ball, there are obtained the three values of $p+w$ for the three charges of powder, which values therefore are 19·06 oz., and 21·19 oz. and 25·47 oz. Then, by calculating the values of the first force n, by the last rule above, with these new data, the whole will be found as in the following table.

The gun.		Charge of Powder.			Weight of ball and charge, or values of $p+w$.	Velocity, or the values of v.	First force or the value of n.
No.	Length or value of b.	Weight in ounces.	Length or value of a.	Length or value of h.			
	Inches.						
1	28·53	4	3·45	2·54	19·06	1100	1155
		8	5·99	5·08	21·19	1340	1377
		16	11·07	10·16	25·47	1430	1456
2	38·43	4	3·45	2·54	19·06	1180	1167
		8	5·99	5·08	21·19	1580	1506
		16	11·07	10·16	25·47	1660	1492
3	57·70	4	3·45	2·54	19·06	1300	1210
		8	5·99	5·08	21·19	1790	1586
		16	11·07	10·16	25·47	2000	1646
4	80·23	4	3·45	2·54	19·06	1370	1203
		8	5·99	5·08	21·19	1940	1627
		16	11·07	10·16	25·47	2200	1648

And here it appears that the values of n, the first force of the charge, are much more uniform and regular than by the former calculations in the preceding problem, at least in all excepting the smallest charge, 4 oz. in each gun; which it would seem must be owing to some general cause or causes. Nor have we long to search, to find out what those causes may be. For when it is considered that these numbers for the value of n, in the last column of the table, ought to exhibit the first force of the fired powder, when it is supposed to occupy the space only in which the bare powder itself lies; and that whereas it is manifest that the condensed fluid of the charge in these experiments occupies the whole space between the ball and the bottom of the gun bore, or the whole space taken up by the powder and the bag or cartridge together, which exceeds the former space, or that of the powder alone, at least in the proportion of the circle of the gun bore, to the same as diminished by the thickness of the surrounding flannel of the bag that contained the powder; it is manifest that the force was diminished on that account. Now by gently compressing a number of folds of the flannel together, it has been found that the thickness of the single flannel was equal to the 40th part of an inch; the double of which, $\frac{1}{20}$ or ·05 of an inch, is therefore the quantity by which the diameter of the circle of the powder within the bag, was less than that of the gun bore. But

the diameter of the gun bores was 2·02 inches; therefore, deducting the ·05, the remainder 1·97 is the diameter of the powder cylinder within the bag; and because the areas of circles are to each other as the squares of their diameters, and the squares of these numbers, 1·97 and 2·02, being to each other as 388 to 408, or as 97 to 102; therefore, on this account alone, the numbers before found, for the value of n, must be increased in the ratio of 97 to 102.

But there is yet another circumstance, which occasions the space at first occupied by the inflamed powder to be larger than that at which it has been taken in the foregoing calculations, and that is the difference between the content of a sphere and cylinder. For, the space supposed to be occupied at first by the elastic fluid, was considered as the length of a cylinder measured to the hinder part of the curve surface of the ball, which is manifestly too little by the difference between the content of half the ball and a cylinder of the same length and diameter, that is, by a cylinder whose length is $\frac{1}{3}$ the semidiameter of the ball. Now that diameter was 1·96 inches; the half of which is 0·98, and $\frac{1}{3}$ of this is 0·33 nearly. Hence then it appears that the lengths of the cylinders, at first filled by the dense fluid, viz. 3·45, and 5·99, and 11·07, have been all taken too little by 0·33; hence it follows that, on this account also, all the numbers before found for the value of the first force n, must be further increased in the ratios of 3·45 and 5·99 and 11·07, to the same numbers increased by 0·33, that is, to the numbers 3·78 and 6·32 and 11·40.

Compounding now these last ratios with the foregoing one, viz. 97 to 102, it prodcues these three, viz. the ratios of 334 and 581 and 1074, respectively to 385 and 647 and 1163. Therefore increasing the last column of numbers, for the value of n, viz. those of the 4 oz. charge in the ratio of 334 to 385, and those of the 8 oz. charge in the ratio of 581 to 647, and those of the 16 oz. charge in the ratio of 1074 to 1163, with every gun, they will be reduced to the numbers in the annexed table; where the numbers are still larger and more regular than before*.

Powder.	The Guns.			
	1	2	3	4
oz.				
4	1372	1387	1438	1430
8	1637	1677	1766	1812
16	1577	1616	1782	1784

* From the experiments of 1815, 1816, it appears that n exceeds 2000 in the best gunpowder.

Thus then at length it appears that the first force of the inflamed gunpowder, when occupying only the space at first filled with the powder, is about 1800, that is, 1800 times the elasticity of the natural air, or pressure of the atmosphere, in the charges with 8 oz. and 16 oz. of powder, in the two longer guns; but somewhat less in the two shorter, probably owing to the gradual firing of gunpowder in some degree; and also less in the lowest charge 4 oz. in all the guns, which may probably be owing to the less degree of heat in the small charge. But besides the foregoing circumstances that have been noticed, or used in the calculations, there are yet several others that might and ought to be taken into the account, in order to a strict and perfect solution of the problem; such as, the counter pressure of the atmosphere, and the resistance of the air on the fore part of the ball while moving along the bore of the gun; the loss of the elastic fluid by the vent and windage of the gun; the gradual firing of the powder; the unequal density of the elastic fluid in the different parts of the space it occupies between the ball and the bottom of the bore; the difference between pressure and percussion when the ball is not laid close to the powder; and perhaps some others: on all which accounts it is probable that, instead of 1800, the first force of the elastic fluid is not less than 2000 times the strength of natural air.

Corol. From the theorem last used for the velocity of the ball and elastic fluid, viz. $v = \surd(\frac{2230hd^2}{p+w} n \times \log. \frac{b}{a}) = \surd \frac{8567hn}{p+w} \times \log. \frac{b}{a}$, we may find the velocity of the elastic fluid alone, viz. by taking w, or the weight of the ball, $= 0$ in the theorem, by which it becomes barely $v = \surd(\frac{8567hn}{p} \times \log. \frac{b}{a})$, for that velocity. And by computing the several preceding examples by this theorem, supposing the value of n to be 2000, the conclusions come out a little various, being between 4000 and 5000, but most of them nearer to the latter number. So that it may be concluded that the velocity of the flame, or of the fired gunpowder, expands itself at the muzzle of the gun, at the rate of about 5000 feet per second nearly.

ON THE MOTION OF BODIES IN FLUIDS.

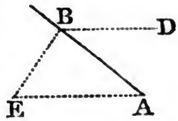
PROBLEM XIX.

To determine the Force of Fluids in Motion; and the Circumstances attending Bodies moving in Fluids.

1. It is evident that the resistance to a plane, moving perpendicularly through an infinite fluid, at rest, is equal to the pressure or force of the fluid on the plane at rest, and the fluid moving with the same velocity, and in the contrary direction, to that of the plane in the former case. But the force of the fluid in motion, must be equal to the weight or pressure which generates that motion; and which, it is known, is equal to the weight or pressure of a column of the fluid, whose base is equal to the plane, and its altitude equal to the height through which a body must fall, by the force of gravity, to acquire the velocity of the fluid: and that altitude is, for the sake of brevity, called the altitude due to the velocity. So that, if a denote the area of the plane, v the velocity, and n the specific gravity of the fluid; then, the altitude due to the velocity v being $\frac{v^2}{2g}$, the whole resistance, or motive force m, will be $a \times n \times \frac{v^2}{2g} = \frac{anv^2}{2g}$; g being, as we have all along assumed it, $= 32\frac{1}{6}$ feet. And hence, *cæteris paribus*, the resistance is as the square of the velocity.

2. This ratio, of the square of the velocity, may be otherwise derived thus. The force of the fluid in motion must be as the force of one particle multiplied by the number of them; but the force of a particle is as its velocity; and the number of them striking the plane in a given time, is also as the velocity; therefore the whole force is as $v \times v$ or v^2, that is, as the square of the velocity.

3. If the direction of motion, instead of being perpendicular to the plane, as above supposed, be inclined to it in any angle, the sine of that angle being s, to the radius 1; then the resistance to the plane, or the force of the fluid against the plane, in the direction of the motion, as assigned above, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination, or in the ratio of 1 to s^3.



For, AB being the direction of the plane, and BD that of the motion, making the angle ABD, whose sine is s; the number of particles, or quantity of the fluid striking the plane, will be diminished in the ratio of 1 to s, or of radius to the sine of the angle B of inclination; and the force of each particle will also be diminished in the same ratio of 1 to s: so that, on both these accounts, the whole resistance will be diminished in the ratio of 1 to s^2, or in the duplicate ratio of radius to the sine of the said angle. But again, it is to be considered that this whole resistance is exerted in the direction BE perpendicular to the plane; and any force in the direction BE, is to its effect in the direction AE, parallel to BD, as AE to BE, that is, as 1 to s. So that finally, on all these accounts, the resistance in the direction of motion, is diminished in the ratio of 1 to s^3, or in the triplicate ratio of radius to the sine of inclination. Hence, comparing this with article 1, the whole resistance, or the motive force on the plane, will be $m = \dfrac{anv^2s^3}{2g}$.

4. Also, if w denote the weight of the body, whose plane face a is resisted by the absolute force m; then the retarding force f, or $\dfrac{m}{w}$, will be $\dfrac{anv^2s^3}{2gw}$.

5. And if the body be a cylinder, whose face or end is a, and diameter d, or radius r, moving in the direction of its axis; because then $s = 1$, and $a = pr^2 = \frac{1}{4}pd^2$, where $p = 3{\cdot}1416$; the resisting force m will be - - - - - - $\dfrac{npd^2v^2}{8g} = \dfrac{npr^2v^2}{2g}$, and the retarding force $f = \dfrac{npd^2v^2}{8gw} = \dfrac{npr^2v^2}{2gw}$.

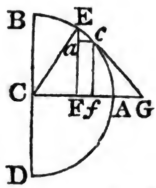
6. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face a conical surface, or an elliptic section, or any other figure every where equally inclined to the axis, the sine of inclination being s: then the number of particles of the fluid striking the face being still the same, but the force of each opposed to the direction of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force m would be $\dfrac{npd^2v^2s^2}{8g} = \dfrac{npr^2v^2s^2}{2g}$.

But if the body were terminated by an end or face of any

other form, as a spherical one, or such like, where every part of it has a different inclination to the axis; then a further investigation becomes necessary, such as in the following proposition.

PROBLEM XX.

To determine the Resistance of a Fluid to any Body, moving in it, of a Curved End; as a Sphere, or a Cylinder, with a Hemispherical End, &c.



1. LET BEAD be a section through the axis CA of the solid, moving in the direction of that axis. To any point of the curve draw the tangent EG, meeting the axis produced in G: also, draw the perpendicular ordinates EF, ef, indefinitely near each other; and draw ae parallel to CG.

Putting CF $= x$, EF $= y$, BE $= z$, $s =$ sine $\angle$ G to radius 1, and $p = 3{\cdot}1416$: then $2py$ is the circumference whose radius is EF, or the circumference described by the point E, in revolving upon the axis CA; and $2py \times$ Ee or $2py\dot{z}$ is the fluxion of the surface, or it is the surface described by Ee, in the said revolution about CA, and which is the quantity represented by a in art. 3 of the last problem: hence $\frac{nv^2s^3}{2g} \times 2py\dot{z}$ or $\frac{pnv^2s^3}{g} \times y\dot{z}$ is the resistance on that ring, or the fluxion of the resistance to the body, whatever the figure of it may be. And the fluent of which will be the resistance required.

2. In the case of a spherical form: putting the radius CA or CB $= r$, we have $y = \sqrt{(r^2 - x^2)}$, $s = \frac{\text{EF}}{\text{EG}} = \frac{\text{CF}}{\text{CE}} = \frac{x}{r}$, and $y\dot{z}$ or EF $\times$ Ee $=$ CE $\times ae = r\dot{x}$; therefore the general fluxion $\frac{pnv^2}{g} \times s^3y\dot{z}$ becomes $\frac{pnv^2}{g} \times \frac{x^3}{r^3} \times r\dot{x} = \frac{pnv^2}{gr^2} \times x^3\dot{x}$; the fluent of which, or $\frac{pnv^2}{4gr^2}x^4$, is the resistance to the spherical surface generated by BE. And when x or CF is $= r$ or CA, it becomes $\frac{pnv^2r^2}{4g}$ for the resistance on the whole

hemisphere; which is also equal to $\frac{pnv^2d^2}{16g}$, where $d = 2r$ the diameter.

3. But the perpendicular resistance to the circle of the same diameter d or BD, by art. 5 of the preceding problem, is $\frac{pnv^2d^2}{8g}$; which, being double the former, shows that the resistance to the sphere, is just equal to half the direct resistance to a great circle of it, or to a cylinder of the same diameter.

4. Since $\frac{1}{6}pd^3$ is the magnitude of the globe; if N denote its density or specific gravity, its weight w will be $= \frac{1}{6}pd^3N$, and therefore the retardive force f or $\frac{m}{w} = \frac{pnv^2d^2}{16g} \times \frac{6}{pNd^3}$ $= \frac{3nv^2}{8gNd}$; which is also $= \frac{v^2}{2gs}$ by art. 8 of the general theorems in page 401; hence then $\frac{3n}{4Nd} = \frac{1}{s}$, and $s = \frac{N}{n}$ $\times \frac{4}{3}d$; which is the space that would be described by the globe, while its whole motion is generated or destroyed by a constant force which is equal to the force of resistance, if no other force acted on the globe to continue its motion. And if the density of the fluid were equal to that of the globe, the resisting force is such, as, acting constantly on the globe without any other force, would generate or destroy its motion in describing the space $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter, by the accelerating or retarding force.

5. Hence the greatest velocity that a globe will acquire by descending in a fluid, by means of its relative weight in the fluid, will be found by making the resisting force equal to that weight. For, after the velocity is arrived at such a degree, that the resisting force is equal to the weight that urges it, it can increase no longer, and the globe will afterwards continue to descend with that velocity uniformly. Now, N and n being the separate specific gravities of the globe and fluid, $N-n$ will be the relative gravity of the globe in the fluid, and therefore $w = \frac{1}{6}pd^3(N-n)$ is the weight by which it is urged; also $m =$ - - - - - $\frac{pnv^2d^2}{16g}$ is the resistance; consequently $\frac{pnv^2d^2}{16g} = \frac{1}{6}pd^3(N-n)$ when the velocity becomes uniform; from which equation

is found $v = \sqrt{(2g \cdot \frac{4}{3}d \cdot \frac{N-n}{n})}$, for the said uniform or greatest velocity.

And, by comparing this form with that in art. 6 of the general theorems in page 400, it will appear that its greatest velocity is equal to the velocity generated by the accelerating force $\frac{N-n}{n}$, in describing the space $\frac{4}{3}d$, or equal to the velocity generated by gravity in freely describing the space $\frac{N-n}{n} \times \frac{4}{3}d$. If $N = 2n$, or the specific gravity of the globe be double that of the fluid, then $\frac{N-n}{n} = 1 =$ the natural force of gravity; and then the globe will attain its greatest velocity in describing $\frac{4}{3}d$ or $\frac{4}{3}$ of its diameter.—It is further evident that if the body be very small, it will very soon acquire its greatest velocity, whatever its density may be.

Exam. If a leaden ball, of 1 inch diameter, descend in water, and in air of the same density as at the earth's surface, the three specific gravities being as $11\frac{1}{3}$, and 1, and $\frac{3}{2500}$. Then $v = \sqrt{(4 \cdot 16\frac{1}{12} \cdot \frac{4}{36} \cdot 10\frac{1}{3})} = \frac{1}{9}\sqrt{(31 \cdot 193)} = 8{\cdot}5944$ feet, is the greatest velocity per second the ball can acquire by descending in water. And $v = \sqrt{(4 \cdot \frac{193}{12} \cdot \frac{4}{36} \cdot \frac{31}{3} \cdot \frac{2500}{3})}$ nearly $= \frac{50}{9}\sqrt{\frac{34 \cdot 193}{3}} = 259{\cdot}82$ is the greatest velocity it can acquire in air.

But if the globe were only $\frac{1}{100}$ of an inch diameter, the greatest velocities it could acquire, would be only $\frac{1}{10}$ of these, namely $\frac{86}{100}$ of a foot in water, and 26 feet nearly in air. And if the ball were still further diminished, the greatest velocity would also be diminished, and that in the subduplicate ratio of the diameter of the ball.

PROBLEM XXI.

To determine the Relations of Velocity, Space, and Time, of a Ball moving in a Fluid, in which it is projected with a given Velocity.

1. Let $a =$ the first velocity of projection, x the space described in any time t, and v the velocity then. Now, by art. 4 of the last problem, the accelerative force $f = \frac{3nv^2}{8gNd}$,

where n is the density of the fluid, N that of the ball, and d its diameter. Therefore the general equation $v\dot{v} = gf\dot{s}$ becomes $v\dot{v} =$ - - - - - - - - - -

$\frac{-3nv^2}{8\text{N}d}\dot{x}$; and hence $\frac{\dot{v}}{v} = \frac{-3n}{8\text{ND}}\dot{x} = -b\dot{x}$, putting b for $\frac{3n}{8\text{N}d}$.

The correct fluent of this, is log. a—log. v or log. $\frac{a}{v} = bx$.

Or, putting $c = 2{\cdot}718281828$, the number whose hyp. log. is 1, then is $\frac{a}{v} = c^{bx}$, and the velocity $v = \frac{a}{c^{bx}} = ac^{-bx}$.

2. The velocity v at any time being the c^{-bx} part of the first velocity, therefore the velocity lost in any time, will be the $1 - c^{-bx}$ part, or the $\frac{c^{bx}-1}{c^{bx}}$ part of the first velocity.

EXAMPLES.

Exam. 1. If a globe be projected, with any velocity, in a medium of the same density with itself, and it describe a space equal to $3d$ or 3 of its diameters. Then $x = 3d$, and $b = \frac{3n}{8\text{N}d} = \frac{3}{8d}$; therefore $bx = \frac{9}{8}$, and $\frac{c^{bx}-1}{c^{bx}} = \frac{2{\cdot}08}{3{\cdot}08}$ is the velocity lost, or nearly $\frac{2}{3}$ of the projectile velocity.

Exam. 2. If an iron ball of 2 inches diameter were projected with a velocity of 1200 feet per second; to find the velocity lost after moving through any space, as suppose 500 feet of air: we should have $d = \frac{2}{12} = \frac{1}{6}$, $a = 1200$, $x = 500$, $\text{N} = 7\frac{1}{3}$, $n = {\cdot}0012$; and therefore $bx =$ - - -

$\frac{3nx}{8\text{N}d} = \frac{3 \,.\, 12 \,.\, 500 \,.\, 3 \,.\, 6}{8 \,.\, 22 \,.\, 10000} = \frac{81}{440}$, and $v = \frac{1200}{c^{\frac{81}{440}}} = 998$ feet

per second: having lost 202 feet, or nearly $\frac{1}{6}$ of its first velocity.

Exam. 3. If the earth revolved about the sun, in a medium as dense as the atmosphere near the earth's surface; and it were required to find the quantity of motion lost in a year. Then, if the earth's mean density be about $4\frac{1}{2}$, and its distance from the sun 12000 of its diameters, we have $24000 \times 3{\cdot}1416 = 75398$ diameters $= x$, and $bx =$ - - -

$\frac{3 \,.\, 75398 \,.\, 12 \,.\, 2}{8 \,.\, 10000 \,.\, 9} = 7{\cdot}5398$; hence $\frac{c^{bx}-1}{c^{bx}} = \frac{1880}{1881}$ parts

are lost of the first motion in the space of a year, and only the $\frac{1}{1886}$ part remains. If the earth's mean density be taken $= 5$, the result will become $\frac{885}{886}$ for the motion lost.

Exam. 4. If it be required to determine the distance moved, x, when the globe has lost any part of its motion, as suppose $\frac{1}{2}$, and the density of the globe and fluid equal; the general equation gives $x = \frac{1}{b} \times \log. \frac{a}{v} = \frac{8d}{3} \times \log.$ of $2 = 1 \cdot 8483925d$. So that the globe loses half its motion before it has described twice its diameter.

3. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{x}}{v} = \frac{c^{bx}\dot{x}}{a}$. Now, to find the fluent of this, put $z = c^{bx}$; then is $bx =$ log. z, and $b\dot{x} = \frac{\dot{z}}{z}$, or $\dot{x} = \frac{\dot{z}}{bz}$; conseq. $\dot{t}$ or $\frac{c^{bx}\dot{x}}{a} = \frac{z\dot{x}}{a} = \frac{\dot{z}}{ab}$; and hence $t = \frac{z}{ab} = \frac{c^{bx}}{ab}$. But as t and x vanish together, and when $x = 0$, the quantity $\frac{c^{bx}}{ab}$ is $= \frac{1}{ab}$; therefore, by correction, $t = \frac{c^{bx}-1}{ab} = \frac{1}{bv} - \frac{1}{ba} = \frac{1}{b}(\frac{1}{v} - \frac{1}{a})$ the time sought; where $b = \frac{3n}{8\text{ND}}$, and $v = \frac{a}{c^{bx}}$ the velocity.

Exam. If an iron ball of 2 inches diameter were projected in the air with a velocity of 1200 feet per second; and it were required to determine in what time it would pass over 500 yards or 1500 feet, and what would be its velocity at the end of that time: We should have, as in exam. 2 above, $b = \frac{3 \,.\, 12 \,.\, 3 \,.\, 6}{8 \,.\, 22 \,.\, 10000} = \frac{1}{2716}$, and $bx = \frac{1500}{2716} = \frac{375}{679}$, hence $\frac{1}{b} = \frac{2716}{1}$, and $\frac{1}{a} = \frac{1}{1200}$, and $\frac{1}{v} = \frac{c^{bx}}{a} = \frac{1 \cdot 7372}{1200} = \frac{1}{690}$ nearly. Consequently $v = 690$ is the velocity; and $t = \frac{1}{b}(\frac{1}{v} - \frac{1}{a}) = 2716 \times (\frac{1}{690} - \frac{1}{1200}) = 1\frac{31}{46}$ seconds is the time required, or 1" and $\frac{2}{3}$ nearly.

PROBLEM XXII.

To determine the Relations of Space, Time, and Velocity, when a Globe descends, by its own Weight, in a Fluid.

The foregoing notation remaining, viz. d = diameter, N and n the density of the ball and fluid, and v, s, t, the velocity, space, and time, in motion; we have $\frac{1}{6}pd^3$ = the magnitude of the ball, and $\frac{1}{6}pd^3 (\text{N} - n)$ = its weight in the fluid, also $m = \frac{pnd^2v^2}{16g}$ = its resistance from the fluid; consequently $\frac{1}{6}pd^3(\text{N}-n) - \frac{pnd^2v^2}{16g}$ is the motive force by which the ball is urged; which being divided by $\frac{1}{6}p\text{N}d^3$, the quantity of matter moved, gives $f = 1 - \frac{n}{\text{N}} - \frac{3nv^2}{8g\text{N}d}$ for the accelerative force.

2. Hence $v\dot{v}, = gf\dot{s}$, and $\dot{s} = \frac{v\dot{v}}{gf} = \frac{\text{N}v\dot{v}}{g(\text{N}-n) - \frac{3n}{8d}v^2} = \frac{1}{b} \times \frac{v\dot{v}}{a - v^2}$, putting $b = \frac{3n}{8\text{N}d}$, and $\frac{1}{a} = \frac{3\text{N}}{g \,.\, 8d(\text{N}-n)}$, or $ab = g$ nearly; the fluent of which is s = - - - - - - - $\frac{1}{2b} \times$ log. of $\frac{a}{a - v^2}$, an expression for the space s, in terms of the velocity v. That is, when s and v begin, or are equal to nothing, both together.

But if the body commence motion in the fluid with a certain given velocity e, or enter the fluid with that velocity, like as when the body, after falling in empty space from a certain height, falls into a fluid like water; then the correct fluent will be $s = \frac{1}{2b} \times$ hyp. log. of $\frac{a-e^2}{a-v^2}$.

3. But now to determine v in terms of s, put c = 2·718281828; then, since the log. of $\frac{a}{a-v^2} = 2bs$, therefore $\frac{a}{a-v^2} = c^{2bs}$, or $\frac{a-v^2}{a} = c^{-2bs}$; hence $v = \sqrt{(a - ac^{-2bs})}$ is the velocity sought.

4. The greatest velocity is to be found, as in art. 5 of prob. 20, by making f or $1 - \frac{n}{N} - \frac{3nv^2}{8gNd} = 0$, which gives $v = \sqrt{(g \cdot 8d \cdot \frac{N-n}{3n})} = \sqrt{a}$. The same value of v is obtained by making the fluxion of v^2, or of $a - ac^{-2bs}, = 0$, And the same value of v is also obtained by making s infinite, for then $c^{-2bs} = 0$. But this velocity $\sqrt{a}$ cannot be attained in any finite time, and it only denotes the velocity to which the general value of v or $\sqrt{(a - ac^{-2bs})}$ continually approaches. It is evident, however, that it will approximate towards it the faster, the greater b is, or the less d is; and that, the diameters being very small, the bodies descend by nearly uniform velocities, which are directly in the subduplicate ratio of the diameters. See also art. 5, prob. 20, for other observations on this head.

5. To find the time t. Now $\dot{t} = \frac{\dot{s}}{v} = \sqrt{\frac{1}{a}} \times \frac{\dot{s}}{\sqrt{(1-c^{-2bs})}}$. Then, to find the fluent of this fluxion, put $z = \sqrt{(1-c^{-2bs})} = \frac{v}{\sqrt{a}}$, or $z^2 = 1 - c^{-2bs}$; hence $z\dot{z} = b\dot{s}c^{-2bs}$, and $\dot{s} = \frac{z\dot{z}}{bc^{-2bs}} = \frac{1}{b} \cdot \frac{z\dot{z}}{1-z^2}$, consequently $\dot{t} = \frac{1}{b\sqrt{a}} \cdot \frac{\dot{z}}{1-z^2}$, ...

and therefore the fluent is $t = \frac{1}{2b\sqrt{a}} \times \log. \frac{1+z}{1-z} = \frac{1}{2b\sqrt{a}} \times \log. \frac{1+\sqrt{(1-c^{-2bs})}}{1-\sqrt{(1-c^{-2bs})}} = \frac{1}{2b\sqrt{a}} \times \log. \frac{\sqrt{a}+v}{\sqrt{a}-v}$, which is the general expression for the time.

Or thus: because $\dot{s} = \frac{1}{b} \cdot \frac{v\dot{v}}{a-v^2}$, theref. $\dot{t} = \frac{1}{b} \cdot \frac{\dot{v}}{a-v^2}$;

and the fluent, by form 10, is $\frac{1}{2b\sqrt{a}} \times \log. \frac{\sqrt{a}+v}{\sqrt{a}-v}$.

Exam. If it were required to determine the time and velocity, by descending in air 1000 feet, the ball being of lead, and 1 inch diameter.

Here $N = 11\frac{1}{3}$, $n = \frac{3}{2500}$, $d = \frac{1}{12}$, and $s = 1000$.

Hence $a = \frac{2 \cdot 16 \cdot \frac{1}{12} \cdot \frac{3}{16} \cdot 11\frac{1}{3}}{3 \cdot \frac{3}{2500}} = \frac{2 \cdot 193 \cdot 8 \cdot 34 \cdot 2500}{3 \cdot 3 \cdot 12 \cdot 12 \cdot 3} = \frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}$, and $b = \frac{3 \cdot \frac{3}{2500}}{8 \cdot 11\frac{1}{3} \cdot \frac{1}{12}} = \frac{3 \cdot 3 \cdot 3 \cdot 12}{8 \cdot 34 \cdot 2500} = \frac{9 \cdot 9}{68 \cdot 50^2}$;

consequently $v = \surd a \times \surd(1 - c^{-2bs}) = \surd\frac{193 \,.\, 34 \,.\, 50^2}{9 \,.\, 27} \times$ $\surd(1 - c^{-\frac{3}{8}\frac{1}{5}}) = 203\frac{2}{3}$ the velocity. And $t = \frac{1}{2b\surd a} \times \log.$ $\frac{2 + \surd(1 - c^{-2bs})}{1 - \surd(1 - c^{-2bs})} = \surd\frac{34 \,.\, 2500}{27 \,.\, 193} \times \log. \frac{1{\cdot}78383}{0{\cdot}21617} = 8{\cdot}5236''$, the time.

Note. If the globe be so light as to ascend in the fluid; it is only necessary to change the signs of the first two terms in the value of f, or the accelerating force, by which it becomes $f = \frac{n}{N} - 1 - \frac{3nv^2}{8gxd}$; and then proceed in all respects as before.

SCHOLIUM.

To compare this theory, contained in the last four problems, with experiment, the few following numbers are here extracted from extensive tables of velocities and resistances, resulting from a course of many hundred very accurate experiments, made in the course of the year 1786.

In the first column are contained the mean uniform or greatest velocities acquired in air, by globes, hemispheres, cylinders, and cones, all of the same diameter, and the altitude of the cone nearly equal to the diameter also, when urged by the several weights expressed in avoirdupois ounces, and standing on the same line with the velocities, each in their proper column. So, in the first line, the numbers show, that, when the greatest or uniform velocity was accurately 3 feet per second, the bodies were urged by these weights, according as their different ends went foremost; namely, by ·028 oz. when the vertex of the cone went foremost; by ·064 oz. when the base of the cone went foremost; by ·027 oz. for a whole sphere; by ·050 oz. for a cylinder; by ·051 oz. for the flat side of the hemisphere; and by ·020 oz. for the round or convex side of the hemisphere. Also, at the bottom of all, are placed the mean proportions of the resistances of these figures in the nearest whole numbers. Note, the common diameter of all the figures was 6·375, or $6\frac{3}{8}$ inches; so that the area of the circle of that diameter is just 32 square inches, or $\frac{2}{9}$ of a square foot; and the altitude of the cone was $6\frac{5}{8}$ inches. Also, the diameter of the small hemisphere was $4\frac{3}{4}$ inches, and conse-

quently the area of its base 17⅗ square inches, or ⅛ of a square foot nearly.

From the given dimensions of the cone, it appears, that the angle made by its side and axis, or direction of the path, is 25° 42′, very nearly.

The mean height of the barometer at the times of making the experiments, was nearly 30·1 inches, and of the thermometer 62°; consequently the weight of a cubic foot of air was equal to 1⅕ oz. nearly, in those circumstances.

Veloc. per sec.	Cone. vertex.	Cone. base.	Whole globe.	Cylinder.	Hemisphere. flat.	Hemisphere. round.	Small Hemis. flat.
feet.	oz.	oz.	oz.	oz.	oz.	oz.	oz.
3	·028	·064	·027	·050	·051	·020	·028
4	·048	·109	·047	·090	·096	·039	·048
5	·071	·162	·068	·143	·148	·063	·072
6	·098	·225	·094	·205	·211	·092	·103
7	·129	·298	·125	·278	·284	·123	·141
8	·168	·382	·162	·360	·368	·160	·184
9	·211	·478	·205	·456	·464	·199	·233
10	·260	·587	·255	·565	·573	·242	·287
11	·315	·712	·310	·688	·698	·297	·349
12	·376	·850	·370	·826	·836	·347	·418
13	·440	1·000	·435	·979	·988	·409	·492
14	·512	1·166	·505	1·145	1·154	·478	·573
15	·589	1·346	·581	1·327	1·336	·552	·661
16	·673	1·546	·663	1·526	1·538	·634	·754
17	·762	1·763	·752	1·745	1·757	·722	·853
18	·858	2·002	·848	1·986	1·998	·818	·959
19	·959	2·260	·949	2·246	2·258	·922	1·073
20	1·069	2·540	1·057	2·528	2·542	1·033	1·196
Propor. Numb.	126	291	124	285	288	119	140

From this table of resistances, several practical inferences may be drawn. As,

1. That the resistance is nearly as the surface; the resistance increasing but a very little above that proportion in the greater surfaces. Thus, by comparing together the numbers

in the 6th and last columns, for the bases of the two hemispheres, the areas of which are in the proportion of $17\frac{3}{4}$ to 32, or as 5 to 9 very nearly; it appears that the numbers in those two columns, expressing the resistances, are nearly as 1 to 2, or as 5 to 10, as far as to the velocity of 12 feet; after which the resistances on the greater surface increase gradually more and more above that proportion. And the mean resistances are as 140 to 288, or as 5 to $10\frac{2}{7}$. This circumstance therefore agrees nearly with the theory.

2. The resistance to the same surface, is nearly as the square of the velocity; but gradually increasing more and more above that proportion, as the velocity increases. This is manifest from all the columns. And therefore this circumstance also differs but little from the theory, in small velocities.

3. When the hinder parts of bodies are of different forms, the resistances are different, though the fore parts be alike; owing to the different pressures of the air on the hinder parts. Thus, the resistance to the fore part of the cylinder, is less than that on the flat base of the hemisphere, or of the cone; because the hinder part of the cylinder is more pressed or pushed, by the following air, than those of the other two figures.

4. The resistance on the base of the hemisphere, is to that on the convex side, nearly as $2\frac{2}{5}$ to 1, instead of 2 to 1, as the theory assigns the proportion. And the experimented resistance, in each of these, is nearly $\frac{1}{4}$ part more than that which is assigned by the theory.

5. The resistance on the base of the cone is to that on the vertex, nearly as $2\frac{3}{10}$ to 1. And in the same ratio is radius to the sine of the angle of the inclination of the side of the cone, to its path or axis. So that, in this instance, the resistance is directly as the sine of the angle of incidence, the transverse section being the same, instead of the square of the sine.

6. Hence we can find the altitude of a column of air, whose pressure shall be equal to the resistance of a body, moving through it with any velocity. Thus,

Let a = the area of the section of the body, similar to any of those in the table, perpendicular to the direction of motion;

r = the resistance to the velocity, in the table; and

x = the altitude sought, of a column of air, whose base is a, and its pressure r.

Then $ax =$ the content of the column in feet,
and $1\frac{1}{5}ax$ or $\frac{6}{5}ax$ its weight in ounces; . . . .

therefore $\frac{6}{5}ax = r$, and $x = \frac{5}{6} \times \frac{r}{a}$ is the altitude sought in feet, namely, $\frac{5}{6}$ of the quotient of the resistance of any body divided by its transverse section; which is a constant quantity for all similar bodies, however different in magnitude, since the resistance r is as the section a, as was found in art. 1. When $a = \frac{2}{9}$ of a foot, as in all the figures in the foregoing table, except the small hemisphere: then, $x = \frac{5}{6} \times \frac{r}{a}$ becomes $x = \frac{15}{4}r$, where r is the resistance in the table, to the similar body.

If, for example, we take the convex side of the large hemisphere, whose resistance is ·634 oz. to a velocity of 16 feet per second, then $r = \cdot634$, and $x = \frac{15}{4}r = 2\cdot3775$ feet, is the altitude of the column of air whose pressure is equal to the resistance on a spherical surface, with a velocity of 16 feet. And to compare the above altitude with that which is due to the given velocity, it will be $32^2 : 16^2 :: 16 : 4$, the altitude due to the velocity 16; which is near double the altitude that is equal to the pressure. And as the altitude is proportional to the square of the velocity, therefore, in small velocities, the resistance to any spherical surface, is equal to the pressure of a column of air on its great circle, whose altitude is $\frac{19}{32}$ or ·594 of the altitude due to its velocity.

But if the cylinder be taken, whose resistance, $r = 1\cdot526$: then $x = \frac{15}{4}r = 5\cdot72$: which exceeds the height, 4, due to the velocity, in the ratio of 23 to 16 nearly. And the difference would be still greater, if the body were larger; and also if the velocity were more.

7. Also, if it be required to find with what velocity any flat surface must be moved, so as to suffer a resistance just equal to the whole pressure of the atmosphere:

The resistance on the whole circle whose area is $\frac{2}{9}$ of a foot, is ·051 oz. with a velocity of 3 feet per second; it is $\frac{1}{9}$ of ·051, or ·0056 oz. only, with a velocity of 1 foot. But $2\frac{1}{2} \times 13600 \times \frac{2}{9} = 7555\frac{5}{9}$ oz. is the whole pressure of the atmosphere. Therefore, as $\sqrt{\cdot0056} : \sqrt{7556} :: 1 : 1162$ nearly, which is the velocity sought. Being almost equal to the velocity with which air rushes into a vacuum.

8. Hence may be inferred the great resistance suffered by military projectiles. For, in the table, it appears, that a globe of $6\frac{3}{8}$ inches diameter, which is equal to the size of an iron ball weighing 36lb. moving with a velocity of only

16 feet per second, meets with a resistance equal to the pressure of $\frac{3}{5}$ of an ounce weight; and therefore, computing only according to the square of the velocity, the least resistance that such a ball would meet with, when moving with a velocity of 1600 feet, would be equal to the pressure of 417lb., and that independent of the pressure of the atmosphere itself on the fore part of the ball, which would be 487lb. more, as there would be no pressure from the atmosphere on the hinder part, in the case of so great a velocity as 1600 feet per second. So that the whole resistance would be more than 900lb. to such a velocity.

9. Having said, in the last article, that the pressure of the atmosphere is taken entirely off the hinder part of the ball moving with a velocity of 1600 feet per second; which must happen when the ball moves faster than the particles of air can follow by rushing into the place quitted and left void by the ball, or when the ball moves faster than the air rushes into a vacuum from the pressure of the incumbent air: let us therefore inquire what this velocity is. Now the velocity with which any fluid issues, depends on its altitude above the orifice, and is indeed equal to the velocity acquired by a heavy body in falling freely through that altitude. But, supposing the height of the barometer to be 30 inches, or $2\frac{1}{2}$ feet, the height of a uniform atmosphere, all of the same density as at the earth's surface, would be $2\frac{1}{2} \times 14 \times 833\frac{1}{3}$ or 29167 feet; therefore $\sqrt{16} : \sqrt{29167} :: 32 : 8\sqrt{29167} = 1366$ feet, which is the velocity sought. And therefore, with a velocity of 1600 feet per second, or any velocity above 1366 feet, the ball must continually leave a vacuum behind it, and so must sustain the whole pressure of the atmosphere on its fore part, as well as the resistance arising from the *vis inertia* of the particles of air struck by the ball.

10. On the whole, we find that the resistance of the air, as determined by the experiments, differs very widely, both in respect to its quantity on all figures, and in respect to the proportions of it on oblique surfaces, from the same as determined by the preceding theory; which accords with that of Sir Isaac Newton, and most modern philosophers. Neither should we succeed better if we have recourse to the theory given by Professor Gravesande, or others, as similar differences and inconsistencies still occur.

We conclude therefore, that all the theories of the resistance of the air hitherto given, are very erroneous. And the preceding one is only laid down, till further experiments, on this important subject, shall enable philosophers to deduce from them another, that shall be more consonant to the true phenomena of nature.

APPENDIX.

TABLES, &c. OF COMPARATIVE STRENGTH;

OR, THE SPECIFIC COHESION OF DIFFERENT SUBSTANCES.

(*From Tables drawn up by Mr. Thomas Tredgold.*)

It is the cohesion of the parts of solid bodies, which not only serves to characterize different substances, but also to determine their relative value in the various uses to which they may be appropriated. The standard degree of cohesion, employed in the following tables, is plate-glass, which is taken as *unity*, and the other substances are stronger or weaker in proportion as they are above or below 1. The strength of woods of the same kinds is, it will be observed, extremely variable, depending on the age, the nature of the soil, and the situation of the climate where they are grown.

Table I.—*Woods.*

	Specific Cohesion.		Specific Cohesion.
Lance-wood -	2·621	Oak - - -	1·836
Locust-tree - -	2·185	Dry, cut 4 years -	1·707
Jujube (Ziziphus)	2·008	Provence, seasoned*	1·559
Ash (Fraxinus).		English, seasoned -	1·509
Red, seasoned -	1·899	Oak - - -	1·481
Ash - - -	1·804	French, seasoned†	1·450
White, seasoned -	1·509	Provence, seasoned‡	1·444
Ash - - -	1·274	Provence, seasoned, young - -	1·363
Oak (Quercus) -	1·891	Oak, dry - -	1·274
——, highest result	1·861		

* Its colour brown, and it was hard and large-veined.

† This specimen lay six months in water after it was cut, and was afterwards dried. When the trial was made, it had been cut four years.

‡ Middle-aged timber, fine-veined, light and pliant.

	Specific Cohesion.
Baltic, seasoned -	1·211
Oak, lowest result	1·146
——, - - -	1·107
English - -	1·085
Oak - - -	1·076
French, unseasoned	1·060
White American, seasoned - -	1·009
Oak - - -	1·009
French, unseasoned	0·960
Oak - - -	0·955
English - -	0·936
Dantzic - -	0·818
Beech (Fagus sylvaticus) - -	1·880
Arbutus, from -	1·845
to - -	0·814
Orange (Aurantium)	1·764
to - -	1·629
Bay (Laurus) -	1·547
to - -	1·085
Teak (Tectona grandis).	
Java, seasoned -	1·509
Pegu, seasoned -	1·400
Malabar, seasoned	1·395
Alber (Bet. Alnus)	1·506
Mulberry (Morus)	1·492
to - -	1·221
Elm (Ulmus) -	1·432
Firs (Pinus).	
Pitch pine - -	1·398
Fir - - -	1·380
Fir (strongest) -	1·318
Pitch pine - -	1·281
Pine (Pin du Nord)	1·264
Larch (Pinus Larix)	1·177
Fir, strong red -	1·172
Fir, Memel, seasoned	1·154
Fir, Russian -	1·062
Fir - - -	1·061
Fir - - -	1·039
Fir, Riga - -	0·963
Fir, American -	0 942
Fir - - -	0·903
Fir, yellow deal -	0·900
Fir, weakest - -	0·879
Larch, Scotch, seasoned - -	0·837
Pitch pine - -	0·830
Larch, Scotch, very dry - - -	0·745
Fir, Scotch (P. sylvestris) - -	0·711
Fir, white deal -	0·455
Sissor, of Bengal -	1·395
Saul, of Bengal -	1·375
Plum, (Prunus) -	1·357
to - -	1·205
Willow, (Salix) -	1·357
Willow, dry - -	0·809
Mahogany (Swictenia).	
Spanish - -	1·283
Citron (Citreum) -	1·357
to - -	0·868
Chestnut, Sweet (Fagus castanea).	
100 years in use -	1·291
Jasmine (Jasminum)	1·276
to - -	1·248
Pomegranate (Punica)	1·221
to - -	0·882
Tamarisk (Tamariscus) - -	1·194
to - -	0·732
Maple (Acer).	
Norway - -	1·123
Elder (Sambucus)	1·086
Lemon (Limon) -	1·004
Quince (Cydonia)	0·841
to - -	0·624
Cypress (Cupressus)	0·732
to - -	0·542
Poplar (Pop. alba)	0·705
to - -	0·488
Poplar (P. nigra) lateral cohesion of the animal rings	0·189
Cedar - - -	0·528

Table II.—*Comparative Strength of Metals.*

(*h*) and (*l*) mark the highest and lowest result obtained from each kind of iron.

	Specific Cohesion. Pl. Glass as 1.
Steel.	
Razor temper	15·927
Soft	12·739
Iron.	
Wire	12·004
German bar, mark BR (*h*)	9·880
Swedish bar (*h*)	9·445
German bar, mark L (*h*)	9·119
Wire	9·108
Bar	8·964
Liege bar (*h*)	8·794
Spanish bar	8·685
Bar	8·581
Bar	8·492
Oosement bar (*h*)	8·142
Cable	7·752
German bar, mark L (*l*)	7·382
German Bar, common	7·339
Swedish bar, Oosement bar } (*l*)	7·296
Bar of best quality	7·006
Liege bar (*l*)	6·621
German bar, mark BR (*l*)	6·514
Bar*	6·480
Bar of good quality	5·839
Cable	5·787
Bar, fine-grained	5·306
——, medium fineness	3·618
——, coarse-grained	2·172

	Specific Cohesion. Pl. Glass as 1.
Cast Iron.	
French	7·470
German	7·250
French, soft	6·754
English	5·520
French	5·412
——	4·540
English, soft	4·334
French gray	4·000
Gray, of Cruzot, 2nd fusion	3·257
Gray, of Cruzot, 1st fusion	3·202
Copper.	
Wire	6·606
Cast, Barbary	2·398
——, Japan	2·152
Platinum.	
Wire	5·995
Wire	5·625
Silver.	
Wire	4·090
Cast	4·342
Gold.	
Wire	3·279
Cast	2·171
Tin.	
Wire	0·7568
Cast, English block	0·706

* This is the mean result of thirty-three experiments.

	Specific Cohesion. Pl. Glass as 1.		Specific Cohesion. Pl. Glass as 1.
Cast, English block	0·565	Cast, Goslar, from	0·3118
——, Banca - -	0·3906	to - -	0·2855
——, Malacca -	0·342		
Bismuth.		**Lead.**	
Cast - - -	0·345	Milled - - -	0·3538
—— - - -	0·3193	Wire - - -	0·334
Zinc.		Wire - - -	0·274
		Wire - - -	0·2704
Wire - - -	2·394	Cast, English -	0·094
Patent sheet - -	1·762	Antimony, cast -	0·1126

Table III.—*Comparative Strength of Marble, Ivory, and other Miscellaneous Substances.*

	Specific Cohesion. Glass as 1.		Specific Cohesion. Glass as 1.
Hemp fibres glued together - -	9·766	Portland stone (compact lime-stone) -	0·083
Paper strips glued together - -	3·184	Soft stone† of Givry - -	0·041
Ivory - - -	1·765	Brick from - -	0·031
Slate, Welsh, (clay slate) - -	1·358	to - -	0·030
Plate-glass - -	1·000	Brick from Dorking	0·029
Marble (white) -	0·955	Stone, homogeneous white, of a fine grain - -	0·022
Horn of an ox -	0·950	Plaster of Paris -	0·0077
Whalebone - -	0·814	Mortar of sand and lime, 16 years made - -	0·0054
Bone of an ox -	0·559		
Hard stone* of Givry - -	0·230		

Comparative Strength of Substances.

Note. If any of the numbers in these tables be multiplied by 9420, the product will express the force in pounds avoirdupois that would tear asunder a bar of the respective substance *an inch square*. By this process, therefore, these

* This stone was hard, of a red colour, and the beds distinctly marked.

† This stone was white, rather soft, and the beds not distinctly marked. These numbers were calculated from experiments on the transverse strength.

numbers will furnish values of c, similar, in nature, and applicable to the same purposes, as the values of c in the table at pa. 392.

Thus, $6{\cdot}754 \times 9420 = 63622{\cdot}68$, value of c for French soft cast iron.

Again, $1{\cdot}123 \times 9420 = 10578{\cdot}66$, value of c for Norway maple.

And $\cdot 3538 \times 9420 = 3332{\cdot}796$, value of c for milled lead. And so with regard to others.

*Practical Rules for ascertaining the Dimensions of Gudgeons, Shafts, &c.**

1. Let w utmost amount of the stress in cwts., l the length of the gudgeon in inches from the shoulder to the extreme point of bearing, d the diameter of the gudgeon in inches; then

$$0{\cdot}42\,(\mathrm{w}l)^{\frac{1}{3}} = d.$$

2. If a cylindrical shaft have no other lateral stress to sustain than its own weight, then the rule is $\sqrt{(\cdot 007l^3)} = d$, diameter of the shaft in inches.

3. Let the stress, supposed to be at the middle, be n times the weight of the shaft; then

$$\sqrt{(\cdot 012l^3n)} = d, \text{ in inches.}$$

4. For *hollow* cylindrical shafts of cast iron, to resist lateral stress, let D, the exterior diameter, and ND, the interior one; then w being, as before, in cwts.

$$\left[\frac{\mathrm{w}l^2}{2(1-\mathrm{N}^4)}\right]^{\frac{1}{4}} = \mathrm{D}, \text{ diameter in inches.}$$

5. If the hollow shaft support n times its own weight; then

$$\sqrt{\frac{\cdot 012l^3n}{1+\mathrm{N}^2}} = \mathrm{D}.$$

6. For *wrought iron* shafts, find the adequate diameter for *cast iron*, and multiply by ·935.

7. For *oak* shafts, multiply the adequate dimension for *cast* iron by 1·83.

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* These are selected, for their obvious utility, from Tredgold's additions to Buchanan's Essays on Millwork, &c. See farther the rules and examples in a subsequent part of this volume.

8. For *fir* shafts, multiply the requisite dimension for *cast* iron by 1·716.

9. For cylindrical shafts of cast iron to resist torsion, let H be the number of horses' power, N the revolutions of the shaft in a minute; then,

$$\sqrt[3]{\frac{240\text{H}}{\text{N}}} = d, \text{ inches.}$$

10. For *wrought* iron multiply the preceding result by 0·963.

11. For *oak*, the multiplier is 2·238.

12. For *fir*, the multiplier is 2·06.

13. If a shaft have to sustain both lateral stress and torsion, the sum of the straining forces must be taken. The practical rule to be then employed, for *cast* iron, is

$$\left(\frac{240\text{H}}{\text{N}} + \frac{\text{W}l^2}{2}\right)^{\frac{1}{3}} = d, \text{ in inches.}$$

EXAMPLES.

1. Let the length of a cast iron shaft be 12 feet, the lateral stress double its own weight. Required the diameter.

Ans. 6·44 inches.

2. Supposing the length the same, and the lateral stress quadruple its own weight. Required the diameter.

Ans. 9·1 inches.

3. Required the corresponding diameters of shafts of *oak*, and of *fir*, in both cases.

Ans. in the first, for *oak* 11·78, for *fir* 11·05.
In the second, for *oak* 16·65, for *fir* 15·62.

4. Let the interior diameter of a hollow cast iron shaft, of 12 feet long, be six-tenths of the exterior diameter, and the stress four times the weight of the shaft: required both diameters.

Ans. exterior 7·9, interior 4·7 } inches.

5. Let the moving force be equal to 7 horses, the number of turns per minute $11\frac{1}{2}$: required the diameter of the shaft to resist the torsion, both for cast iron and fir.

Ans. cast iron 5·267, fir 10·850 } inches.

6. Suppose that a cylindrical shaft of cast iron is to make 34 revolutions per minute, the power of the first mover being

equal to three horses, the length of the shaft 8 feet, and the lateral stress 3 cwts. when reduced to the middle point. Required the diameter of the shaft.

Ans. $\left(\frac{240\times3}{34}+\frac{3 \,.\, 8^2}{2}\right)^{\frac{1}{3}}=\sqrt[3]{(21 \cdot 18+96)}=4 \cdot 893$ inches.

ON MODELS.

FROM an experiment made to ascertain the firmness of the model of a machine, or of an edifice, certain precautions are necessary before we can infer the firmness of the structure itself.

The classes of forces must be distinguished; as, whether they tend to *draw* asunder the parts, to *break* them transversely, or to *crush* them by compression. To the first class belongs the stretching suffered by key-stones, or bonds of vaults, &c.; to the second, the load which tends to bend or break horizontal or inclined beams; to the third, the weight which presses vertically upon walls and columns.

PROP. 1. If the side of a model be to the corresponding side of the structure as 1 to n, the stress which tends to *draw* asunder, or to *break transversely*, the parts, increases from the smaller to the greater scale as 1 to n^3; while the resistance to those ruptures increases only as 1 to n^2.

The structure, therefore, will have so much less firmness than the model, as n is greater.

If W be the greatest weight which one of the beams of the model can bear, and w the weight or stress which it actually sustains, then the limit of n will be $n=\frac{W}{w}$.

PROP. 2. The side of a model being to the corresponding side of the structure as 1 to n, the stress which tends to crush the parts by compression, increases from the smaller to the greater scale, as 1 to n^3, while the resistance increases only in the ratio of 1 to n.

Hence, if W were the greatest load which a modular wall, or column, could carry, and w the weight with which it is actually loaded; then the greatest limit of increased dimensions would be found from the expression $n=\sqrt{\frac{W}{w}}$.

If, retaining the length or height nh, and the breadth nb, we wished to give to the solid such a thickness xt, as that it should not break in consequence of its increased dimensions, we should have $x = n^2 \sqrt{\frac{w}{W}}$.

In the case of a pilaster with a square base, or of a cylindrical column, if the dimension of the model were d, and of the largest pillar, which should not crush with its own weight when n times as high, xd, we should have

$$x = n \sqrt[3]{\frac{n^2 w}{W}}.$$

These theorems will often find their application in the profession of an engineer, whether civil or military.

ON THE MOTION OF MACHINES AND THEIR MAXIMUM EFFECTS.

Art. 1. When forces acting in contrary directions, or in any such directions as produce contrary effects, are applied to machines, there is, with respect to every simple machine (and of consequence with respect to every combination of simple machines) a certain relation between the powers and the distances at which they act, which, if subsisting in any such machine when at rest, will always keep it in a state of rest, or of *statical* equilibrium; and for this reason, because the efforts of these powers, when thus related, with regard to magnitude and distance, being equal and opposite, annihilate each other, and have no tendency to change the state of the system to which they are applied. So also, if the same machine have been put into a state of *uniform* motion, whether rectilinear or rotatory, by the action of any power distinct from those we are now considering, and these two powers be made to act upon the machine in such motion in a similar manner to that in which they acted upon it when at rest, their simultaneous action will preserve it in that state of uniform motion, or of *dynamical* equilibrium; and this for the same reason as before, because their contrary effects destroy each other, and have therefore no tendency to change the *state* of the machine. But, if at the time a machine is in a state of balanced rest, any one of the opposite forces be increased while it continues to act at the same distance, this

excess of force will disturb the statical equilibrium, and produce motion in the machine; and if the same excess of force continues to act in the same manner, it will, like every constant force, produce an accelerated motion; or, if it should undergo particular modifications when the machine is in different positions, it may occasion such variations in the motion as will render is alternately accelerated and retarded. Or the different species of resistance to which a moving machine is subjected, as the rigidity of ropes, friction, resistance of the air, &c. may so modify a motion, as to change a regular or irregular variable motion into one which is uniform.

2. Hence then the motion of machines may be considered as of *three* kinds. 1. That which is gradually accelerated, which obtains commonly in the first instants of the communication. 2. That which is entirely uniform. 3. That which is alternately accelerated and retarded. Pendulum clocks, and machines which are moved by a balance, are related to the third class. Most other machines, a short time after their motion is commenced, fall under the second. Now though the motion of a machine is alternately accelerated and retarded, it may, notwithstanding, be measured by a uniform motion, because of the periodical and regular repetition which may exist in the acceleration and retardation. Thus the motion of a seconds' pendulum, considered in respect to a single oscillation, is accelerated during the first half second, and retarded during the next: but the same motion taken for many oscillations may be considered as uniform. Suppose, for example, that the extent of each oscillation is 5 inches, and that the pendulum has made 10 oscillations: its total effect will be to have run over 50 inches in 10 seconds; and, as the space described in each second is the same, we may compare the effect to that produced by a moveable which moves for 10 seconds with a velocity of 5 inches per second. We see, therefore, that the theory of machines whose motions are uniform, conduces naturally to the estimation of the effects produced by machines whose motion is alternately accelerated and retarded: so that the problems comprised in this chapter will be directed to those machines whose motions fall under the first two heads; such problems being of far the greatest utility in practice.

Defs. 1. When in a machine there is a system of forces or of powers mutually in opposition, those which produce or tend to produce a certain effect are called *movers* or *moving powers*; and those which produce or tend to produce an effect which opposes those of the moving powers, are called *resistances*. If various movers act at the same time, their equivalent (found by means of prop. 7, Motion and Forces)

is called individually *the moving force*; and, in like manner, the resultant of all the resistances reduced to some one point, *the resistance*. This reduction in all cases simplifies the investigation.

2. The *impelled point* of a machine is that to which the action of the moving power may be considered as immediately applied; and the *working point* is that where the resistance arising from the work to be performed immediately acts, or to which it ought all to be reduced. Thus, in the wheel and axle, (Mechan. prop. 62), where the moving power P is to overcome the weight or resistance W, by the application of the cords to the wheel and to the axle, B is the impelled point, and A the working point.

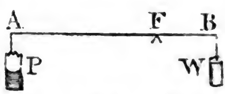
3. The *velocity of the moving power* is the same as the velocity of the impelled point; the *velocity of the resistance* the same as that of the working point.

4. The *performanee* or *effect* of a machine, or the *work done*, is measured by the product of the resistance into the velocity of the working point; the *momentum of impulse* is measured by the product of the moving force into the velocity of the impelled point.

These definitions being established, we may now exhibit a few of the most useful problems, giving as much variety in their solutions as may render one or other of the methods of easy application to any other cases which may occur.

PROPOSITION I.

If R *and* r *be the distances of the power* P, *and the weight or resistance* W, *from the fulcrum* F *of a straight lever; then will the velocity of the power and of the weight at the end of any time* t *be* $\frac{R^2P-RrW}{R^2P+r^2W}gt$, *and* $\frac{RrP-r^2W}{R^2P+r^2W}gt$, *respectively, the weight and inertia of the lever itself not being considered.*



If the effort of the power balanced that of the resistance, P would be equal to $\frac{rW}{R}$. Consequently, the difference between this value of P and its actual value, or $P-\frac{r}{R}W$, will be the force which tends to move the lever. And because this power applied to the point A accelerates the masses P and W, the mass to be substituted

for W, in the point A, must be $\frac{r^2}{R^2}W$, (Dynam. art. 225), in order that this mass at the distance R may be equally accelerated with the mass W at the distance R. Hence the power $P - \frac{r}{R}W$ will accelerate the quantity of matter $P + \frac{r^2}{R^2}W$; and the accelerating force $F = (P - \frac{r}{R}W) \div (P + \frac{r^2}{R^2}W) = \frac{PR^2 - RrW}{PR^2 + r^2W}$. But (page 400) $v \propto Ft$ or is $= gtF$ (g being $= 32\frac{1}{6}$ feet); which in this case $= \frac{RP - RrW}{R^2P + r^2W} \cdot gt$, the velocity of P. And because veloc. of P : veloc. of W :: R : r, therefore veloc. of $W = \frac{r}{R}$ veloc. of $P = \frac{r}{R} \times \frac{R^2P - RrW}{R^2P + r^2W}gt = \frac{RrP - r^2W}{R^2P + r^2W} \cdot gt$.

Cor. 1. The space described by the power in the time t, will be $= \frac{R^2P - RrW}{R^2P + r^2W} \cdot \frac{1}{2}gt^2$; the space described by W in the same time will be $= \frac{RrP - r^2W}{R^2P + r^2W} \cdot \frac{1}{2}gt^2$.

Cor. 2. If R : r :: n : 1, then will the force which accelerates A be $= \frac{Pn^2 - Wn}{Pn^2 + W}$.

Cor. 3. If at the same time the inertia of the moving force P be $= 0$, as in muscular action, the force accelerating A will be $= \frac{Pn^2 - Wn}{W}$.

Cor. 4. If the mass moved have no weight, but possesses inertia only, as when a body is moved along a horizontal plane, the force which accelerates A will be $= \frac{Pn^2}{Pn^2 + W}$. And either of these values may be readily introduced into the investigation.

Cor. 5. The work done in the time t, if we retain the original notation, will be $= \frac{RrP - r^2W}{R^2P + r^2W}gt \times W = \frac{RrPW - r^2W^2}{R^2P + r^2W} \cdot gt$.

Cor. 6. When the work done is to be a maximum, and we wish to know the weight when P is given, we must make the fluxion of the last expression $= 0$. Then we shall have $rR^3P^2 - 2r^2R^2PW - r^4W^2 = 0$ and $W = P \times [\sqrt{\frac{R^4}{r^4} + \frac{R^3}{r^3}} - \frac{R^2}{r^2}]$.

Cor. 7. If $R : r :: n : 1$, the preceding expression will become $W = P \times [\sqrt{(n^4+n^3)} - n^2]$.

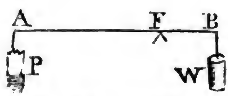
Cor. 8. When the arms of the lever are equal in length, that is, when $n = 1$, then is $W = P \times (\sqrt{2}-1) = \cdot 414214P$, or nearly $\frac{5}{12}$ of the moving force.

Scholium.

If we in like manner investigate the formula relating to motion on the axis in peritrochio, it will be seen that the expressions correspond exactly. Hence it follows, that when it is required to proportion the power and weight so as to obtain a maximum effect on the wheel and axle, (the weight of the machinery not being considered), we may adopt the conclusions of cors. 6 and 7 of this prop. And in the extreme case where the wheel and axle becomes a pulley, the expression in cor. 8 may be adopted. The like conclusions may be applied to machines in general, if R and r represent the distances of the impelled and working points from the axis of motion; and if the various kinds of resistance arising from friction, stiffness of ropes, &c., be properly reduced to their equivalents at the working points, so as to be comprehended in the character W for resistance overcome.

PROPOSITION II.

Given R *and r, the arms of a straight lever,* M *and m their respective weights, and* P *the power acting at the extremity of the arm* R; *to find the weight raised at the extremity of the other arm when the effect is a maximum.*



In this case $\frac{1}{2}m$ is the weight of the shorter end reduced to B, and conseq. $\frac{mr}{2R}$ is the weight which, applied at A, would balance the shorter end: therefore $\frac{mr}{2R} + \frac{r}{R}W$, would sustain both the shorter end and the weight W in equilibrio. But $P + \frac{1}{2}M$ is the power really acting at the longer end of the lever; consequently $P + \frac{1}{2}M - \frac{mr}{2R} + \frac{r}{R}W$, is the absolute *moving* power. Now the distance of the centre of gyration of the beam from F*

* The distance of R, the centre of gyration, from C the centre or axis of motion, in some of the most useful cases, is as below:

is $= \sqrt{\dfrac{R^3+r^3}{3(R+r)}}$, which let be denoted by ϱ; then (Dynam. art. 225) $\dfrac{\varrho^2}{R^2} \cdot (M+m)$ will represent the mass equivalent to the beam or lever when reduced to the point A; while the weight equivalent to W, when referred to that point, will be $\dfrac{r^2}{R^2}W$. Hence, proceeding as in the last prop. we shall have $\dfrac{\varrho^2}{R^2} \cdot (M+m) + P + \dfrac{r^2}{R^2}W$ for the inertia to be overcome; and $(P+\frac{1}{2}M-\dfrac{mr}{2R}-\dfrac{r}{R}W) \div \dfrac{\varrho^2}{R^2}(M+m)+P+\dfrac{r^2}{R^2}W$ = the accelerating force of P, or of W reduced to A. Multiply this by W; and, for the sake of simplifying the process, put q for $P+\frac{1}{2}M-\dfrac{rm}{2R}$, and n for $P+\dfrac{\varrho^2}{R^2}(M+m)$, then will $\dfrac{qW-\dfrac{rW^2}{R}}{n+\dfrac{r^2}{R^2}W}$ be a quantity which varies as the effect varies, and which, indeed, when multiplied by gt, denotes the effect itself. Putting the fluxion of this equal to nothing, and reducing, we at length find

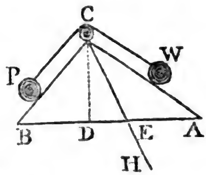
$$W = \frac{R}{r}\sqrt{\left(\frac{nqR}{r}+\frac{n^2R^2}{r^2}\right)}-\frac{nR^2}{r^2}.$$

Cor. When $R=r$, and $M=m$, if we restore the values of n and q, the expression will become $W=\sqrt{(2P^2+2mP+\frac{4}{5}m^2)}-(P+\frac{2}{3}m)$.

In a circular wheel of uniform thickness . . .	$CR = \text{rad.}\sqrt{\frac{1}{2}}$.
In the periphery of a circle revolving about the diam.	$CR = \text{rad.}\sqrt{\frac{1}{2}}$.
In the plane of a circle . . . ditto	$CR = \frac{1}{2}\text{rad}$.
In the surface of a sphere . . ditto	$CR = \text{rad}\sqrt{\frac{2}{3}}$.
In a solid sphere . . . . . . ditto	$CR = \text{rad}\sqrt{\frac{2}{5}}$.
In a plane ring formed of circles whose radii are R, r, revolving about centre	$CR = \sqrt{\dfrac{R^4-r^4}{2R^2-2r^2}}$
In a cone revolving about its vertex	$CR = \frac{1}{2}\sqrt{\frac{12}{5}a^2-\frac{3}{5}r^2}$.
In a cone . . . . . . its axis	$CR = r\sqrt{\frac{3}{10}}$.
In a straight lever whose arms are R and r . .	$CR = \sqrt{\dfrac{R^3+r^3}{3(R+r)}}$.

PROPOSITION III.

Given the length l and angle e of elevation of an inclined plane BC; *to find the length* L *of another inclined plane* AC, *along which a given weight* W *shall be raised from the horizontal line* AB *to the point* C, *in the least time possible, by means of another given weight* P *descending along the given plane* CB: *the two weights being connected by an inextensible thread* PCW *running always parallel to the two planes.*



Here we must, as a preliminary to the solution of this proposition, deduce expressions for the motion of bodies connected by a thread, and running upon double inclined planes. Let the angle of elevation CAD be E, while e is the elevation CBD. Then at the end of the time t, P will have a velocity v; and gravity would impress upon it, in the instant $\dot{t}$ following, a new velocity $= g \sin e \,.\, \dot{t}$, provided the weight P were then entirely free: but, by the disposition of the system, $\dot{v}$ will be the velocity which obtains in reality. Then, estimating the spaces in the direction CP, as the body W moves with an equal velocity but in a contrary sense, it is obvious that, by applying the 3d Law of Motion, the decomposition may be made as follows. At the end of the time $t+\dot{t}$ we have, for the velocity impressed on,

$$\text{P} \ldots v + g \sin \text{E} \,.\, \dot{t}, \text{ where } \begin{cases} v+\dot{v} \ldots\ldots \text{effective veloc. from C towards B.} \\ g \sin e \,.\, \dot{t} - \dot{v} \ldots\ldots \text{velocity destroyed.} \end{cases}$$

$$\text{W} \,.\, -v + g \sin \text{E} \,.\, \dot{t}, \text{ where } \begin{cases} -v-\dot{v} \ldots \text{effective veloc. from C towards A.} \\ \dot{v} + g \sin \text{E} \,.\, \dot{t}, \ldots\ldots \text{velocity destroyed.} \end{cases}$$

If, therefore, gravity impresses, during the time $\dot{t}$, upon the masses P, W, the respective velocities $g \sin e \,.\, \dot{t} - \dot{v}$, and $g \sin \text{E} \,.\, \dot{t} + \dot{v}$, the system will be in equilibrio. The quantities of motion being therefore equal, it will be

$$\text{P}g \sin e \,.\, \dot{t} - \text{P}\dot{v} = \text{W}g \sin \text{E} \,.\, \dot{t} + \text{W}\dot{v}.$$

Whence the effective accelerating force is found, i. e.

$$\varphi = \frac{\dot{v}}{\dot{t}} = \frac{\text{P} \sin e - \text{W} \sin \text{E}}{\text{P}+\text{W}} \times g.$$

Thus it appears that the motion is uniformly varied, and we readily find the equations for the velocity and space from which the conditions of the motion are determined, viz.

$$v = \frac{\text{P} \sin e - \text{W} \sin \text{E}}{\text{P} + \text{W}} \ldots s = \frac{\text{P} \sin e - \text{W} \sin \text{E}}{\text{P} + \text{W}} \cdot \tfrac{1}{2} g t^2.$$

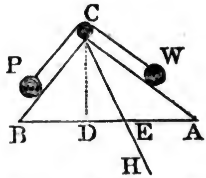
The latter of these two equations gives $t^2 =$
$\frac{s(\text{P}+\text{W})}{\frac{1}{2}g(\text{P} \sin e - \text{W} \sin e)}$. But in the triangle ABC it is AC : BC : : sin B : sin A, that is, L : l : : sin e : sin E ; hence $\frac{1}{m}\text{L} = \sin e$, and $\frac{1}{m}l = \sin \text{E}$; m being constant quantity always determinable from the data given. And t^2 becomes $\frac{s(\text{P}+\text{W})}{\frac{1}{2}g\frac{1}{m}(\text{PL} - \text{W}l)}$.

Now when any quantity, as t, is a minimum, its square is manifestly a minimum: so that substituting for s its equal L, and striking out the constant factors, we have $\frac{\text{L}^2}{\text{PL} - \text{W}l} =$ a min. or its fluxion $\frac{2\text{L}\dot{\text{L}}(\text{PL} - \text{W}l) - \text{PL}^2\dot{\text{L}}}{(\text{PL} - \text{W}l)^2} = 0$. Here, as in all similar cases, since the fraction vanishes, its numerator must be equal to 0; consequently $2\text{PL}^2 - 2\text{W}l\text{L} - \text{PL}^2 = 0$, $\text{PL} = 2\text{W}l$, or L : l : : 2W : P.

Cor. 1. Since neither sin e nor sin E enters the final equation, it follows, that if the elevation of the plane BC is not given, the problem is unlimited.

Cor. 2. When sin $e = 1$, BC coincides with the perpendicular CD, and the power P acts with all its intensity upon the weight W. This is the case of the present problem which has commonly been considered.

Scholium.



This proposition admits of a neat *geometrical* demonstration. Thus, let CE be the plane upon which, if W were placed, it would be sustained in equilibrio by the power P on the plane CB, or the power P′ hanging freely in the vertical CD; then (Mechan. prop. 193) BC : CD : CE : : P : P′ : W. But W is to the force with which it tends to descend along the plane CA, as CA to CD; consequently, the weight P′ is to that force, as CA : CE ; or the weight P on the plane BC, is to

the same force in the same ratio; because either of these weights in their respective positions would sustain W on CE. Therefore the excess of P above that force (which excess is the power accelerating the motions of P and W) is to P, as CA — CE to CA; or, taking CH = CA, as EH to CA. Now, the motion being uniformly accelerated, we have $S \propto FT^2$, or $T^2 \propto \frac{S}{F}$: consequently, the square of the time in which AC is described by W, will be as AC directly, and as $\frac{EH}{AC}$ inversely; and will be least when $\frac{CA^2}{EH}$ is a *minimum*; that is when $\frac{CE^2}{EH} + EH + 2CE$, or (because 2CE is invariable) when $\frac{CE^2}{EH} + EH$ is a minimum. Now, as, when the sum of two quantities is given, their product is a *maximum* when they are equal to each other; so it is manifest that when their product is given, their sum must be a *minimum* when they are equal. But the product of $\frac{CE^2}{EH}$ and EH is CE^2, and consequently given; therefore the sum of $\frac{EC^2}{EH}$ and EH is least, when those parts are equal; that is, when EH = CE, or CA = 2CE. So that the length of the plane CA is double the length of that on which the weight W would be kept in equilibrio by P acting along CB.

When CD and CB coincide, the case becomes the same as that considered by Maclaurin, in his *View of Newton's Philosophical Discoveries*, pa. 183, 8vo. edit.

PROPOSITION IV.

Let the given weight P *descend along* CB, *and by means of the thread* PCW (*running parallel to the planes*) *draw a weight* W *up the plane* AC: *it is required to find the value of* W, *when its momentum is a maximum, the lengths and positions of the planes being given.* (*See the preceding fig.*).

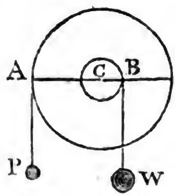
The general expression for the vel. is $v = \frac{P \cdot \sin c - W \sin E}{P + W} gt$, which, by substitut. $\frac{1}{m} L$ for sin c, and $\frac{1}{m} l$ for sin E, becomes

$v = \dfrac{\frac{1}{m}(\text{PL}-\text{W}l)}{\text{P}+\text{W}}gt$. This mul. into W, gives $\dfrac{\frac{1}{m}(\text{PWL}-\text{W}^2l)}{\text{P}+\text{W}}gt$; which, by the prop. is to be a maximum. Or, striking out the constant factors, $\frac{1}{m}$, gt, then is $\dfrac{\text{PWL}-\text{W}^2l}{\text{P}+\text{W}} =$ a max. Putting this into fluxions, and reducing, we have $\text{P}^2\text{L} - 2\text{PW}l - \text{W}^2l = 0$, or $\text{W} = \text{P}\sqrt{(\frac{\text{L}}{l}+1)} - \text{P}$.

Cor. When the inclinations of the planes are equal, L and l are equal, and $\text{W} = \text{P}\sqrt{2} - \text{P} = \text{P} \times (\sqrt{2}-1) = \cdot 4142\text{P}$; agreeing with the conclusion of the lever of equal arms, or the extreme case of the wheel and axle, i. e. the pulley.

PROPOSITION V.

Given the radius R *of a wheel, and the radius r, of its axle, the weight of both, w, and the distance of the centre of gyration from the axis of motion, ϱ; also a given power* P *acting at the circumference of the wheel; to find the weight* W *raised by a cord folding about the axle, so that its momentum shall be a maximum.*



The force which absolutely impels the point A is P, while W acts in a direction contrary to P, with a force $= \frac{r\text{W}}{\text{R}}$; this therefore subducted from P, leaves $\text{P} - \frac{r\text{W}}{\text{R}} = \frac{\text{RP}-r\text{W}}{\text{R}}$, for the reduced force impelling the point A. And the inertia which resists the communication of motion to the point A will be the same as if the mass $\dfrac{\varrho^2w+r^2\text{W}+\text{R}^2\text{P}}{\text{R}^2}$ were concentrated in the point A. (Dynam. art. 225). If the former of these be divided by the latter, the quotient $\dfrac{\text{R}(\text{RP}-\text{R}^2\text{W})}{\varrho^2w+r^2\text{W}+\text{R}^2\text{P}}$ is the force accelerating A: multiplying this by $\frac{r}{\text{R}}$, we have $\dfrac{\text{R}r\text{P}-r^2\text{W}}{\varrho^2\text{W}+r^2\text{W}+\text{R}^2\text{P}}$ for the force which accelerates the weight W in its ascent. Consequently the velocity of W will be $= \dfrac{\text{R}r\text{P}-r^2\text{W}}{\varrho^2w+r^2\text{W}+\text{R}^2\text{P}}gt$; which multiplied into W gives $\dfrac{\text{R}r\text{PW}-r^2\text{W}^2}{\varrho^2w+r^2\text{W}+\text{R}^2\text{P}}gt$ for the momentum. As

this is to be a maximum, its fluxion will $= 0$; whence we shall obtain

$$w = \frac{\sqrt{(R^4P^2 + 2R^2P\rho^2 w + \rho^4 w^2 + PwRr\rho^2 + P^2R^2r^2)} - R^2P - \rho^2 w}{r^2}.$$

Cor. 1. When $R = r$, as in the case of the single fixed pulley, then $w = \sqrt{(2P^2R^3 + 2RP\rho^2 w + \frac{\rho^4}{R} w^2 + PwR\rho^2)} - \frac{\rho^2}{R^2} w - P$.

Cor. 2. When the pulley is a cylinder of uniform matter $\rho^2 = \frac{1}{2}R^2$, and the express. becomes $w = \sqrt{[R^2(2P^2 + \frac{3}{2}Pw + \frac{1}{4}w^2)]} - \frac{1}{2}w - P$.

Cor. 3. If, in the first general expression for the momentum of w, Q be put $= R^2P + \rho^2 w$, we shall have $\frac{RrPw - r^2w^2}{Q + r^2w}$ = a maximum. Which, in fluxions and reduced, gives $w = \frac{1}{r^2}\sqrt{[Q \cdot (Q + RrP)]} - \frac{1}{r^2}Q$.

Cor. 4. If the moving force be destitute of inertia, then will $Q = \rho^2 w$, and w, as in the last corollary.

PROPOSITION VI.

Let a given power P *be applied to the circumference of a wheel, its radius* R, *to raise a weight* W *at its axle, whose radius is r, it is required to find the ratio of* R *and r when* W *is raised with the greatest momentum; the characters* w *and* ρ *denoting the same as in the last proposition.*

Here we suppose r to vary in the expression for the momentum of W, $\frac{WRrP - r^2W^2}{\rho^2 w + r^2W + R^2P} gt$. And we suppose, that by the conditions of any specified instance, we can ascertain what quantity of matter q shall make $r^2 q = \rho^2 w$, which, in fact, may always be done as soon as we can determine ρ. The expression for the work will then become $\frac{RrPW - r^2W^2}{R^2P + r^2(q + W)} gt$. The fluxion of which being made $= 0$, gives, after a little reduction, $r = \frac{R\sqrt{[P^2W^2 + P^3(q + W)]} - PW}{P(q + W)}$.

Cor. When the inertia of the machine is evanescent, with respect to that of $P + W$, then is $r = R\sqrt{(1 + \frac{P}{W})} - 1$.

PROPOSITION VII.

In any machine whose motion accelerates, the weight will be moved with the greatest velocity, when the velocity of the power is to that of the weight, as $1+\text{P}\surd(1+\frac{\text{P}}{\text{W}})$ *to* 1; *the inertia of the machine being disregarded.*

For any such machine may be considered as reduced to a lever, or to a wheel and axle whose radii are R and r: in which the velocity of the weight $\frac{\text{R}r\text{P}-r^2\text{W}}{\text{R}^2\text{P}+r^2\text{W}}gt$ (prop. 1) is to be a maximum, r being considered as variable. Hence then, following the usual rules, we find $\text{PR}=r[\text{W}+\surd(\text{W}^2+\text{PW})]$. From which, since the velocities of the power and weight are respectively as R and r, the ratio in the proposition immediately flows.

Cor. *When the weight moved is equal to the power, then is* $\text{R}:r::1+\surd2:1::2\cdot4142:1$ *nearly.*

PROPOSITION VIII.

If in any machine whose motion accelerates, the descent of one weight causes another to ascend, and the descending weight be given, the operation being supposed continually repeated, the effect will be greatest in a given time when the ascending weight is to the descending weight as 1 *to* 1·618, *in the case of equal heights; and in other cases, when it is to the exact counterpoise in a ratio which is always between* 1 *to* $1\frac{1}{2}$ *and* 1 *to* 2.

Let the space descended be 1, that ascended s; the descending weight 1, the ascending weight $\frac{1}{w}$: then would the equilibrium require $w=s$; and $1-\frac{s}{w}$ will be the force acting on 1. Now the mass $\frac{1}{w}$, reduced to the point at which the mass 1 acts, will be $=\frac{1}{w}s^2=\frac{s^2}{w}$; consequently the whole mass moved is equivalent to $1+\frac{s^2}{w}$, and the relative force is $(1-\frac{s}{w})\div(1+\frac{s^2}{w})=\frac{w-s}{w+s^2}$. But, the space be-

ing given, the time is as the root of the accelerating force inversely, that is, as $\sqrt{\frac{w+s^2}{w-s}}$: and the whole effect in a given time, being directly as the weight raised, and inversely as the time of ascent, will be as $\frac{1}{w}\sqrt{\frac{w-s}{w+s^2}}$; which must be a maximum. Consequently its square $\frac{w-s}{w^3+s^2w^2}$ must be a max. likewise. This latter expression, in fluxions and reduced, gives $w = \frac{s}{4}[\sqrt{(s^2+10s+9)} - a + 3]$.

Here if $s = 1$, $w = \frac{1+\sqrt{5}}{2}$: but if s be diminished without limit, $w = \frac{3}{2}s$; if it be augmented without limit, then will $\sqrt{(s^2+10s+9)}$ approach indefinitely near to $s+5$, and consequently $w = 2s$. Whence the truth of the proposition is manifest. See Dr. Gregory's Mechanics, or Dr. Young's Philosophy.

PROPOSITION IX.

Let φ denote the absolute effort of any moving force, when it has no velocity; and suppose it not capable of any effort when the velocity is W; *let* F *be the effort answering to the velocity* V; *then, if the force be uniform,* F *will be* $= \varphi(1-\frac{V}{W})^2$.

For it is the difference between the velocities W and V which is efficient, and the action, being constant, will vary as the square of the efficient velocity. Hence we shall have this analogy, $\varphi : F :: (W-0)^2 : (W-V)^2$: consequently, $F = \varphi(\frac{W-V}{W})^2 = \varphi\,(1-\frac{V}{W})^2$.

Though the pressure of an animal is not actually uniform during the whole time of its action, yet it is nearly so; so that in general we may adopt this hypothesis in order to approximate to the true nature of animal action. On which supposition the preceding prop. as well as the following one will apply to animal exertion.

Cor. Retaining the same notation, we have $W = \frac{V\sqrt{\varphi}}{\sqrt{\varphi}-\sqrt{F}}$. This, applied to the motion of animals, gives this theorem: *The utmost velocity with which an animal not impeded can*

move, is to the velocity with which it moves when impeded by a given resistance, as the square root of its absolute force to the difference of the square roots of its absolute and efficient forces.

PROPOSITION X.

To investigate expressions by means of which the maximum effect, in machines whose motion is uniform, may be determined.

I. It follows, from the observations made in art. 1, and the definitions in this chapter, that when a machine, whether simple or compound, is put into motion, the velocities of the impelled and working points, are inversely as the forces which are in equilibrio, when applied to those points in the direction of their motion. Consequently, if f denote the resistance when reduced to the working point, and v its velocity; while F and V denote the force acting at the impelled point, and its velocity; we shall have $FV = fv$, or introducing t the time, $FVt = fvt$. Hence, *in all working machines which have acquired an uniform motion, the performance of the machine is equal to the momentum of impulse.*

II. Let F be the effort of a force on the impelled point of a machine when it moves with the velocity V, the velocity being W when $F = 0$, and let the relative velocity $W - V = u$. Then since (prop. IX.) $F = \varphi\left(\frac{W-V}{W}\right)^2$, the momentum of impulse FV will become $V\varphi\left(\frac{u}{W}\right)^2 = \varphi \cdot \frac{u^2}{W^2}(W-u)$; because $V = W-u$. Making this expression for FV a maximum, or suppressing the constant quantities, and making $u^2(W-u)$ a max. or its flux. $= 0$, when u is variable, we find $2W = 3u$, or $u = \frac{2}{3}W$. Whence $V = W - u = W - \frac{2}{3}W = \frac{1}{3}W$.

Consequently, *when the ratio of* V *to* v *is given by the construction of the machine, and the resistance is susceptible of variation, we must load the machine more or less till the velocity of the impelled point is one-third of the greatest velocity of the force; then will the work done be a maximum.*

Or, the work done by an animal is greatest, when the velocity with which it moves is one-third of the greatest velocity with which it is capable of moving when not impeded.

III. Since $F = \varphi\frac{u^2}{W^2} = \varphi\left(\frac{\frac{4}{9}W^2}{W^2}\right) = \frac{4}{9}\varphi$, in the case of the maximum, we have $FV = \frac{4}{9}\varphi V = \frac{4}{9}\varphi \cdot \frac{1}{3}W = \frac{4}{27}\varphi W$, for the

momentum of impulse, or for the work done, when the machine is in its best state. *Consequently, when the resistance is a given quantity, we must make* V : v : : $9f$: 4φ ; *and this structure of the machine will give the maximum effect* = $\frac{4}{27}\varphi$W.

IV. If we inquire the greatest effect on the supposition that φ only is variable, we must make it infinite in the above expression for the work done, which would then become

WF, or W $\frac{V}{v}$ f, or W$\frac{V}{v}ft$, including the time in the formula.

Hence we see, *that the sum of the agents employed to move a machine may be infinite, while the effect is finite :* for the variations of φ, which are proportional to this sum, do not influence the above expression for the effect.

Scholium.

The propositions now delivered contain the most material principles in the theory of machines. The manner of applying several of them is very obvious : the application of some, being less manifest, may be briefly illustrated, and the chapter concluded with two or three observations.

The last theorem may be applied to the actions of men and of horses, with more accuracy than might at first be supposed. Observations have been made on men and horses drawing a lighter along a canal, and working several days together. The force exerted was measured by the curvature and weight of the track-rope, and afterwards by a spring steelyard. The product of the force thus ascertained, into the velocity per hour, was considered as the momentum. In this way the action of *men* was found to be very nearly as $(\text{W}—\text{V})^2$: the action of horses loaded so as not to be able to trot was nearly as $(\text{W}-\text{V})^{1\cdot 7}$, or as $(\text{W}-\text{V})^{\frac{9}{5}}$. Hence the hypothesis we have adopted may in many cases be safely assumed.

According to the best observations, the force of a man at rest is on the average about 70 pounds ; and the utmost velocity with which he can walk is about 6 feet per second, taken at a medium. Hence, in our theorems, $\varphi = 70$, and W = 6. Consequently F$=\frac{4}{9}\varphi = 31\frac{1}{9}$lbs., the greatest force a man can exert when in motion : and he will then move at the rate of $\frac{1}{3}$W, or 2 feet per second, or rather less than a mile and a half per hour.

The strength of a horse is generally reckoned about 6 times that of a man ; that is, nearly 420lbs., at a dead pull. His utmost walking velocity is about 10 feet per second. There-

fore his maximum action will be $\frac{4}{9}$ of $420=186\frac{2}{3}$lbs., and he will then move at the rate of $\frac{1}{3}$ of 10, or $3\frac{1}{3}$ feet, per second, or nearly $2\frac{1}{3}$ miles per hour. In both these instances we suppose the force to be exerted in drawing a weight along a horizontal plane; or by raising a weight by a cord running over a pulley, which makes its direction horizontal *.

2. The theorems just given may serve to show, in what points of view machines ought to be considered by those who would labour beneficially for their improvement.

The first object of the utility of machines consists in furnishing the means of *giving to the moving force the most commodious direction*; and, when it can be done, of causing its action to be applied immediately to the body to be moved. These can rarely be united: but the former can be accomplished in most instances; of which the use of the simple lever, pulley, and wheel and axle, furnish many examples. The second object gained by the use of machines, is *an accommodation of the velocity of the work to be performed, to the velocity with which alone a natural power can act*. Thus, whenever the natural power acts with a certain velocity which cannot be changed, and the work must be performed with a greater velocity, a machine is interposed moveable round a fixed support, and the distances of the impelled and working points are taken in the proportion of the two given velocities.

But the essential advantage of machines, that, in fact, which properly appertains to the *theory* of mechanics, consists in augmenting, or rather in modifying, the energy of the moving power, in such manner that it may produce effects of which it would have been otherwise incapable. Thus a man might carry up a flight of steps 20 pieces of stone, each weighing 30 pounds (one by one) in as small a time as he could (with the same labour) raise them altogether by a piece of machinery, that would have the velocities of the impelled and working points as 20 to 1; and, in this case, the instrument would furnish no real advantage, except that of saving his steps. But if a large block of 20 times 30, or 600 lbs. weight, were to be raised to the same height, it would far surpass the utmost efforts of the man, without the intervention of some such contrivance.

The same purpose may be illustrated somewhat differently; confining the attention all along to machines whose motion is uniform. The product fv represents, during the unit of

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* See, for more on this subject, *Mr. Tredgold's Treatise on Rail-roads*, and *Gregory's Mathematics for Practical Men*, pp. 369—385.

time, the effect which results from the motion of the resistance; this motion being produced in any manner whatever. If it be produced by applying the moving force immediately to the resistance, it is necessary not only that the products FV and fv should be equal; but that at the same time $F = f$, and $V = v$: if, therefore, as most frequently happens, f be greater than F, it will be absolutely impossible to put the resistance in motion by applying the moving force immediately to it. Now machines furnish the means of disposing the product FV in such a manner that it may always be equal to fv, however much the factors of FV may differ from the analogous factors in fv; and, consequently, of putting the system in motion, whatever is the excess of f over F.

Or, generally, as M. Prony remarks (Archi. Hydraul. art. 504), machines enable us to dispose the factors of FVt in such a manner, that while that product continues the same, its factors may have to each other any ratio we desire. If, for instance, time be precious, the effect must be produced in a very short time, and yet we should have at command a force capable of little velocity but of great effort, a machine must be found to supply the velocity necessary for the intensity of the force: if, on the contrary, the mechanist has only a weak power at his disposition, but capable of a great velocity, a machine must be adopted that will compensate, by the velocity the agent can communicate to it, for the force wanted: lastly, if the agent is capable neither of great effort, nor of great velocity, a convenient machine may still enable him to accomplish the effect desired, and make the product FVt of force, velocity, and time, as great as is requisite. Thus, to give another example: Suppose that a man, exerting his strength immediately on a mass of 25 lbs., can raise it vertically with a velocity of 4 feet per second; the same man acting on a mass of 1000lbs., cannot give it any vertical motion though he exerts his utmost strength, unless he has recourse to some machine. Now he is capable of producing an effect equal to $25 \times 4 \times t$: the letter t being introduced because, if the labour is continued, the value of t will not be indefinite, but comprised within assignable limits. Thus we have $25 \times 4 \times t = 1000 \times v \times t$; and consequently $v = \frac{1}{10}$ of a foot. This man may therefore with a machine, as a lever, or axis in peritrochio, cause a mass of 1000lbs, to raise $\frac{1}{10}$ of a foot, in the same time that he could raise 25lbs. 4 feet without a machine; or he may raise the greater weight as far as the less, by employing 40 times as much time.

From what has been said on the extent of the effects which may be attained by machines, it will be seen that, so long as a moving force exercises a determinate effort, with a velocity

also determinate, or so long as the product of these is constant, the effect of the machine will remain the same : thus, under this point of view, supposing the preponderance of the effort of the moving power, and abstracting from inertia and friction of materials, the convenience of application, &c., all machines are equally perfect. But, from what has been shown, (props. 9, 10) a moving force may, by diminishing its velocity, augment its effort, and reciprocally. There is therefore a certain effort of the moving force, such that its product by the velocity which comports to that effort, is the greatest possible. Admitting the truth of the law assumed in the propositions just referred to, we have, when the effect is a *maximum*, $V = \frac{1}{3}W$, or $F = \frac{4}{9}\varphi$; and these two values obtaining together, their product $\frac{4}{27}\varphi W$ expresses the value of the greatest effect with respect to the unit of time. In practice it will always be advisable to approach as nearly to these values as circumstances will admit ; for it cannot be expected that they can always be exactly attained. But a small variation will not be of much consequence : for, by a well known property of those quantities which admit of a proper maximum and minimum, a value assumed at a moderate distance from either of these extremes will produce no sensible change in the effect.

If the relation of F to V followed any other law than that which we have assumed, we should find from the expression of *that law* values of F, V, &c., different from the preceding. The general method however would be nearly the same.

With respect to practice, the grand object in all cases should be to procure an *uniform motion*, because it is that from which (*cæteris paribus*) the greatest effect always results. Every irregularity in the motion wastes some of the impelling power ; and it is the greatest only of the varying velocities which is equal to that which the machine would acquire if it moved uniformly throughout : for, while the motion accelerates, the impelling force is greater than what balances the resistance at that time opposed to it, and the velocity is less than what the machine would acquire if moving uniformly ; and when the machine attains its greatest velocity, it attains it because the power is not then acting against the whole resistance. In both these situations, therefore, the performance of the machine is less than if the power and resistance were exactly balanced ; in which case it would move uniformly (art. 1). Besides this, when the motion of a machine, and particularly a very ponderous one, is irregular, there are continued repetitions of strains, and jolts which soon derange and ultimately destroy the whole structure. Every attention should therefore be paid to the removal of all causes of irregularity.

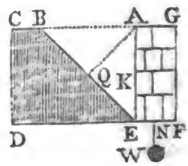
PRESSURE OF EARTH AND FLUIDS AGAINST WALLS AND FORTIFICATIONS, THEORY OF MAGAZINES, &c.

PROBLEM I.

To determine the pressure of earth against walls.

When new-made earth, such as is used in forming ramparts, &c.; is not supported by a wall as a facing, or by counterforts and land-ties, &c., but left to the action of its weight and the weather; the particles loosen and separate from each other, and form a sloping surface, nearly regular; which plane surface is called the natural slope of the earth; and is supposed to have always the same inclination or deviation from the perpendicular, in the same kind of soil. In common earth or mould, being a mixture of all sorts thrown together, the natural slope is commonly at about half a right angle, or 45 degrees; but clay and stiff loam stand at a greater angle above the horizon, while sand and light mould will only stand at a much less angle. The engineer or builder must therefore adopt his calculations accordingly.—It may be observed that the triangle of earth, supposed to act against the wall, is considered as a rigid solid, to simplify the problem, and obtain an outline of a practical near solution, for the purpose of teaching, in the absence of good experiments.—But for an essay on the theory of the pressure of soft or semifluid earth by Dr. T. Young, see Hutton's Dictionary, 2nd edit. vol. 2, page 229.

Now, we have already given, at page 386, &c. the general theory and determination of the force with which the triangle of the earth (which would slip down if not supported) presses against the wall. But it is often found a convenient approximation, to conceive the triangle of earth acting perpendicularly against AE at K, or $\frac{1}{3}$ of the altitude AE above the foundation at E; the expression for which force is found to be $\frac{AE^3 \cdot AB^2}{6BE^2}m$; where m denotes the specific gravity of the earth of the triangle ABE.—It may be remarked that this is deduced from using the area only of the profile, or transverse triangular section ABE, instead of the prismatic solid of any given length,



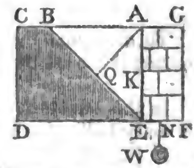
having that triangle for its base. And the same thing is done in determining the power of the wall to support the earth, viz. using only its profile or transverse section in the same plane or direction as the triangle ABE. This it is evident will produce the same result as the solids themselves, since, being both of the same given length, these have the same ratio as their transverse sections.

In addition to this determination, we may here further observe, that this pressure ought to be diminished in proportion to the cohesion of the matter in sliding down the inclined plane BE. Now it has been found by experiments, that a body requires about one-third of its weight to move it along a plane surface. The above expression must therefore be reduced in the ratio of 3 to 2; by which means it becomes $\frac{AE^3 \cdot AB^2}{9BE^2}m$ for the true practical efficacious pressure of the earth against the wall.

Since $\frac{AB}{BE}$, which occurs in this expression of the force of the earth, is equal to the sine of the $\angle AEB$ to the radius 1, put the sine of that $\angle E = e$; also put $a =$ AE the altitude of the triangle; then the above expression of the force, viz. $\frac{AE^3 \cdot AB^2}{9BE^2}m$, becomes $\frac{1}{9}a^3e^2m$, for the perpendicular pressure of the earth against the wall. And if that angle be 45°, as is usually the case in common earth, then is $e^2 = \frac{1}{2}$, and the pressure becomes $\frac{1}{18}a^3m$.

PROBLEM II.

To determine the thickness of wall to support the earth.



In the first place suppose the section of the wall to be a rectangle, or equally thick at top and bottom, and of the same height as the rampart of earth, like AEFG in the annexed figure. Conceive the weight W, proportional to the area GE, to be appended to the base directly below the centre of gravity of the figure. Now the pressure of the earth determined in the first problem, being in a direction parallel to AG, to cause the wall to overset and turn back about the point f, the effort of the wall to oppose that effect, will be the weight W drawn into FN the length of the lever by

which it acts, that is, W $\times$ FN or AEFG $\times$ FN in general, whatever be the figure of the wall.

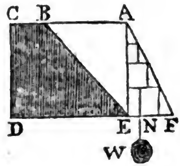
But now in case of the rectangular figure, the area GE $=$ AE $\times$ EF $= ax$, putting $a =$ AE the altitude as before, and $x =$ EF the required thickness; also in this case FN $= \frac{1}{2}$EF $= \frac{1}{2}x$, the centre of gravity being in the middle of the rectangle. Hence then $ax \times \frac{1}{2}x = \frac{1}{2}ax^2$, or rather $\frac{1}{2}ax^2n$, is the effort of the wall to prevent its being overturned, n denoting the specific gravity of the wall.

Now to make this effort a due balance to the pressure of the earth, we put the two opposing forces equal, that is $\frac{1}{2}ax^2n = \frac{1}{6}a^3e^2m$, or $\frac{1}{2}x^2n = \frac{1}{6}a^2e^2m$, an equation which gives $x = \frac{1}{3}ae\sqrt{\frac{2m}{n}}$, for the requisite thickness of the wall, just to sustain it in equilibrio.

Corol. 1. The factor ae, in this expression, is $=$ the line AQ drawn perp. to the slope of earth BE: theref. the breadth x becomes $= \frac{1}{3}$AQ $\sqrt{\frac{2m}{n}}$, which conseq. is directly proportional to the perp. AQ.—When the angle at E is $= 45°$, or half a right angle, as is commonly the case, its sine e is $= \sqrt{\frac{1}{2}}$, and the breadth of the wall $x = \frac{1}{3}a\sqrt{\frac{m}{n}}$. Further, when the wall is of brick, its specific gravity is nearly the same as the earth, or $m = n$, and then its thickness $x = \frac{1}{3}a$, or one-third of its height.—But when the wall is of stone, of the specific gravity $2\frac{1}{2}$, that of earth being nearly 2, that is, $m = 2$, and $n = 2\frac{1}{2}$; then $\sqrt{\frac{m}{n}} = \sqrt{\frac{4}{5}} = \cdot 895$, $\frac{1}{3}$ of which is $\cdot 298$. and the breadth $x = \cdot 298a = \frac{3}{10}a$ nearly. That is, the thickness of the stone wall must be $\frac{3}{10}$ of its height.

PROBLEM III.

To determine the thickness of the wall at the bottom, when its section is a triangle, or coming to an edge at top.



In this case, the area of the wall AEF is only half of what it was before, or only $\frac{1}{2}$AE $\times$ EF $= \frac{1}{2}ax$, and the weight W $= \frac{1}{2}axn$. But now, the centre of gravity is at only $\frac{1}{3}$ of FE from the line AE, or FN $= \frac{2}{3}$FE $= \frac{2}{3}x$. Consequently FN $\times$ W $= \frac{2}{3}x \times \frac{1}{2}axn = \frac{1}{3}ax^2n$. This, as before, being put $=$ the pressure of

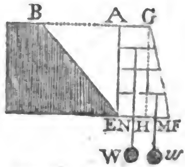
the earth, gives the equation $\frac{1}{3}ax^2n = \frac{1}{6}a^3e^2m$, or $x^2n = \frac{1}{3}a^2e^2m$, and the root x, or thickness EF $= ae\sqrt{\frac{m}{3n}} = a\sqrt{\frac{m}{6n}}$ for the slope of 45°.

Now when the wall is of brick, or $m = n$ nearly, this becomes $x = a\sqrt{\frac{1}{6}} = \cdot 408a = \frac{2}{5}a$, or $\frac{4}{10}$ of the height nearly.

But when the wall is of stone, or m to n as 2 to $2\frac{1}{2}$, then $\sqrt{\frac{m}{n}} = \sqrt{\frac{4}{5}}$, and the thickness x or $a\sqrt{\frac{m}{6n}} = a\sqrt{\frac{2}{15}} = \cdot 365a = \frac{3}{8}a$ nearly, or nearly $\frac{3}{8}$ of the height.

PROBLEM IV.

To determine the thickness of the wall at the top, when the face is not perpendicular, but inclined as the front of a fortification wall usually is.



Here GF represents the outer face of a fort, AEFG the profile of the wall, having AG the thickness at top, and EF that at the bottom. Draw GH perp. to EF; and conceive the two weights W, w, to be suspended from the centres of gravity of the rectangle AH and the triangle GHF, and to be proportional to their areas respectively. Then the two momenta of the weights W, w, acting by the levers FN, FM, must be made equal to the pressure of the earth in the direction perp. to AE.

Now put the required thickness AG or EH $= x$, and the altitude AE or GH $= a$ as before. And because in such cases the slope of the wall is usually made equal to $\frac{1}{5}$ of its altitude, that is, FH $= \frac{1}{5}$AE or $\frac{1}{5}a$, the lever FM will be $\frac{2}{3}$ of $\frac{1}{5}a = \frac{2}{15}a$, and the lever FN $=$ FH $+ \frac{1}{2}$EH $= \frac{1}{5}a + \frac{1}{2}x$. But the area of GHF $=$ GH $\times \frac{1}{2}$HF $= a \times \frac{1}{10}a = \frac{1}{10}a^2 = w$, and the area AH $=$ AE $\times$ AG $= ax =$ W; these two drawn into the respective levers FM, FN, give the two momenta, $\frac{2}{15}aw = \frac{2}{15}a \times \frac{1}{10}a^2 = \frac{1}{75}a^3$, and $(\frac{1}{5}a + \frac{1}{2}x) \times ax = \frac{1}{5}a^2x + \frac{1}{2}ax^2$; theref. the sum of the two, $(\frac{1}{2}ax^2 + \frac{1}{5}a^2x + \frac{2}{75}a^3)n$ must be $= \frac{1}{18}a^3m$, or dividing by $\frac{1}{2}an$, $x^2 + \frac{2}{5}ax + \frac{2}{75}a^2 = \frac{1}{9}a^2 \times \frac{m}{n}$; now adding $\frac{1}{75}a^2$ to both sides to complete the square, the equation becomes $x^2 + \frac{2}{5}ax + \frac{1}{25}a^2 = \frac{1}{9}a^2 \cdot \frac{m}{n} + \frac{1}{25}a^2$, the root of which

is $x+\frac{1}{5}a=a\surd(\frac{1}{25}+\frac{m}{9n})$, and hence $x=a\surd(\frac{1}{25}+\frac{m}{9n})-\frac{1}{5}a$. And the base EF $=a\surd(\frac{1}{25}+\frac{m}{9n})$.

Now, for a brick wall, $m=n$ nearly, and then the breadth $x=a\surd(\frac{1}{25}+\frac{1}{9})-\frac{1}{5}a=\frac{1}{15}a\surd34-\frac{1}{5}a=\cdot189a$, or almost $\frac{1}{5}a$ in brick walls.—But in stone walls, $\frac{m}{n}=\frac{4}{5}$, and $x=a\surd(\frac{1}{25}+\frac{4}{45})-\frac{1}{5}a=\frac{1}{15}a\surd29-\frac{1}{5}a=\cdot159a=\frac{4}{25}a$ nearly, for the thickness AG at top, in stone walls.

In the same manner we may proceed when the slope is supposed to be any other part of the altitude, instead of $\frac{1}{5}$ as used above. Or a general solution might be given, by assuming the thickness $=\frac{1}{c}$ part of the altitude.

REMARK.

Thus then we have given all the calculations that may be necessary in determining the thickness of a wall, proper to support the rampart or body of earth, in any work. If it should be objected, that our determination gives only such a thickness of wall, as makes it an exact mechanical balance to the pressure or push of the earth, instead of giving the former a decided preponderance over the latter, as a security against any failure or accidents: To this we answer, that what has been done is sufficient to insure stability, for the following reasons and circumstances. First, it is usual to build several counterforts of masonry, behind and against the wall, at certain distances or intervals from one another; which contribute very much to strengthen the wall, and to resist the pressure of the rampart. 2dly. We have omitted to include the effect of the parapet raised above the wall; which must add somewhat, by its weight, to the force or resistance of the wall. It is true we could have brought these two auxiliaries to exact calculation, as easily as we have done for the wall itself: but we have thought it as well to leave these two appendages, thrown in as indeterminate additions, above the exact balance of the wall as before determined, to give it an assured stability. Besides these advantages in the wall itself, certain contrivances are also usually employed to diminish the pressure of the earth against it: such as land-ties and branches, laid in the earth, to diminish its force and push against the wall. For all these reasons then, we think the practice of making the walls of the thickness assigned by

this theory, may be safely depended on, and profitably adopted; as the additional circumstances, just mentioned, will sufficiently insure stability; and its expense will be less than is incurred by any former theory.

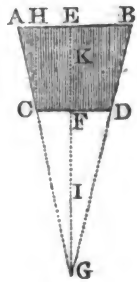
PROBLEM V.

To determine the quantity of pressure sustained by a dam or sluice, made to pen up a body of water.

By art. 247, Hydrostatics, page 248, the pressure of a fluid against any upright surface, as the gate of a sluice or canal, is equal to half the weight of a column of the fluid, whose base is equal to the surface pressed, and its altitude the same as that of the surface. Or, by art. 249 of the same, the pressure is equal to the weight of a column of the fluid, whose base is equal to the surface pressed, and its altitude equal to the depth of the centre of gravity below the top or surface of the water; which comes to the same thing as the former article, when the surface pressed is a rectangle, because its centre of gravity is at half the depth.

Ex. 1. Suppose the dam or sluice be a rectangle, whose length, or breadth of the canal, is 20 feet, and the depth of water 6 feet. Here $20 \times 6 = 120$ feet, is the area of the surface pressed; and the depth of the centre of gravity being 3 feet, viz. at the middle of the rectangle; therefore $120 \times 3 = 360$ cubic feet is the content of the column of water. But each cubic foot of water weighs 1000 ounces, or $62\frac{1}{2}$ pounds; therefore $360 \times 1000 = 360000$ ounces, or 22500 pounds, or 10 tons and 100lb. is the weight of the column of water, or the quantity of pressure on the gate or dam.

Ex. 2. Suppose the breadth of a canal at the top, or surface of the water, to be 24 feet, but at the bottom only 16 feet, the depth of water being 6 feet, as in the last example: required the pressure on a gate which, standing across the canal, dams the water up?



Here the gate is in form of a trapezoid, having the two parallel sides AB, CD, viz. $AB = 24$, and $CD = 16$, and depth 6 feet. Now, by mensuration, problem 3, vol. 1, $\frac{1}{2}(AB+CD) \times 6 = 20 \times 6 = 120$ the area of the sluice, the same as before in the 1st example: but the centre of gravity cannot be so low down as before, because the figure is wider above and narrower below, the whole depth being the same.

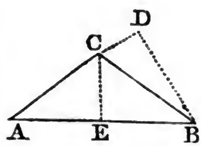
Now, to determine the centre of gravity K of the trapezoid AD, produce the two

sides AC, BD, till they meet in G; also draw GKE and CH perp. to AB: then AH : CH :: AE : GE, that is, 4 : 6 :: 12 : 18 = GE; and EF being = 6, theref. FG = 12. Now, by Statics, art. 111, EF = 6 = $\frac{1}{3}$EG gives F the centre of gravity of the triangle ABG, and FI = 4 = $\frac{1}{3}$FG gives I the centre of gravity of the triangle CDG. Then assuming K to denote the centre of AD, it will be, by art. 96, as the trap. AD : △ CDG :: IF : FK, or △ ABC − △ CDG : △ CDG :: IF : FK, or by theor. 88, Geom. $GE^2 - GF^2 : GF^2 :: IF : FK$, that is $18^2 - 12^2$ to 12^2 or $3^2 - 2^2$ to 2^2 or 5 : 4 :: IF = 4 : $\frac{16}{5} = 3\frac{1}{5}$ = FK; and hence EK = $6 - 3\frac{1}{5} = 2\frac{4}{5} = \frac{14}{5}$ is the distance of the centre K below the surface of the water. This drawn into 120 the area of the dam-gate, gives 336 cubic feet of water = the pressure, = 336000 ounces = 21000 pounds = 9 tons 80 lb. the quantity of pressure against the gate, as required, being a 15th part less than in the first case.

Ex. 3. Find the quantity of pressure against a dam or sluice, across a canal, which is 20 feet wide at top, 14 at bottom, and 8 feet depth of water?

PROBLEM VI.

To determine the strongest angle of position of a pair of gates for the lock on a canal or river.



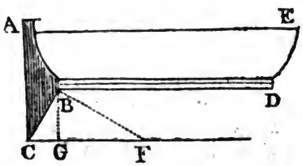
Let AC, BC be the two gates, meeting in the angle C, projecting out against the pressure of the water, AB being the breadth of the canal or river. Now the pressure of the water on a gate AC, is as the quantity, or as the extent or length of it, AC. And the mechanical effect of that pressure, is as the length of lever to the middle of AC, or as AC itself. On both these accounts then the pressure is as AC^2. Therefore the resistance or the strength of the gate must be as the reciprocal of this AC^2.

Now produce AC to meet BD, perp. to it, in D; and draw CE to bisect AB perpendicularly in E; then, by similar triangles, as AC : AE :: AB : AD; where, AE and AB being given lengths, AD is reciprocally as AC, or AD^2 reciprocally as AC^2; that is, AD^2 is as the resistance of the gate AC. But the resistance of AC is increased by the pressure of the other gate in the direction BC. Now the force in BC is resolved into the two BD, DC; the latter of which, DC, being parallel to AC, has no effect upon it; but the former, BD, acts perpendicularly on it. Therefore the whole effective strength or resistance of the gate is as the prodect $AD^2 \times BD$.

If now there be put AB $= a$, and BD $= x$, then $AD^2 = AB^2 - BD^2 = a^2 - x^2$; conseq. $AD^2 \times BD = (a^2 - x^2) \times x = a^2x - x^3$ for the resistance of either gate. And, if we would have this to be the greatest, or the resistance a maximum, its fluxion must vanish, or be equal to nothing: that is, $a^2\dot{x} - 3x^2\dot{x} = 0$; hence $a^2 = 3x^2$, and $x = a\sqrt{\frac{1}{3}} = \frac{1}{3}a\sqrt{3} = \cdot57735a$, the natural sine of 35° 16′: that is, the strongest position for the lock gates, is when they make the angle A or B = 35° 16′, or the complemental angle ACE or BCE = 54° 44′, or the whole salient angle ACB = 109° 28′.

Scholium.

Allied to this problem, are several other cases in mechanics: such as, the action of the water on the rudder of a ship, in sailing, to turn the ship about, to alter her course; and the action of the wind on a ship's sails, to impel her forward; also the action of water on the wheels of water-mills, and of the air on the sails of wind-mills, to cause them to turn round.



Thus, for instance, let ABC be the rudder of a ship ABDE, sailing in the direction BD, the rudder placed in the oblique position BC, and consequently striking the water in the direction CF, parallel to BD. Draw BF perp. to BC, and BG perp. to CF. Then the sine of the angle of incidence, of the direction of the stroke of the rudder against the water, will be BF, to the radius CF; therefore the force of the water against the rudder will be as BF^2, by art 3, page 425. But the force BF resolves into the two BG, GF, of which the latter is parallel to the ship's motion, and therefore has no effect to change it; but the former BG, being perp. to the ship's motion, is the only part of the force to turn the ship about and change her course. But $BF : BG :: CF : CB$, therefore $CF : CB :: BF^2 : \frac{BC \cdot BF^2}{CF}$ the force upon the rudder to turn the ship about.

Now put $a =$ CF, $x =$ BC; then $BF^2 = a^2 - x^2$, and the force $\frac{BC \cdot BF^2}{CF} = \frac{x(a^2 - x^2)}{a} = \frac{a^2x - x^3}{a}$; and, to have this a maximum, its flux. must be made to vanish, that is, $a^2\dot{x} - 3x^2\dot{x} = 0$; and hence $x = a\sqrt{\frac{1}{3}} =$ BC $=$ the natural sine of 35° 16′ = angle F; therefore the complemental angle C = 54° 44′ as

before, for the obliquity of the rudder, when it is most efficacious.

The case will be also the same with respect to the wind acting on the sails of a wind-mill, or of a ship, viz. that the sails must be set so as to make an angle of 54° 44′ with the direction of the wind; at least at the beginning of the motion, or nearly so when the velocity of the sail is but small in comparison with that of the wind; but when the former is pretty considerable in respect of the latter, then the angle ought to be proportionally greater, to have the best effect, as shown in Maclaurin's Fluxions, p. 734, &c.

A consideration, somewhat related to the same also, is the greatest effect produced on a mill-wheel, by a stream of water striking its sails or float-boards. The proper way in this case seems to be, to consider the whole of the water as acting on the wheel, but striking it only with the relative velocity, or the velocity with which the water overtakes and strikes upon the wheel in motion, or the difference between the velocities of the wheel and the stream. This then is the power or force of the water; which multiplied by the velocity of the wheel, the product of the two, viz. of the relative velocity and the absolute velocity of the wheel, that is $(\mathrm{v}-v)v = \mathrm{v}v - v^2$, will be the effect of the wheel; where v denotes the given velocity of the water, and v the required velocity of the wheel. Now to make the effect $\mathrm{v}v - v^2$ a maximum, or the greatest, its fluxion must vanish, that is, $\mathrm{v}\dot{v} - 2v\dot{v} = 0$, hence $v = \frac{1}{2}\mathrm{v}$; or the velocity of the wheel will be equal to half the velocity of the stream, when the effect is the greatest; and this agrees best with experiments.

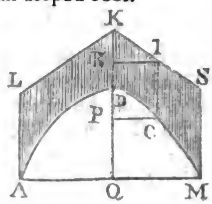
A former way of resolving this problem was, to consider the water as striking the wheel with a force as the square of the relative velocity, and this multiplied by the velocity of the wheel, to give the effect; that is, $(\mathrm{v} - v)^2 v =$ the effect. Now the flux. of this product is $(\mathrm{v}-v)^2\dot{v} - (\mathrm{v}-v) \times 2v\dot{v} = 0$; hence $\mathrm{v} - v = 2v$, or $\mathrm{v} = 3v$, and $v = \frac{1}{3}\mathrm{v}$, or the velocity of the wheel equal only to $\frac{1}{3}$ of the velocity of the water.

PROBLEM VII.

To determine the form and dimensions of gunpowder magazines.

In the practice of engineering, with respect to the erection of powder-magazines, the exterior shape is usually made like the roof of a house, having two sloping sides, forming two inclined planes, to throw of the rain, and meeting in an

angle or ridge at the top; while the interior represents a vault, more or less extended, as the occasion may require; and the shape, or transverse section, in the form of some arch, both for strength and commodious room, for placing the powder barrels. It has been usual to make this interior curve a semicircle. But, against this shape, for such a purpose, we must enter our decided protest; as it is an arch the farthest of any from being in equilibrium in itself, and the weakest of any, by being unavoidably much thinner in one part than in others. Besides, it is constantly found, that after the centering of semicircular arches is struck, and removed, they settle at the crown, and rise up at the flanks, even with a straight horizontal form at top, and still much more so in powder magazines with a sloping roof; which effects are exactly what might be expected from a contemplation of the true theory of arches. Now this shrinking of the arches must be attended with other additional bad effects, by breaking the texture of the cement, after it has been in some degree dried, and also by opening the joints of the voussoirs at one end. Instead of the circular arch, therefore, we shall in this place give an investigation, founded on the true principles of equilibrium, of the only just form of the interior, which is properly adapted to the usual sloped roof.



For this purpose put $a =$ DK the thickness of the arch at the top, $x =$ any absciss DP of the required arch ADCM, $u =$ KR the corresponding absciss of the given exterior line KI, and $y =$ PC $=$ RI their equal ordinates. Then by the principles of arches, in my tracts on that subject, it is found that CI or $w = a + x -$

$u = Q \times \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3}$, or $= Q \times \frac{\ddot{x}}{\dot{y}^2}$, supposing $\dot{y}$ a constant quantity, and where Q is some certain quantity to be determined hereafter. But KR or u is $= ty$, if t be put to denote the tangent of the given angle of elevation KIR, to radius 1; and then the equation is $w = a + x - ty = \frac{Q\ddot{x}}{\dot{y}^2}$.

Now the fluxion of the equation $w = a + x - ty$, is $\dot{w} = \dot{x} - t\dot{y}$, and the 2d fluxion is $\ddot{w} = \ddot{x}$; therefore the foregoing general equation becomes $w = \frac{Q\ddot{w}}{\dot{y}^2}$; and hence $w\dot{w}$

$= \frac{Qw\ddot{w}}{\dot{y}^2}$, the fluent of which gives $\dot{w}^2 = \frac{Qw^2}{\dot{y}^2}$: but at D the value of w is $= a$, and $\dot{w} = 0$, the curve at D being parallel to KI; therefore the correct fluent is $w^2 - a^2 = \frac{Q\dot{w}^2}{\dot{y}^2}$. Hence then $\dot{y}^2 = \frac{Q\dot{w}^2}{w^2-a^2}$, or $\dot{y} = \frac{\dot{w}\sqrt{Q}}{\sqrt{(w^2-a^2)}}$; the correct fluent of which gives $y = \sqrt{Q} \times$ hyp. log. of $\frac{w+\sqrt{(w^2-a^2)}}{a}$.

Now, to determine the value of Q, we are to consider that when the vertical line CI is in the position AL or MS, then $w =$ CI becomes $=$ AL or MS $=$ the given quantity c suppose, and $y =$ AQ or QM $= b$ suppose, in which position the last equation becomes $b = \sqrt{Q} \times$ hyp. log. $\frac{c+\sqrt{(c^2-a^2)}}{a}$; and hence it is found that the value of the constant quantity $\sqrt{Q}$, is $\frac{b}{\text{h.l.}\frac{c+\sqrt{(c^2-a^2)}}{a}}$; which being substituted for it, in the above general value of y, that value becomes

$$y = b \times \frac{\text{log. of } \frac{w+\sqrt{(w^2-a^2)}}{a}}{\text{log. of } \frac{c+\sqrt{(c^2-a^2)}}{a}} = b \times \frac{\text{log. of } w+\sqrt{(w^2-a^2)}-\text{log.}\,a}{\text{log. of } c+\sqrt{(c^2-a^2)}-\text{log.}\,a};$$

from which equation the value of the ordinate PC may always be found, to every given value of the vertical CI.

But if, on the other hand, PC be given, to find CI, which will be the more convenient way, it may be found in the following manner: Put A $=$ log. of a, and C $= \frac{1}{b} \times$ log. of $\frac{c+\sqrt{(c^2-a^2)}}{a}$; then the above equation gives $\text{C}y + \text{A} =$ log. of $w + \sqrt{(w^2-a^2)}$; again, put $n =$ the number whose log. is $\text{C}y + \text{A}$; then $n = w + \sqrt{(w^2 - a^2)}$; and hence $w = \frac{a^2+n^2}{2n} =$ CI.

Now, for an example in numbers, in a real case of this nature, let the foregoing figure represent a transverse vertical section of a magazine arch balanced in all its parts, in which the span or width AM is 20 feet, the pitch or height DQ is 10 feet, thickness at the crown DK $=$ 7 feet, and the angle of the ridge LKS 112° 37′, or the half of it LKD $=$ 56° 18½, the complement of which, or the elevation KIR, is 33° 41′½, the tangent of which is $= \frac{2}{3}$, which will therefore be the value of t in the foregoing investigation. The values of the other letters will be as follows, viz. DK$=a=7$; AQ $= b =$ 10;

DQ $= h = 10$; AL $= c = 10\frac{1}{3} = \frac{31}{3}$; A$=$log. of $7 = \cdot8450980$; C $= \frac{1}{b} \times$ log. of $\frac{c+\sqrt{(c^2-a^2)}}{a} = \frac{1}{10}$ log. of $\frac{31+\sqrt{520}}{21} = \frac{1}{10}$ log. of $2\cdot56207 = \cdot0408591$; $cy +$ A $= \cdot0408591y + \cdot8450980 =$ log. of n. From the general equation then, viz.

CI $= w = \frac{a^2+n^2}{2n} = \frac{a^2}{2n} + \frac{1}{2}n$, by assuming y successively equal to 1, 2, 3, 4, &c., thence finding the corresponding values of $cy +$ A or $\cdot0408591y + \cdot8450980$, and to these, as common logs, taking out the corresponding natural numbers, which will be the values of n; then the above theorem will give the several values of w or CI, as they are here arranged in the annexed table, from which the figure of the curve is to be constructed, by thus finding so many points in it.

Val. of y or CP.	Val. of w or CI.
1	7·0309
2	7·1243
3	7·2806
4	7·5015
5	7·7888
6	8·1452
7	8·5737
8	9·0781
9	9·6623
10	10·3333

Otherwise. Instead of making n the number of the log. $cy +$ A, if we put $m =$ the natural number of the log. cy only; then $m = \frac{w+\sqrt{(w^2-a^2)}}{a}$, and $am - w = \sqrt{(w^2-a^2)}$, or by squaring, &c., $a^2m^2 - 2amw + w^2 = w^2 - a^2$ and hence $w = \frac{m^2+1}{2m} \times a$: to which the numbers being applied, the very same conclusions result as in the foregoing calculation and table.

PROBLEM VIII.

To construct Powder Magazines with a Parabolical Arch.

It has been shown, in my tract on the Principles of Arches of Bridges, that a parabolic arch is an arch of equilibration, when its extrados, or form of its exterior covering, is the very same parabola as the lower or inside curve. Hence then a parabolic arch, both for the inside and outer form, will be very proper for the structure of a powder magazine. For, the inside parabolic shape will be very convenient as to room for stowage: 2dly, the exterior parabola, every where parallel to the inner one, will be proper enough to carry off the rain water: 3dly, the structure will be in perfect equilibrium: and 4thly, the parabolic curve is easily constructed, and the fabric erected.

$a = 50 =$ AD the height above the horizontal line, $t =$ tang. $\angle$DBC or 75° the complement of the plane's inclination, $\tau =$ tang. HBI or $\angle$H$=60°$ the comp. of $2\angle$C, $s=$sine of $2\angle$HBI $= 120°$ the double elevation, or $=$ sine of $4\angle$C; also $x=$AB the impetus or height fallen through. Then,

BI$=4$KH$=2sx$, by the projectiles prop. 176, p. 213.

and $\left\{\begin{array}{l}\text{BK}=\tau\times\text{KH}=\frac{1}{2}s\tau x\\ \text{CD}=t\times\text{BD}=t(x-a)\end{array}\right\}$ by trigonometry.

also, KD $=$ BK $-$ BD $= \frac{1}{2}s\tau x - x + a$, and KE $= \frac{1}{2}$BI $= sx$; then, by the parabola, $\sqrt{\text{BK}} : \sqrt{\text{DK}} :: \text{KE} : \text{FG} = \text{KE} \times \sqrt{\frac{\text{KD}}{\text{KB}}} = \sqrt{\frac{\tau s^2x^2 - 2sx^2 + 2asx}{\tau}} = \sqrt{[\frac{2s}{\tau}ax - (\frac{2s}{\tau} - s^2)x^2]} = 2b\sqrt{(ax - b^2x^2)}$, putting $b =$ sine of $2\angle$C $=$ sine of 30°. Hence CG $=$ CD $+$ DF $\pm$ FG $= tx - ta + sx \pm 2b\sqrt{(ax - b^2x^2)}$ a maximum, the fluxion of which made $= 0$, and the equation reduced, gives $x = \frac{a}{2b^2} \times (1 \pm \sqrt{\frac{n^2}{(n^2+4b^4)}}$, where $n = s + t$, and the double sine $\pm$ answers to the two roots or values of x, or to the two points G, G, where the parabolic path cuts the horizontal line CG, the one in ascending and the other in descending.

Now, in the present case, when the $\angle$ C $= 15°$, $t =$ tang. $75° = 2 + \sqrt{3}$, $\tau =$ tan. $60° = \sqrt{3}$, $s=$sin. $60° = \frac{1}{2}\sqrt{3}$, $b =$ sin. $30° = \frac{1}{2}$, $n = s + t = 2 + \frac{3}{2}\sqrt{3}$; then $\frac{a}{2b^2} = 2a = 100$, and $\frac{n^2}{n^2+4b^4} = \frac{n^2}{n^2+\frac{1}{4}} = \frac{41+6\sqrt{3}}{52}$; theref. $x = \frac{a}{2b^2} \times (1 \pm \sqrt{\frac{n^2}{n^2+4b^4}})$ $= 100 \times (1 \pm \frac{1}{2}\sqrt{\frac{41+6\sqrt{3}}{13}}) = 100 \times (1 \pm \cdot 99414) = 199\cdot 414$ or $\cdot 586$; but the former must be taken. Hence the body must strike the inclined plane at 149·414 feet below the horizontal line; and its path after reflection will cut the said line in two points; or it will touch it when $x = \frac{a}{bb}$. Hence also the greatest distance CG required is 826·9915 feet.

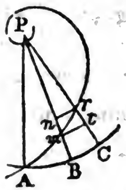
Corol. If it were required to find CG or $tx - ta + sx \pm 2b\sqrt{(ax - b^2x^2)} = g$ a given quantity, this equation would give the value of x by solving a quadratic.

PROBLEM III.

Suppose a ship to sail from the Orkney Islands, in latitude 59° 3′ north, on a N. N. E. *course, at the rate of* 10 *miles an hour; it is required to determine how long it will be before she arrives at the pole, the distance she will have sailed,*

and the difference of longitude she will have made when she arrives there?

Let ABC represent part of the equator; P the pole; AmrP a loxodromic or rhumb line, or the path of the ship continued to the equator; PB, PC, any two meridians indefinitely near each other; nr, or mt, the part of a parallel of latitude intercepted between them.



Put c for the cosine, and t for the tangent of the course, or angle nmr to the radius r; Am, any variable part of the rhumb from the equator, $= v$; the latitude Bm $= w$; its sine x, and cosine y; and AB, the dif. of longitude from A, $= z$. Then, since the elementary triangle mnr may be considered as a right-angled plane triangle, it is, as rad. $r : c =$ sin. $\angle\, mrn : : \dot{v} = mr : \dot{w} = mn : : v : w$; theref. $cv = rw$, or $v = \frac{rw}{c} = \frac{sw}{r}$, by putting s for the secant of the $\angle nmr$ the ship's course. In like manner, if W be any other latitude, and V its corresponding length of the rhumb; then $\text{V} = \frac{r\text{W}}{c}$; and hence $\text{V} - v = r \times \frac{\text{W}-w}{c}$, or $\text{D} = \frac{rd}{c}$, by putting $\text{D} = \text{V}-v$ the distance, and $d = \text{W}-w$ the dif. of latitude; which is the common rule.

The same is evident without fluxions: for since the $\angle mrn$ is the same in whatever point of the path AmrP the point m is taken, each indefinitely small particle of AmrP, must be to the corresponding indefinitely small part of Bm, in the constant ratio of radius to the cosine of the course; and therefore the whole lines, or any corresponding parts of them, must be in the same ratio also, as above determined. In the same manner it is proved that radius : sine of the course : : distance : the departure.

Again, as radius, $r : t =$ tang. $nmr : : \dot{w} = mn : nr$ or mt, and as $r : y : :$ PB : Pm : : $\dot{z} =$ BC : mt; hence, as the extremes of these proportions are the same, the rectangles of the means must be equal, viz. $y\dot{z} = t\dot{w} = \frac{tr\dot{x}}{y}$ because $\dot{w} = \frac{r\dot{x}}{y}$ by the property of the circle; theref. $\dot{z} = \frac{tr\dot{x}}{y^2} = \frac{tr\dot{x}}{r^2-x^2}$; the gene-

ral fluents of these are $z = t \times$ hyp. log. $\sqrt{\frac{r+x}{r-x}} + c$; which corrected by supposing $z = 0$, when $x = a$, are $z = t \times$ (hyp. log. $\sqrt{\frac{r+x}{r-x}}$ − hyp. log. $\sqrt{\frac{r+a}{r-a}}$) ; but $r \times$ (hyp. log. $\sqrt{\frac{r+x}{r-x}}$ − hyp. log. $\sqrt{\frac{r+a}{r-a}}$) is the meridional parts of the dif. of the latitudes whose sines are x and a, which call b ; then is $z = \frac{tb}{r}$, the same as it is by Mercator's sailing.

Further, putting $m = 2{\cdot}71828$ the number whose hyp. log. is 1, and $n = \frac{2z}{t}$; then, when z begins at A, $m^n = \frac{r+x}{r-x}$, and theref. $x = r \times \frac{m^n-1}{m^n+1} = r - \frac{2r}{m^n+1}$: hence it appears that as m, or rather n or z increases (since m is constant), that x approximates to an equality with r, because $\frac{2r}{m^n+1}$ decreases or converges to 0, which is its limit ; consequently r is the limit or ultimate value of x ; but when $x = r$, the ship will be at the pole ; theref. the pole must be the limit, or evanescent state, of the rhumb or course : so that the ship may be said to arrive at the pole after making an infinite number of revolutions round it ; for the above expression $\frac{2r}{m^n+1}$ vanishes when n, and consequently z, is infinite, in which case x is $= r$.

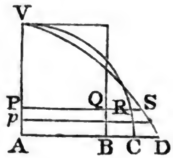
Now, from the equation $D = \frac{rd}{c} = \frac{sd}{r}$, it is found, that when $d = 30° \; 57'$ the comp. of the given lat. $59° \; 3'$ and $c =$ sine of $67° \; 30'$ the comp. of the course, D will be = 2010 geographical miles, the required ultimate distance ; which, at the rate of 10 miles an hour, will be passed over in 201 hours or $8\frac{3}{8}$ days. The dif. of long. is shown above to be infinite. When the ship has made one revolution, she will be but about a yard from the pole, considering her as a point.

When the ship has arrived infinitely near the pole, she will go round in the manner of a top, with an infinite velocity ; which at once accounts for this paradox, viz. that though she make an infinite number of revolutions round the pole, yet her distance run will have an ultimate and definite value, as

above determined : for it is evident that however great the number of revolutions of a top may be, the space passed over by its pivot or bottom point, while it continues on or nearly on the same point, must be infinitely small, or less than a certain assignable quantity.

PROBLEM IV.

A current of water is discharged by three equal openings or sluices, in the following shapes : the first a rectangle, the second a semicircle, and the third a parabola, having their altitudes equal, and their bases in the same horizontal line, and the water level with the tops of the arches : on this supposition it is required to show what may be the proportion of the quantities discharged by these sluices.



Let VB be half the parallelogram, AVC half the semicircle, and AVD half the parabola, that is, the halves of the respective sluices or gates. Put $a =$ AV the common altitude, and $c = \cdot7854$: then is ca^2 the area of each of the figures ; also $ca =$ AB, $a =$ AC, and $\frac{3}{2}ca =$ AD ; also put $x =$ VP any variable depth, and $\dot{x} =$ Pp. Then, the water discharged at any depth x, being as the velocity and aperture, and the velocity being in all the figures as $\sqrt{x}$, therefore $\dot{x}\sqrt{x} \times$ PQ, and $\dot{x}\sqrt{x} \times$ PR, and $\dot{x}\sqrt{x} \times$ PS, or $cax^{\frac{1}{2}}\dot{x}$, and $x\dot{x}\sqrt{(2a - x)}$, and $\frac{3}{2}c\sqrt{a} \times x\dot{x}$, are proportional to the fluxions of the quantity of water discharged by the said figures or sluices respectively ; the correct fluents of which, when $x = a$, are $\frac{2}{3}ca^{\frac{5}{2}}$, and $\frac{2}{15}a^{\frac{5}{2}}(8\sqrt{2} - 7)$, and $\frac{3}{4}ca^{\frac{5}{2}}$, the 2d fluent being found by art. 65, page 338 of this vol. Hence the quantities of water discharged by the rectangle, the semicircle, and the parabola, are respectively as $\frac{2}{3}c$, and $\frac{2}{15}(8\sqrt{2}-7)$, and $\frac{3}{4}c$, or as 1, and $\dfrac{2}{5c}(8\sqrt{2}-7)$, and $\frac{9}{8}$, or as 1, and 1·09847, and $1\frac{1}{8}$.

PROBLEM V.

The initial velocity of a 24 *lb. ball of cast iron, which is projected in a direction perpendicular to the horizon, being supposed* 1200 *feet per second ; and that the resistance of the medium is constantly as the square of the velocity, and everywhere of the same density : required the time of flight, and the height to which it will ascend.*

Answer. By problems 5 and 6, of the last chapter, the

ascent will be found = 5337 feet, and the time of the ascent 28 seconds.

PROBLEM VI.

To determine the same as in the last question, supposing the density of the atmosphere to decrease in ascending after the usual way?

Ans. By probs. 7 and 8, the height will be 5614 feet, and the time 34 seconds.

PROBLEM VII.

It is required to find the diameter of a circular parachute, by means of which a man of 150lb. weight may descend on the earth, from a balloon at a height in the air, with the velocity of only 10 feet in a second of time, being the velocity acquired by a body freely descending through a space of only 1 foot $6\frac{3}{4}$ inches, or of a man jumping down from a height of $18\frac{3}{4}$ inches: the parachute being made of such materials and thickness, that a circle of it of 50 feet diameter, weighs only 150lb., and so in proportion more or less according to the area of the circle.

If a falling body descend with a uniform velocity, it must necessarily meet with a resistance, from the medium it descends in, equal to the whole weight that descends. Let x denote the diameter of the parachute, and $a = \cdot 7854$; then ax^2 will be its area, and as $50^2 : x^2 :: 150 : \frac{3}{50}x^2$ the weight of the same, to which adding 150 lb., the man's weight, the sum $\frac{3}{50}x^2+150$ will be the whole descending weight. Again, in the table of resistances, page 435, we find that a circle of $\frac{2}{9}$ of a square foot area, moving with 10 feet velocity, meets with a resistance of $\cdot 57$ ounces $= \cdot 0475$lb.; and the resistances, with the same velocity, being as the surfaces, therefore as $\frac{2}{9} : \cdot 0475 :: ax^2 : \cdot 21375ax^2 = \cdot 16788x^2$ the resistance of the air to the parachute, to which the descending weight must be equal; that is, $\cdot 16788x^2 = \frac{3}{50}x^2+150$; hence $\cdot 10788x^2 = 150$, or $x^2 = 1390\cdot 5$, and hence $x = 37\frac{2}{7}$ feet, the diameter of the parachute required.

PROBLEM VIII.

To determine how far a man, who pushes with the force of 100lb. can force a sponge into a piece of ordnance, whose diameter is 5 inches, and length 10 feet, when the barometer stands at 30 inches; the vent, or touch-hole, being stopped, and

the sponge having no windage, that is, fitting the bore quite close?

A column of quicksilver 30 inches high, and 5 in diameter, is $5^2 \times 30 \times \cdot7854 = 589\cdot05$ inches; which, at 8·102 oz. each inch, weighs 4772·48 oz. or 298·28lb., which is the pressure of the atmosphere alone, being equal to the elasticity of the air in its natural state; to this adding to 100lb. gives 398·28lb., the whole external pressure. Then, as the spaces which a quantity of air possesses, under different pressures, are in the reciprocal ratio of those pressures, it will be, as 398·28 : 298·28 : : 10 feet or 120 inches : 90 inches nearly, the space occupied by the air; theref. $120 - 90 = 30$ inches, is the distance sought.

PROBLEM IX.

To assign the cause of the deflection of military projectiles.

It having been surmised that, in the practice of artillery, the deflection of the shot in its flight, to the right or left, from the line or direction the gun is laid in, chiefly arises from the motion of the gun during the time the shot is passing out of the piece; it is required to determine what space an 18 pounder will recoil or fly back, while the shot is passing out of the gun; supposing its weight to be 4800lb. that of the carriage 2400 lb., the quantity of powder 8 lb., the length of the cylinder 108 inches, that of the charge 13 inches, and the diameter of the bore 5·13 inches; supposing also that the resistance from the friction between the platform and carriage is equal to 3600 lb.?

It is well known that confined gunpowder, when fired, immediately changes in a great measure into an elastic air, which endeavours to expand in all directions. Now, in the question, the action of this fluid is exerted equally on the bottom of the bore of the gun and on the ball, during the passage of the latter through the cylinder; the two bodies therefore move in opposite directions, with velocities which are at all times in the inverse ratio of the quantities of matter moved. Now let x be the space through which the gun recoils; then, as the charge occupies 13 inches of the barrel, and the semidiameter of the barrel is 2·565, the space moved through by the ball when it quits the piece, is $108 - 13 - 2\cdot565 - x = 92\cdot435 - x$: and as the elastic fluid expands in both directions, the quantity which advances towards the muzzle, is to that which retreats from it, as $92\cdot435 - x$ to x:

conseq. $\frac{8x}{92 \cdot 435}$ and $\frac{92 \cdot 435 - x}{92 \cdot 435} \times 8$ are the quantities of the powder which move, the former with the gun, and the latter with the ball; besides these, the weight of ball that moves forwards being 18 lb., and of the weights and resistance backwards $4800 + 2400 + 3600 = 10800$ lb., hence the whole weights moved in the two directions are $10800 + \frac{8x}{92 \cdot 435}$ and $18 + \frac{92 \cdot 435 - x}{92 \cdot 435} \times 8$, or $\frac{998298 + 8x}{92 \cdot 435}$ and $\frac{2403 \cdot 31 - 8x}{92 \cdot 435}$, or as the numerators of these only. But when the time and moving force are given, or the same, then the spaces are inversely as the quantities of matter; therefore $x : 92 \cdot 435 - x :: 2403 \cdot 31 - 8x : 998298 + 8x$, or by composition, $x : 92 \cdot 435 :: 2403 \cdot 31 - 8x : 1000701 \cdot 31$, and by div. $x : 1 :: 2403 \cdot 31 - 8x : 10826$, theref. $10826x = 2403 \cdot 31 - 8x$, or $10834x = 2403 \cdot 31$, and hence $x = \cdot 2218$ inc. $= \frac{2}{9}$ of an inch nearly, or the recoil of the gun is less than a quarter of an inch.

Hence it may be concluded, that so small a recoil, straight backwards, can have no effect in causing the ball to deviate from the pointed line of direction: and that it is very probable we are to seek for the cause of this effect in the ball striking or rubbing against the sides of the bore, in its passage through it, especially near the exit at the muzzle; by which it must happen, that if the ball strike against the right side, the ball will deviate to the left; if it strike on the left side, it must deviate to the right; if it strike against the under side, it must throw the ball upwards, and make it to range farther; but if it strike against the upper side, it must beat the ball downwards, and cause a shorter range: all which irregularities are found to take place, especially in guns that have much windage, or which have the balls too small for the bore.

PROBLEM X.

A ball of lead of 4 inches diameter, is dropped from the top of a tower, of 65 yards high, and falls into a cistern full of water at the bottom of the tower, of $20\frac{1}{4}$ yards deep: it is required to determine the times of falling, both to the surface and to the bottom of the water.

The fall in air is 195 feet, and in water $60\frac{3}{4}$ feet by the common rules of descent, as $\sqrt{16} : \sqrt{195} :: 1'' : \frac{1}{4}\sqrt{195} =$ 3·49 seconds, the time of descending in air. And as $\sqrt{16}$:

$\sqrt{195}$: : 32 : 8 $\sqrt{195}=111\cdot71$ feet, the velocity at the end of that time, or with which the ball enters the water.

Again, by prob. 22, art. 2, p. 432, the space $s = \frac{1}{2b} \times$ hyp. log. of $\frac{a-e^2}{a-v^2}$, or rather $\frac{1}{2b} \times$ hyp. log. of $\frac{e^2-a}{v^2-a}$ (the velocity being decreasing, and e^2 greater than a) $= \frac{m}{2b} \times$ com. log. of $\frac{e^2-a}{v^2-a}$, where $N=11325$ the density of lead, $n=1000$ that of water, $a = \frac{256d(N-n)}{3n}$, $b = \frac{3}{8dN}$, $e = 111\cdot71$ the velocity at entering the water, and v the velocity at any time afterwards, also d the diameter of the ball $= 4$ inches, and $m = 2\cdot302585$ the hyp. log. of 10.

Hence then $N = 11325$, $n = 1000$, $N - n = 10325$, $d = \frac{4}{12} = \frac{1}{3}$; then $a = \frac{256d(N-n)}{3n} = \frac{256 \;.\; 10325}{9000} = 293\frac{1}{2}$, and $b = \frac{3n}{8dn} = \frac{9n}{8N} = \frac{9000}{90600} = \frac{15}{151} = \frac{1}{10}$ nearly. Also $e=111\cdot71$; therefore $s = 60\frac{3}{4} = \frac{m}{2b} \times$ log. of $\frac{e^2-a}{v^2-a} = 5m \times$ log. $\frac{e^2-a}{v^2-a}$. This theorem will give s when v is given, and by reverting, it will give v in terms of s in the following manner.

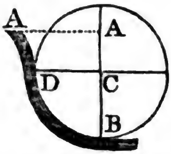
Dividing by $5m$, gives $\frac{s}{5m} =$ log. of $\frac{e^2-a}{v^2-a} = ns$, by putting $n = \frac{1}{5m}$; therefore, the natural number is $10^{ns} = \frac{e^2-a}{v^2-a}$; hence $v^2 - a = \frac{e^2-a}{10^{ns}}$, and $v = \sqrt{(a + \frac{e^2-a}{10^{ns}})}$, which, by substituting the numbers above mentioned for the letters, gives $v = 17\cdot134$ for the last velocity, when the space $s = 60\frac{3}{4}$, or when the ball arrives at the bottom of the water.

But now to find the time of passing through the water, putting $t =$ any time in motion, and s and v the corresponding space and velocity, the general theorem for variable forces gives $\dot{t} = \frac{\dot{s}}{v}$. But the above general value of s being $\frac{1}{2b} \times$ hyp. log. $\frac{e^2-a}{v^2-a}$ or $5 \times$ hyp. log. $\frac{e^2-a}{v^2-a}$, therefore its fluxion $\dot{s} = \frac{-10v\dot{v}}{v^2-a}$, conseq. $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{-10\dot{v}}{v^2-a}$, the correct fluent of

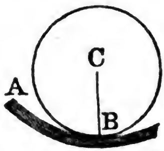
which is $\frac{5}{\sqrt{a}} \times$ hyp. log. $(\frac{e-\sqrt{a}}{e+\sqrt{a}} \times \frac{v+\sqrt{a}}{v-\sqrt{a}}) = t$ the time, which when $v = 17 \cdot 134$, or $s = 60\frac{3}{4}$, gives $2 \cdot 6542$ seconds, for the time of descent through the water.

PROBLEM XI.

Required to determine what must be the diameter of a water-wheel, so as to receive the greatest effect from a stream of water of 12 feet fall?



In the case of an undershot wheel, put the height of the water AB $= 12$ feet $= a$, and the radius BC or CD of the wheel $= x$, the water falling perpendicularly on the extremity of the radius CD at d. Then AC or AD $= a - x$, and the velocity due to this height, or with which the water strikes the wheel at D, will be as $\sqrt{(a-x)}$, and the effect on the wheel being as the velocity and as the length of the lever CD, will be denoted by $x\sqrt{(a-x)}$ or $\sqrt{(ax^2-x^3)}$, which therefore must be a maximum, or its square $ax^2 - x^3$ a maximum. In fluxions, $2ax\dot{x} - 3x^2\dot{x} = 0$; and hence $x = \frac{2}{3}a = 8$ feet the radius.



But if the water be considered as conducted so as to strike on the bottom of the wheel, as in the annexed figure, it will then strike the wheel with its greatest velocity, and there can be no limit to the size of the wheel, since the greater the radius or lever BC, the greater will be the effect.

In the case of an overshot wheel, $a - 2x$ will be the fall of water, $\sqrt{(a-2x)}$ as the velocity, and $x\sqrt{(a-2x)}$ or $\sqrt{(ax^2 - 2x^3)}$ the effect, then $ax^2 - 2x^3$ is a maximum, and $2ax\dot{x} - 6x^2\dot{x} = 0$; hence $\dot{x} = \frac{1}{3}a = 4$ feet is the radius of the wheel.

But all these calculations are to be considered as independent of the resistance of the wheel, and of the weight of the water in the buckets of it.

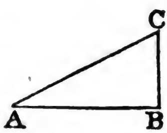
PROBLEM XII.

What angle must a projectile make with the plane of the horizon, discharged with a given velocity v, so as to describe in its flight a parabola including the greatest area possible?

By the set of theorems in art. 175, page 213, for any proposed angle, there can be assigned expressions for the horizontal range and the greatest height the projectile rises to, that is the base and axis of the parabolic trajectory. Thus, putting s and c for the sine and cosine of the angle of elevation; then, by the first line of those theorems, the velocity being v, the horizontal range R is $= \frac{1}{16}scv^2$; and, by the 4th or last line of theorems, the greatest height H is $= \frac{1}{64}s^2v^2$. But, by the parabola, $\frac{2}{3}$ of the product of the base or range and the height is the area, which is now required to be the greatest possible. Therefore $R \times H = \frac{1}{16}scv^2 \times \frac{1}{64}s^2v^2$ must be a maximum, or, rejecting the constant factors, s^3c a maximum. But the cosine c, of the angle whose sine is s, is $\surd(1-s^2)$; therefore $s^3c = s^3\surd(1-s^2) = \surd(s^6 - s^8)$ is the maximum, or its square $s^6 - s^8$ a maximum. In fluxions $6s^5\dot{s} - 8s^7\dot{s} = 0 = 3 - 4s^2$; hence $4s^2 = 3$, or $s^2 = \frac{3}{4}$, and $s = \frac{1}{2}\surd 3 = \cdot 8660254$, the sine of 60°, which is the angle of elevation to produce a parabolic trajectory of the greatest area.

PROBLEM XIII.

Suppose a cannon were discharged at the point A; *it is required to determine how high in the air the point* C *must be raised above the horizontal line* AB, *so that a person at* C *letting fall a leaden bullet at the moment of the cannon's explosion, it may arrive at* B *at the same instant as he hears the report of the cannon, but not till* $\frac{1}{10}$*th of a second after the sound arrives at* B: *supposing the velocity of sound to be* 1140 *feet per second, and that the bullet falls freely without any resistance from the air?*

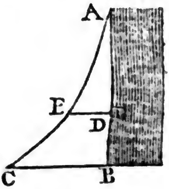


Let x denote the time in which the sound passes to C; then will $x - \frac{1}{10}$ be the time in passing to B, and x the time also the bullet is falling through CB. Then, by uniform motion, $1140x =$ AC, and $1140x - 114 =$ AB, also by descents of gravity, $1^2 : x^2 :: 16 : 16x^2 =$ BC. Then, by right-angled triangles, $AC^2 - BC^2 = AB^2$, that is $1140^2x^2 - 16^2x^4 - 1140^2x^2 -$

$224 \times 1140x + 114^2$, hence $224 \times 1140x - 16^2x^4 = 114^2$, or $1015{\cdot}3x - x^4 = 50{\cdot}77$, the root of which equa. is $x = 10{\cdot}03$ second, or nearly 10 seconds; conseq. $BC = 16x^2 = 1610$ feet nearly, the height required.

PROBLEM XIV.

Required the quantity, in cubic feet, of light earth, necessary to form a bank on the side of a canal, which will just support a pressure of water 5 feet deep, and 300 feet long. And what will the carriage of the earth cost, at the rate of 1 shilling per ton?

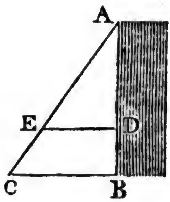


This question may be considered as relating either to water sustained by a solid wall, or by a bank of loose earth. In the former case, let ABC denote the wall, sustaining the pressure of the water behind it. Put the whole altitude AB $= a$, the base BC or thickness at bottom $= b$, any variable depth $AD = x$, and the thickness there $DE = y$. Now the effect which any number of particles of the fluid pressing at D have to break the wall at B, or to overturn it there, is as the number of particles AD or x, and as the lever $BD = a - x$; therefore the fluxion of the effect of all the forces is $(a - x)\, x\dot{x} = ax\dot{x} - x^2\dot{x}$, the fluent of which is $\frac{1}{2}ax^2 - \frac{1}{3}x^3$, which, when $x = a$, is $\frac{1}{6}a^3$ for the whole effect to break or overturn the wall at B; and the effects of the pressure to break at B and D will be as AB^3 and AD^3. But the strength of the wall at D, to resist the fracture there, like the lateral strength of timber, is as the square of the thickness, DE^2. Hence the curve line AEC, bounding the back of the wall, so as to be every where equally strong, is of such a nature, that x^3 is always proportional to y^2, or y as $x^{\frac{3}{2}}$, and is therefore what is called the semicubical parabola.

Now, to find the area ABC, or content of the wall bounded by this convex curve, the general fluxion of the area $y\dot{x}$ becomes $x^{\frac{3}{2}}\dot{x}$, the fluent of which is $\frac{2}{5}x^{\frac{5}{2}} = \frac{2}{5}\, xx^{\frac{3}{2}} = \frac{2}{5}xy$, that is $\frac{2}{5}$ of the rectangle $AB \times BC$; and is therefore less than the triangle ABC, of the same base and height, in the proportion of $\frac{2}{5}$ to $\frac{1}{2}$, or of 4 to 5.

But in the case of a bank of made earth, it would not stand with that concave form of outside, if it were necessary, but would dispose itself in a straight line AC, forming a triangular bank ABC. And even if this were not the case naturally, it would be proper to make it such by art; because now neither

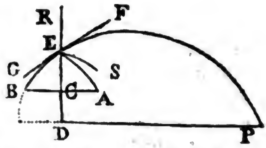
is the bank to be broken as with the effect of the lever, or overturned about the pivot or point C, nor does it resist the fracture by the effect of a lever, as before; but on the contrary, every point is attempted to be pushed horizontally outwards, by the horizontal pressure of the water, and it is resisted by the weight or resistance of the earth at any part, DE. Here then, by hydrostatics, the pressure of the water against any point D, is as the depth AD; and, in the triangle of earth ADE, the resisting quantity in DE is as DE, which is also proportional to AD by similar triangles. So that, at every point D in the depth, the pressure of the water and the resistance of the soil, by means of this triangular form, increase in the same proportion, and the water and the earth will every where mutually balance each other, if at any one point, as B, the thickness BC of earth be taken such as to balance the pressure of the water at B, and then the straight line AC be drawn, to determine the outer shape of the earth. All the earth that is afterwards placed against the side AC, for a convenient breadth at top for a walking path, &c. will also give the whole a sufficient security.



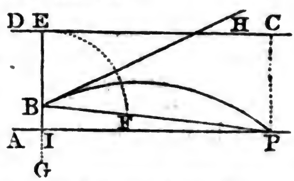
But now to adapt these principles to the numeral calculation proposed in the question; the pressure of water against the point B being denoted by the side AB = 5 feet, and the weight of water being to earth as 1000 to 1984, therefore as 1984 : 1000 :: 5 : 2·52 = BC, the thickness of earth which will just balance the pressure of the water there; hence the area of the triangle ABC = $\frac{1}{2}$AB × BC = $2\frac{1}{2}$ × 2·52 = 6·3; this mult. by the length 300, gives 1890, cubic feet for the quantity of earth in the bank; and this multiplied by 1984 ounces, the weight of 1 cubic foot, gives for the weight of it, 3749760 ounces = 234360 lbs. = 104·625 tons; the expense of which, at 1 shilling the ton, is 5l. 4s. $7\frac{1}{2}$d.

PROBLEM XV.

A person standing at the distance of 10 feet from the bottom of a wall, which is supposed perfectly smooth and hard, desires to know in what direction he must throw an elastic ball against it, with a velocity of 80 feet per second, so that, after reflection from the wall, it may fall at the greatest distance possible from the bottom, on the horizontal plane, which is $2\frac{1}{2}$ feet below the hand discharging the ball?



In the annexed figure let DR be the wall against which the ball is thrown, from the point A, in such direction, that it shall describe the parabolic curve AE before striking the wall, and afterwards be so reflected as to describe the curve EP. Now if ES be the tangent at the point E, to the curve AE described before the reflection, and EF the tangent at the same point to the curve which the ball will describe after reflection, then will the angle REF be = CES; and if the curve FE be produced, so as to have GF for its tangent, it will meet AC produced in B, making BC = AC, and the curve AE will be similar and equal to the portion BE of the parabola BEP, but turned the contrary way. Conceiving either the two curves AE and EP, or the continued curve BEP, to be described by a projectile in its motion, it is manifest that, whether the greater portion of the curve be described before or after the ball reaches the wall DR, will depend on its initial velocity, and on the distance AC or BC, and on the angle of projection. The problem then is now reduced to this, viz. To find the angle at which a ball shall be projected from B, with a given impetus, so that the distance DP, at which it falls, from the given point D, on the plane DP, parallel to the horizon, shall be a maximum.



Now this problem may be constructed in the following manner: From any point E in the horizontal line DC, let fall the indefinite perp. EG, on which set off EB = the impetus corresponding to the given velocity, and BI = 2½ the distance of the horizontal plane below the point of projection; also, through I drawn AP parallel to DC. From the point B set off BP = BE + EI, and bisect the angle EBP by the line BH: then will BH be the required direction of the ball, and IP the maximum range on the plane AP.

For, since the ball moves from the point B, with the velocity acquired by falling through EB, it is manifest, from page 210, that DC is the directrix of the parabola described by the ball. And since both B and P are points in the curve, each of them must, from the nature of the parabola, be as far from the focus as it is from the directrix; therefore B and P will be the greatest distance from each other when the focus F is directly between them, that is, when BP = BE + CP. And when BP is a maximum, since BI is constant, it is ob-

vious that IP is a maximum too. Also, the angle FBH being = EBH, the line BH is a tangent to the parabola at the point B, and consequently it is the direction necessary to give the range IP.

Cor. 1. When B coincides with I, IP will be = BP = BE + EI = 2EI, and the angle EBH will be 45° : as is also manifest from the common modes of investigation.

Cor. 2. When the impetus corresponding to the initial velocity of the ball is very great compared with AC or BC (fig. 1), then the part AE of the curve will very nearly coincide with its tangent, and the direction and velocity at A may be accounted the same as those at E without any sensible error. In this case too the impetus BE (fig. 2) will be very great compared with BI, and consequently, B and I nearly coinciding, the angle EBH will differ but little from 45°.

Calcul. From the foregoing construction the calculation will be very easy. Thus, the first velocity being 80 feet = v, then (page 213,) $\frac{v^2}{2g} = \frac{80 \times 80}{64\frac{1}{3}} = 99{\cdot}48186$ = BE the impetus; hence EI = FP = 101·98186, and BP = BE + EI = 201·46372. Now, in the right-angled triangle BIP, the sides BI and BP are known, hence IP = 201·4482, and the angle IBP = 89° 17′ 20″ : half the suppl. of this angle is 45° 21′ 20″ = EBH. And, in fig. 1, IP — ID = 201·4482 — 10 = 191·4482 = DP, the distance the ball falls from the wall after reflection.

PROBLEM XVI.

From what height above the given point A *must an elastic ball be suffered to descend freely by gravity, so that, after striking the hard plane at* B, *it may be reflected back again to the point* A, *in the least time possible from the instant of dropping it?*

Let C be the point required ; and put AC = x, and AB = a ; then $\frac{1}{4}\sqrt{\text{CB}} = \frac{1}{4}\sqrt{(a+x)}$ is the time in CB, and $\frac{1}{4}\sqrt{\text{CA}} = \frac{1}{4}\sqrt{x}$ is the time in CA : therefore $\frac{1}{4}\sqrt{(a+x)} - \frac{1}{4}\sqrt{x}$ is the time down AB, or the time of rising from B to A again : hence the whole time of falling through CB and returning to A, is $\frac{1}{2}\sqrt{(a+x)} - \frac{1}{4}\sqrt{x}$, which must be a min. or $2\sqrt{(a+x)} - \sqrt{x}$ a minimum, in fluxions $\frac{\dot{x}}{\sqrt{(a+x)}} - \frac{\dot{x}}{2\sqrt{x}} = 0$, and hence $x = a$, that is, AC = $\frac{1}{3}$AB.

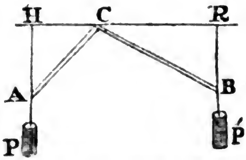
PROBLEM XVII.

Given the height of an inclined plane; required its length, so that a given power acting on a given weight, in a direction parallel to the plane, may draw it up in the least time possible.

Let a denote the height of the plane, x its length, p the power, and w the weight. Now the tendency down the plane is $= \frac{aw}{x}$, hence $p - \frac{aw}{x} =$ the motive force, and $\frac{p - \frac{aw}{x}}{p + w} = \frac{px - aw}{(p + w)x} =$ the accelerating force f; hence, by the theorems for constant forces, page 401, $t^2 = \frac{s}{\frac{1}{2}gf} = \frac{2(p + w)x^2}{(px - aw)g}$ must be a minimum, or $\frac{x^2}{px - aw}$ a min.; in fluxions, $2(px - aw)x\dot{x} - px^2\dot{x} = 0$, or $px = 2aw$, and hence $p : w :: 2a : x ::$ double the height of the plane to its length. (See, also, prob. 3, ch. xi.)

PROBLEM XVIII.

Given the lengths of the two arms of an angular lever, the weight of each arm, the angle made by the arms, and the weight that hangs at the extremity of each; to find the position of the lever when at rest.



Let the respective lengths of the arms AC, BC, be denoted by a, a', their weights by $2w$, $2w'$, and let P, P′, be the weights hanging at A and B respectively. Then, if the arms AC, CB, be regarded as prismatic and of uniform density throughout, instead of regarding their whole weights as acting at their centres, we may conceive their *half*-weights to act in conjunction with the weights P and P′, at the extremities, and investigate the position of the whole in the case of equilibrium, by means of the trigonometrical formulæ at the beginning of this volume.

Thus, HR being horizontal, $(P' + w')$ CR $= (P + w)$ CH, that is, $(P' + w')a' \cos \text{BCR} = (P + w)a \cos \text{ACH}$
or, $-(P' + w')a' \cos(\text{ACH} + \text{ACB}) = (P + w)a \cos \text{ACH}$

or, $-(\text{P}'+w')a'\cos\text{ACH}\cos\text{ACB} - \sin\text{ACH}\sin\text{ACB} = (\text{P}+w)a \cos\text{ACH} : (\text{P}'+w')a\sin\text{ACH}\sin\text{ACB} = [(\text{P}+w)a+(\text{P}'+w')a' \cos\text{ACB}]\cos\text{ACH}$:

$$\frac{\sin\text{ACH}}{\cos\text{ACH}} = \tan\text{ACH} = \frac{(\text{P}+w)a + (\text{P}'+w')a'\cos\text{ACB}}{(\text{P}'+w')a'\sin\text{ACB}}$$

$$= \frac{(\text{P}+w)a}{(\text{P}'+w')a'\sin\text{ACB}} + \cot\text{ACB}$$

$$= \frac{(\text{P}+w)a\,\text{cosec}\,\text{ACB}}{(\text{P}'+w')a'} + \cot\text{ACB}.$$

Cor. 1. If P be given, and it be proposed to find P′ so that AC shall become horizontal: then, putting the first value of tan ACH, above, $= 0$, we shall have its numerator $= 0$; whence $(\text{P}'+w')a'\cos\text{ACB} = -(\text{P}+w)a$,

and $\text{P}'+w' = \dfrac{-(\text{P}+w)a}{a'\cos\text{ACB}} = -(\text{P}+w)\dfrac{a}{a'}\sec\text{ACB}$.

Cor. 2. In order that this value of $\text{P}'+w'$ may be positive, and P a real weight, the factor sec ACB must be negative, that is (p. 11) the angle ACB must be obtuse, and CB directed aslant either above or below the horizon.

Cor. 3. If the arms be of equal length and equal weight, and one of them be required to be horizontal,

then $\text{P}'+w' = \text{P}'+w = -(\text{P}+w)\sec\text{ACB}$,
and $\text{P}' = -w-(\text{P}+w)\sec\text{ACB}$.

PROBLEM XIX.

A cylinder of wood is depressed in water till its top is just level with the surface, and then is suffered to ascend; it is required to determine the greatest altitude to which it will rise, and the other circumstances of its motion.

Let $a =$ the length, and b the area or base of the cylinder, m its specific gravity, that of water being 1, also $a-x$ any variable height through which the cylinder has ascended, or x being the part still immersed in the water. Then bx is the mass and force of the water upwards to raise the cylinder; and $a \times b \times m = abm$ is the weight of the cylinder opposing its ascent; therefore the motive force to raise the cylinder is $bx - abm$; also, the mass of the cylinder being abm, and that of the displaced water bx, the whole matter in motion is $bx + abm$; by which dividing the motive force, we have $\dfrac{bx-abm}{bx+abm} = \dfrac{x-am}{x+am} = f$ the accelerating force. Then

the well known theorem $v\dot{v} = 32f \cdot -\dot{x}$, gives $v\dot{v} = 32\dot{x} \cdot \frac{am-x}{am+x}$, v being the velocity; and the correct fluent is $v^2 =$

$64\,(a - x - 2am \times \text{hyp. log. of } \frac{am+a}{am+x})$ and hence $v =$

$8\sqrt{(a-x-2am \times \text{h. l. } \frac{am+a}{am+x})}$ the general state of the velocity when the part x is immersed, or when the part $a - x$ is out of the water.

Now when the velocity arrives at its greatest state, by the opposite forces bx and abm becoming equal, then $x = am$, or $1 : m :: a : x$, that is, the whole length is to the part immersed, as the specific gravity of the fluid is to that of the cylinder. And if the latter be equal to half the former, which is nearly the case of fir timber, then $x = \frac{1}{2}a$ when the velocity is at the greatest. And the quantity of the greatest velocity is then equal to 7·786 feet per second nearly, taking 10 feet for the length of the cylinder.

After this state, the resistance gradually increasing more and more over the urging force, the velocity always decreases till it quite ceases, and the body becomes for an instant stationary. In that case the above expression for the velocity v becomes equal to 0, which consequently gives

$a - x = 2am \times \text{h. l. } \frac{am+a}{am+x}$ for the part out of the water

when the motion ceases. Or if $m = \frac{1}{2}$ as before, and the length of the cylinder be $a = 10$ feet for instance, the last

equation becomes $10 - x = 10 \times \text{h. l. } \frac{15}{5+x}$, and the root of

this equation is $x = 1\frac{1}{4}$ very nearly, or $8\frac{3}{4}$ feet of the cylinder is out of the water when the upward motion ceases.

After the cylinder has arrived at its greatest height $8\frac{3}{4}$, where the upward motion ceases, the cylinder descends again to the same depth as at first, after which it again returns ascending as before; and so on, continually playing up and down to the same highest and lowest points, like the vibrations of a pendulum, the motion ceasing in both cases in a similar manner at the extreme points, then returning, it gradually accelerates till arriving at the middle point, where it is the greatest, then gradually retarding all the way to the next extremity of the vibration, thus making all the vibrations in equal times, to the same extent between the highest and lowest points, except that, by the small tenacity and friction, &c. of the water against the sides of

the cylinder, it will be gradually and slowly retarded in its motion, and the extent of the vibrations decrease till at length the cylinder, like the pendulum, come to rest in the middle point of its vibrations, where it naturally floats in the quiescent state, with the part *am* or half its length above the water.

PROBLEM XX.

Required to determine the quantity of matter in a sphere, the density varying as the nth power of the distance from the centre?

Let r denote the radius of the sphere, d the density at its surface, $a = 3{\cdot}1416$ the area of a circle whose radius is 1, and x any distance from the centre. Then $4ax^2$ will be the surface of a sphere whose radius is x, which may be considered by expansion as generating the magnitude of the solid; therefore $4ax^2\dot{x}$ will be the fluxion of the magnitude; but as $r^n : x^n :: d : \frac{dx^n}{r^n}$ the density at the distance x, therefore $4ax^2\dot{x} \times \frac{dx^n}{r^n} = \frac{4adx^{n+2}\dot{x}}{r^n}$ = the fluxion of the mass, the fluent of which $\frac{4adx^{n+3}}{(n+3)r^n}$, when $x = r$, is $\frac{4adr^3}{n+3}$, the quantity of the matter in the whole sphere.

Corol. 1. The magnitude of a sphere whose radius is r, being $\frac{4}{3}ar^3$, which call m; then the mass or solid content will be $\frac{3d}{n+3} \times m$, and the mean density is $\frac{3d}{n+3}$.

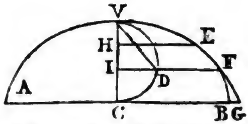
Corol. 2. It having been computed, from actual experiments, that the medium density of the whole mass of the earth is about $\frac{9}{5}$ times the density d at the surface, we can now determine what is the exponent of the decreasing ratio of the density from the centre to the circumference, supposing it to decrease by a regular law, viz. as x^n; for then it will be $\frac{9}{5}d = \frac{3d}{n+3}$, and hence $n = -\frac{4}{3}$. So that, in this case, the law of decrease is as $x^{-\frac{4}{3}}$, or as $\frac{1}{x^{\frac{4}{3}}}$, that is, inversely as the $\frac{4}{3}$ power of the distance from the centre.

Corol. 3. Hence the distance from the centre, where the

density is equal to the mean density, is $x = (\frac{9}{5}) - \frac{3}{4} = \cdot 6435$, or $\cdot 3565$ below the surface, the whole radius being 1. And thus also may be found the place answering to any given density, or the density at any given place.

PROBLEM XXI.

Required to determine where a body, moving down the convex side of a cycloid, will fly off and quit the curve.



Let AVEB represent the cycloid, and VDC its generating semicircle. Let E be the point where the motion commences, whence it moves along the curve, its velocity increasing both on the curve, and also in the horizontal direction DF, till it come to such a point, F suppose, that the velocity in the latter direction is become a constant quantity, then that will be the point where it will quit the cycloid, and afterwards describe a parabola FG, because the horizontal velocity in the latter curve is always the same constant quantity.

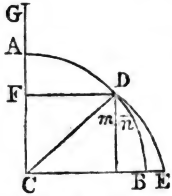
Put the diameter $VC = d$, $VH = a$, $VI = x$; then $VD = \sqrt{dx}$, and $ID = \sqrt{(dx - x^2)}$. Now the velocity in the curve at F in descending down EF, being the same as by falling through HI or $x - a$, by art. 154, p. 204, will be $= 8\sqrt{(x - a)}$; but this velocity in the curve at F, is to the horizontal velocity there, as VD to ID, because VD is parallel to the curve or to the tangent at F, that is $\sqrt{dx} : \sqrt{(dx - x^2)} :: 8\sqrt{(x - a)} : \dfrac{8\sqrt{(x-a)} \times \sqrt{(d-x)}}{\sqrt{d}}$, which is the horizontal velocity at F, where the body is supposed to have that velocity a constant quantity; therefore also $\sqrt{(x - a)} \times \sqrt{(d - x)}$, as well as $(x - a) \times (d - x) = ax + dx - ad - x^2$ is a constant quantity, and also $ax + dx - x^2$; but the fluxion of a constant quantity is equal to nothing, that is $a\dot{x} + d\dot{x} - 2x\dot{x} = 0 = a + d - 2x$, and hence $x = \frac{1}{2}a + \frac{1}{2}d = $ VI, the arithmetical mean between VH and VC.

If the motion should commence at V, then x or VI would be $= \frac{1}{2}d$, and I would be the centre of the semicircle.

PROBLEM XXII.

If a body begin to move from A, with a given velocity, along the quadrant of a circle AB; it is required to show at what point it will fly off from the curve.

Let D denote the point where the body quits the circle ADB, and then describes the parabola DE. Draw the ordinate DF, and let GA be the height producing the velocity at A. Put GA $= a$, AC or CD $= r$, AF $= x$; then the velocity in the curve at D will be the same as that acquired by falling through GF or $a+x$, which is, as before, $8\sqrt{(a+x)}$;



but the velocity in the curve is to the horizontal velocity as Dn to mn or as CD to CF by similar triangles, that is, as

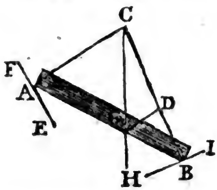
$r : r-x :: 8\sqrt{(x+a)} : 8\sqrt{(x+a)} \times \frac{r-x}{r}$, which is to be a constant quantity where the body leaves the circle, therefore also $(r-x)\sqrt{(x+a)}$ and $(r-x)^2 \times (x+a)$ a constant quantity; the fluxion of which made to vanish, gives $x = \frac{r-2a}{3} =$ AF.

Hence, if $a = 0$, or the body only commence motion at A, then $x = \frac{1}{3}r$, or AF $= \frac{1}{3}$AC when it quits the circle at D. But if a or GA were $= \frac{1}{2}r$ or $\frac{1}{2}$AC, then $r - 2a = 0$, and the body would instantly quit the circle at the vertex A, and describe a parabola circumscribing it, and having the same vertex A.

PROBLEM XXIII.

To determine the position of a bar or beam AB, *being supported in equilibrio by two cords* AC, BC, *having their two ends fixed in the beam, at* A *and* B.

By art. 94, p. 174, the position will be such, that its centre of gravity G will be in the perpendicular or plumb line CG.



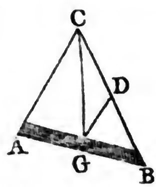
Corol. 1. Draw GD parallel to the cord AC. Then the triangle CGD, having its three sides in the directions of, or parallel to, the three forces, viz. the weight of the beam, and the tensions of the two cords AC, BC, these three forces will be proportional to the three sides CG, GD, CD, respectively, by p. 155; that is, CG is as the weight of the beam, GD as the tension or force of AC, and CD as the tension or force of BC.

Corol. 2. If two planes EAF, HBI, perpendicular to the two cords, be substituted instead of these, the beam will be

still supported by the two planes, just the same as before by the cords, because the action of the planes is in the direction perpendicular to their surface ; and the pressure on the planes will be just equal to the tension or force of the respective cords. So that it is the very same thing, whether the body is sustained by the two cords AC, BC, or by the two planes EF, HI ; the directions and quantities of the forces acting at A and B being the same in both cases. — Also, if the body be made to vibrate about the point C, the points A, B will describe circular arcs coinciding with the touching planes at A, B ; and moving the body up and down the planes, will be just the same thing as making it vibrate by the cords ; consequently the body can only rest, in either case, when the centre of gravity is in the perpendicular CG.

PROBLEM XXIV.

To determine the position of the beam AB, *hanging by one cord* ACB, *having its ends fastened at* A *and* B, *and sliding freely over a tack or pulley fixed at* C.



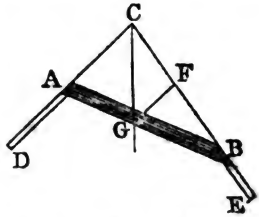
G being the centre of gravity of the beam, CG will be perpendicular to the horizon, as in the last problem. Now as the cord ACB moves freely about the point C, the tension of the cord is the same in every part, or the same both in AC and BC. Draw GD parallel to AC : then the sides of the triangle CGD are proportional to the three forces, the weight and the tensions of the string ; that is, CD and DG are as the forces or tensions in CB and CA. But these tensions are equal ; therefore CD = DG, and consequently the opposite angles DCG and DGC are also equal : but the angle DGC is = the alternate angle ACG ; therefore the angle ACG = BCG ; hence the line CG bisects the vertical angle ACB, and consequently AC : CB : : AG : GB.

PROBLEM XXV.

To determine the position of the two posts AD *and* BE, *supporting the beam* AB, *so that the beam may rest in equilibrio.*

Through the centre of gravity G of the beam, draw CG perp. to the horizon ; from any point C in which draw CAD, CBE through the extremities of the beam ; then AD and BE

will be the positions of the two posts or props required, so as AB may be sustained in equilibrio; because the three forces sustaining any body in such a state, must be all directed to the same point C.



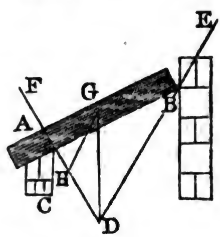
Corol. If GF be drawn parallel to CD; then the quantities of the three forces balancing the beam, will be proportional to the three sides of the triangle CGF, viz. CG as the weight of the beam, CF as the thrust or pressure in BE, and FG as the thrust or pressure in AD.

Scholium. The equilibrium may be equally maintained by the two posts or props AD, BE, as by the two cords AC, BC, or by two planes at A and B perp. to those cords.—It does not always happen that the centre of gravity is at the lowest place to which it can get, to make an equilibrium: for here when the beam AB is supported by the posts DA, EB, the centre of gravity is at the highest it can get; and being in that position, it is not disposed to move one way more than another, and therefore it is as truly in equilibrio, as if the centre was at the lowest point. It is true this is only a tottering equilibrium, and any the least force will destroy it; and then, if the beam and posts be moveable about the angles A, B, D, E, which is all along supposed, the beam will descend till it is below the points D, E, and gain such a position as that G will be at the lowest point, coming there to an equilibrium again. In planes, the centre of gravity G may be either at its highest or lowest point. And there are cases, when that centre is neither at its highest nor lowest point.

PROBLEM XXVI.

Supposing the beam AB *hanging by a pin at* B, *and lying on the wall* AC; *it is required to determine the forces or pressures at the points* A *and* B, *and their directions.*

Draw AD perp. to AB and through G, the centre of gravity of the beam, draw GD perp. to the horizon, to meet AD in D; and join BD. Then the weight of the beam, and the two forces or pressures at A and B, will be in the directions of, and proportional to, the three sides of the triangle GDH, having drawn GH parallel to BD; viz. the weight of the



beam as GD, the pressure at A as HD, and the pressure at B as GH, and in these directions.

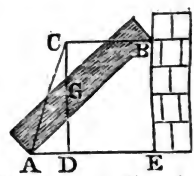
For, the action of the beam is in the direction GD; and the action of the wall at A, is in the perp. AD; conseq. the stress on the pin at B must be in the direction HD, because all the three forces sustaining a body in equilibrio, must tend to the same point, as D.

Corol. 1. If the beam were supported by a pin at A, and laid upon the wall at B; the like construction must be made at B, as has been done at A, and then the forces and their directions will be obtained.

Corol. 2. It is all the same thing, whether the beam is sustained by the pin B and the wall AC, or by two cords BE, AF, acting in the directions DB, DA, and with the forces HG, HD.

PROBLEM XXVII.

To determine the quantities and directions of the forces, exerted by a heavy beam AB, *at its two extremities and its centre of gravity, bearing against a perp. wall at its upper end* B.



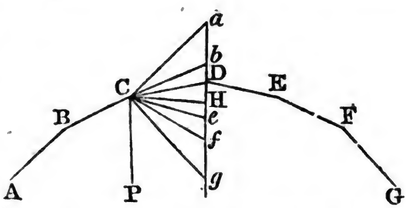
From B draw BC perp. to the face of the wall BE, which will be the direction of the force at B; also through G, the centre of gravity, draw CGD perp. to the horizontal line AE, then CD is the direction of the weight of the beam; and because these two forces meet in the point C, the third force or push A, must be in CA, directly from C; so that the three forces are in the directions CD, BC, CA, or in the directions CD, DA, CA; and, these last three forming a triangle, the three forces are not only in those directions, but are also proportional to these three lines; viz. the weight in or on the beam, as the line CD; the push against the wall at B, as the horizontal line AD; and the thrust at the bottom, as the line AC.

Some of the foregoing problems will be found useful in different cases of carpentry, especially in adapting the framing of the roofs of buildings, so as to be nearest in equilibrio in all their parts. And the last problem, in particular, will be very useful in determining the push or thrust of any arch against its piers or abutments, and thence to assign their thickness necessary to resist that push. The follow-

ing problem will also be of great use in adjusting the form of a mansard roof, or of an arch, and the thickness of every part, so as to be truly balanced in a state of just equilibrium.

PROBLEM XXVIII.

Let there be any number of lines, or bars, or beams, AB, BC, CD, DE, *&c. all in the same vertical plane, connected together and freely moveable about the joints or angles* A, B, C, D, E, *&c., and kept in equilibrio by weights laid on the angles : it is required to assign the proportion of those weights ; as also the force or push in the direction of the said lines ; and the horizontal thrust at every angle.*



Through any point, as D, draw a vertical line *a*DH*g*, &c. ; to which, from any point, as C, draw lines in the direction of, or parallel to, the given lines or beams, viz. C*a* parallel to AB, and C*b* parallel to BC, and C*e* to DE, and C*f* to EF, and C*g* to FG, &c. ; also CH parallel to the horizon, or perpendicular to the vertical line *adg*, in which also all these parallels terminate.

Then will all those lines be exactly proportional to the forces acting or exerted in the directions to which they are parallel, and of all the three kinds, viz. vertical, horizontal, and oblique. That is, the oblique forces or thrusts in direction of the bars . . . . AB, BC, CD, DE, EF, FG,
are proportional to their parallels C*a*, C*b*, CD, C*e*, C*f*, C*g* ;
and the vertical weights on the angles B, C, D, E, F, &c.
are as the parts of the vertical . *ab*, *b*D, D*e*, *ef*, *fg*,
and the weight of the whole frame ABCDEFG,
is proportional to the sum of all the verticals, or to *ag* ; also the horizontal thrust at every angle, is every where the same constant quantity, and is expressed by the constant horizontal line CH.

Demonstration. All these proportions of the forces evidently follow immediately from the general well-known property, in Statics, that when any forces balance and keep each other in equilibrio, they are respectively in proportion as the lines drawn parallel to their directions, and terminating each other.

Thus, the point or angle B is kept in equilibrio by three

forces, viz. the weight laid and acting vertically downward on that point, and by the two oblique forces or thrusts of the two beams AB, CB, and in these directions. But *ca* is parallel to AB, and *cb* to BC, and *ab* to the vertical weight; these three forces are therefore proportional to the three lines *ab*, *ca*, *cb*.

In like manner, the angle C is kept in its position by the weight laid and acting vertically on it, and by the two oblique forces or thrusts in the direction of the bars BC, CD: consequently these three forces are proportional to the three lines *b*D, *cb*, CD, which are parallel to them.

Also, the three forces keeping the point D in its position, are proportional to their three parallel lines, D*e*, CD, *ce*. And the three forces balancing the angle E, are proportional to their three parallel lines *ef*, *ce*, *cf*. And the three forces balancing the angle F, are proportional to their three parallel lines *fg*, *cf*, *cg*. And so on continually, the oblique forces or thrust in the directions of the bars or beams, being always proportional to the parts of the lines parallel to them, intercepted by the common vertical line; while the vertical forces or weights, acting or laid on the angles, are proportional to the parts of this vertical line intercepted by the two lines parallel to the lines of the corresponding angles.

Again, with regard to the horizontal force or thrust: since the line DC represents, or is proportional to the force in the direction DC, arising from the weight or pressure on the angle D; and since the oblique force DC is equivalent to, and resolves into, the two DH, HC, and in those directions, by the resolution of forces, viz. the vertical force DH, and the horizontal force HC; it follows, that the horizontal force or thrust at the angle D, is proportional to the line CH; and the part of the vertical force or weight on the angle D, which produces the oblique force DC, is proportional to the part of the vertical line DH.

In like manner, the oblique force *cb*, acting at C, in the direction CB, resolves into the two *b*H, HC; therefore the horizontal force or thrust at the angle C, is expressed by the line CH, the very same as it was before for the angle D; and the vertical pressure at C, arising from the weights on both D and C, is denoted by the vertical line *b*H.

Also, the oblique force *ac*, acting at the angle B, in the direction BA, resolves into the two *a*H, HC; therefore again the horizontal thrust at the angle B, is represented by the line CH, the very same as it was at the points C and D; and the vertical pressure at B, arising from the weights on B, C, and D, is expressed by the part of the vertical line *a*H.

Thus also, the oblique force *ce*, in direction DE, resolves

into the two CH, He, being the same horizontal force, with the vertical He; and the oblique force cf, in direction EF, resolves into the two CE, Hf; and the oblique force cg, in direction FG, resolves into the two CH, Hg; and so on continually, the horizontal force at every point being expressed by the same constant line CH; and the vertical pressures on the angles by the parts of the verticals. viz. aH the whole vertical pressure at B, from the weights on the angle B, C, D; and bH the whole pressure on C from the weights on C and D: and DH the part of the weight on D causing the oblique force DC; and He the other part of the weight on D causing the oblique pressure DE; and Hf the whole vertical pressure at E from the weights on D and E; and Hg the whole vertical pressure on F arising from the weights laid on D, E, and F. And so on.

So that, on the whole, aH denotes the whole weight on the points from D to A; and Hg the whole weight on the points from D to G; and ag the whole weight on all the points on both sides; while ab, bD, De, ef, fg, express the several particular weights, laid on the angles B, C, D, E, F.

Also, the horizontal thrust is every where the same constant quantity, and is denoted by the line CH.

Lastly, the several oblique forces or thrusts, in the directions AB, BC, CD, DE, EF, FG, are expressed by, or are proportional to, their corresponding parallel lines, ca, cb, cD, ce, cf, cg.

Corol. 1. It is obvious, and remarkable, that the lengths of the bars AB, BC, &c. do not affect or alter the proportions of any of these loads or thrusts; since all the lines, ca, cb, ab, &c. remain the same, whatever be the lengths of AB, BC, &c. The positions of the bars, and the weights on the angles depending mutually on each other, as well as the horizontal and oblique thrusts. Thus, if there be given the position of DC, and the weights or loads laid on the angles D, C, B; set these on the vertical, DH, Db, ba, the cb, ca give the directions or positions of CB, BA, as well as the quantity or proportion CH of the constant horizontal thrust.

Corol. 2. If CH be made radius: then it is evident that Ha is the tangent, and ca the secant of the elevation of ca or AB above the horizon; also Hb is the tangent and cb the secant of the elevation of cb or CB; also HD and CD the tangent and secant of the elevation of CD; also He and ce the tangent and secant of the elevation of ce or DE; also Hf and cf the tangent and secant of the elevation of EF; and so on; also the parts of the vertical ab, bD, ef, fg, denoting the weights laid

on the several angles, are the differences of the said tangents of elevations. Hence then in general,

1st. The oblique thrusts, in the directions of the bars, are to one another, directly in proportion as the secants of their angles of elevation above the horizontal directions; or, which is the same thing, reciprocally proportional to the cosines of the same elevations, or reciprocally proportional to the sines of the vertical angles, a, b, D, e, f, g, &c. made by the vertical line with the several directions of the bars; because the secants of any angles are always reciprocally in proportion as their cosines.

2. The weight or load laid on each angle, is directly proportional to the difference between the tangents of the elevations above the horizon, of the two lines which form the angle.

3. The horizontal thrust at every angle, is the same constant quantity, and has the same proportion to the weight on the top of the uppermost bar, as radius has to the tangent of the elevation of that bar. Or, as the whole vertical ag, is to the line CH, so is the weight of the whole assemblage of bars, to the horizontal thrust. Other properties also, concerning the weights and the thrusts, might be pointed out, but they are less simple and elegant than the above, and are therefore omitted; the following only excepted, which are inserted here on account of their usefulness.

Corol. 3. It may hence be deduced also, that the weight or pressure laid on any angle, is directly proportional to the continual product of the sine of that angle and of the secants of the elevations of the bars or lines which form it. Thus, in the triangle bCD, in which the side bD is proportional to the weight laid on the angle C, because the sides of any triangle are to one another as the sines of their opposite angles, therefore as sin. D : Cb : : sin. bCD : bD; that is, bD is as $\frac{\text{sin. } b\text{CD}}{\text{sin. D}} \times \text{C}b$; but the sine of angle D is the cosine of the elevation DCH, and the cosine of any angle is reciprocally proportional to the secant, therefore bD is as sin bCD $\times$ sec. DCH $\times$ Cb; and Cb being as the secant of the angle bCH of the elevation of bC or BC above the horizon, therefore bD is as sin. bCD $\times$ sec. bCH $\times$ sec. DCH; and the sine of bCD being the same as the sine of its supplement BCD; therefore the weight on the angle C, which is as bD, is as the sin. BCD $\times$ sec. DCH $\times$ sec. bCH, that is, as the continual product of the sine of that angle, and the secants of the elevations of its two sides above the horizon.

Corol. 4. Further, it easily appears also, that the same weight on any angle C, is directly proportional to the sine of that angle BCD, and inversely proportional to the sines of the two parts BCP, DCP, into which the same angle is divided by the vertical line CP. For the secants of angles are reciprocally proportional to their cosines or sines of their complements: but BCP = *c*bH, is the complement of the elevation *b*CH, and DCP is the complement of the elevation DCH; therefore the secant of *b*CH × secant of DCH is reciprocally as the sin. BCP × sin. DCP; also the sine of *b*CD is = the sine of its supplement BCD; consequently the weight on the angle C, which is proportional to sin. *b*CD × sec. *b*CH × sec. DCH, is also proportional to $\frac{\text{sin. BCD}}{\text{sin. BCP} \times \text{sin. DCP}}$, when the whole frame or series of angles is balanced, or kept in equilibrio, by the weights on the angles; the same as in the preceding proposition.

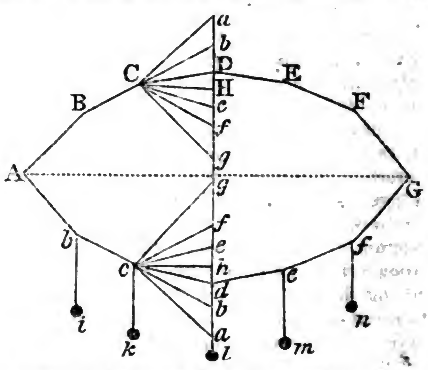
Scholium. The foregoing proposition is very fruitful in its practical consequences, and contains the whole theory of arches, which may be deduced from the premises by supposing the constituting bars to become very short, like arch stones, so as to form the curve of an arch. It appears too, that the horizontal thrust, which is constant and uniformly the same throughout, is a proper measuring unit, by means of which to estimate the other thrusts and pressures, as they are all determinable from it and the given positions; and the value of it, as appears above, may be easily computed from the uppermost or vertical part alone, or from the whole assemblage together, or from any part of the whole, counted from the top downwards.

The solution of the foregoing proposition depends on this consideration, viz. that an assemblage of bars or beams, being connected together by joints at their extremities, and freely moveable about them, may be placed in such a vertical position, as to be exactly balanced, or kept in equilibrio, by their mutual thrusts and pressures at the joints; and that the effect will be the same if the bars themselves be considered as without weight, and the angles be pressed down by laying on them weights which shall be equal to the vertical pressures at the same angles, produced by the bars in the case when they are considered as endued with their own natural weights. And as we have found that the bars may be of any length, without affecting the general properties and proportions of the thrusts and pressures, therefore by supposing them to become short, like arch stones, it is plain that we shall then have the same principles and properties accommodated to a real arch of equilibration, or one that supports itself in a per.

fect balance. It may be further observed, that the conclusions here derived, in this proposition and its corollaries, exactly agree with those derived in a very different way, in my principles of bridges, viz. in propositions 1 and 2, and their corollaries.

PROBLEM XXIX.

If the whole figure in the last problem be inverted, or turned round the horizontal line AG *as an axis, till it be completely reversed, or in the same vertical plane below the first position, each angle* D, *d, &c. being in the same plumb line; and if weights i, k, l, m, n, which are respectively equal to the weights laid on the angles* B, C, D, E, F, *of the first figure, be now suspended by threads from the corresponding angles b, c, d, e, f, of the lower figure; it is required to show that those weights keep this figure in exact equilibrio, the same as the former, and all the tensions or forces in the latter case, whether vertical or horizontal or oblique, will be exactly equal to the corresponding forces of weight or pressure or thrust in the like directions of the first figure.*

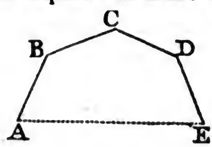


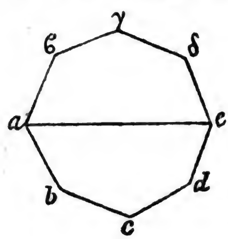
This necessarily happens, from the equality of the weights, and the similarity of the positions, and actions of the whole in both cases. Thus, from the equality of the corresponding weights, at the like angles, the ratios of the weights, *ab*, *bd*, *dh*, *he*, &c. in the lower figure, are the very same as those, *ab*, *b*D, DH, H*e*, &c. in the upper figure; and from the equality of the constant horizontal forces CH, *ch*, and the similarity of the positions, the corresponding vertical lines, denoting the weights, are equal, namely, $ab = ab$, $b\text{D} = bd$, $\text{DH} = dh$, &c. The same may be said of the oblique lines also, *ca*, *cb*, &c. which being parallel to the beams A*b*, *bc*, &c. will denote

the tensions of these, in the direction of their length, the same as the oblique thrusts or pushes in the upper figures. Thus, all the corresponding weights and actions, and positions, in the two situations, being exactly equal and similar, changing only drawing and tension for pushing and thrusting, the balance and equilibrium of the upper figure is still preserved the same in the hanging festoon or lower one.

Scholium. The same figure, it is evident, will also arise, if the same weights, *i*, *k*, *l*, *m*, *n*, be suspended at like distances A*b*, *bc*, &c. on a thread, or cord, or chain, &c. having in itself little or no weight. For the equality of the weights, and their directions and distances, will put the whole line, when they come to equilibrium, into the same festoon shape of figure. So, that, whatever properties are inferred in the corollaries to the foregoing prob. will equally apply to the festoon or lower figure hanging in equilibrio.

This is a most useful principle in all cases of equilibriums, especially to the mere practical mechanist, and enables him in an experimental way to resolve problems, which the best mathematicians have found it no easy matter to effect by mere computation. For thus, in a simple and easy way he obtains the shape of an equilibrated arch or bridge; and thus also he readily obtains the positions of the rafters in the frame of an equilibrated curb or mansard roof; a single instance of which may serve to show the extent and uses to which it may be applied. Thus, if it should be required to make a curb frame roof having a given width AE, and consisting of four rafters AB, BC, CD, DE, which shall either be equal or in any given proportion to each other. There can be no doubt but that the best form of the roof will be that which puts all its parts in equilibrio, so that there may be no unbalanced parts which may require the aid of ties or stays to keep the frame in its position. Here the mechanic has nothing to do, but to take four like but small pieces, that are either equal or in the same given proportions as those proposed, and connect them closely together at the joints A, B, C, D, E, by pins or strings, so as to be freely moveable about them; then suspend this from two pins, *a*, *e*, fixed in a horizontal line, and the chain of the pieces will arrange itself in such a festoon or





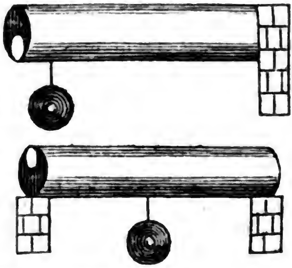
form, *abcde*, that all its parts will come to rest in equilibrio. Then, by inverting the figure, it will exhibit the form and frame of a curb roof $a\beta\gamma\delta e$, which will also be in equilibrio, the thrusts of the pieces now balancing each other, in the same manner as was done by the mutual pulls or tensions of the hanging festoon *a b c d e*. By varying the distance *ae*, of the points of suspension, moving them nearer to, or farther off, the chain will take different forms; then the frame ABCDE may be made similar to that form which has the most pleasing or convenient shape, found above as a model.

Indeed this principle is exceeding fruitful in its practical consequences. It is easy to perceive that it contains the whole theory of the construction of arches: for each stone of an arch may be considered as one of the rafters or beams in the foregoing frames, since the whole is sustained by the mere principle of equilibration, and the method, in its application, will afford some elegant and simple solutions of the most difficult cases of this important problem.

PROBLEM XXX.

Of all hollow cylinders, whose lengths and the diameters of the inner and outer circles continue the same, it is required to show what will be the position of the inner circle when the cylinder is the strongest laterally.

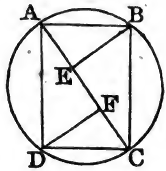
Since the magnitude of the two circles are constant, the area of the solid space, included between their two circumferences, will be the same, whatever be the position of the inner circle, that is, there is the same number of fibres to be broken, and in this respect the strength will be always the same. The strength then can only vary according to the situation of the centre of gravity of the solid part, and this again will depend on the place where the cylinder must first break, or on the manner in which it is fixed.



Now, the cylinder is strongest when the hollow, or inner circle, is nearest to that side where the fracture is to end, that is, at the bottom when it breaks first at the upper side, or when the cylinder is fixed only at one end as in the first figure. But the reverse will be the case when the cylinder is fixed at both ends; and consequently when it opens first below, or ends above, as in the 2d figure annexed.

PROBLEM XXXI.

To determine the dimensions of the strongest rectangular beam that can be cut out of a given cylinder.



Let AB, the breadth of the required beam, be denoted by b, AD the depth by d, and the diameter AC of the cylinder by D. Now when AB is horizontal, the lateral strength is denoted by bd^2 (by page 391), which is to be a maximum. But $AD^2 = AC^2 - AB^2$, or $d^2 = D^2 - b^2$; theref. $bd^2 = (D^2 - b^2)b = D^2 b - b^3$ is a maximum: in fluxions $D^2\dot{b} - 3b^2\dot{b} = 0 = D^2 - 3b^2$, or $D^2 = 3b^2$; also $d^2 = D^2 - b^2 = 3b^2 - b^2 = 2b^2$. Conseq. $b^2 : d^2 : D^2 :: 1 : 2 : 3$, that is, the squares of the breadth, and of the depth, and of the cylinder's diameter, are to one another respectively as the three numbers 1, 2, 3.

Corol. 1. Hence results this easy practical construction: divide the diameter AC into three equal parts, at the points E, F; erect the perpendiculars EB, FD; and join the points, B, D to the extremities of the diameter: so shall ABCD be the rectangular end of the beam as required. For, because AE, AB, AC are in continued proportion (theor. 87, Geom.), theref. $AE : AC :: AB^2 : AC^2$; and in like manner $AF : AC :: AD^2 : AC^2$; hence $AE : AF : AC :: AB^2 : AD^2 : AC^2 :: 1 : 2 : 3$.

Corol. 2. The ratios of the three b, d, D, being as the three $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, or as 1, 1·414, 1·732, are nearly as the three 5, 7, 8·6, or more nearly as 12, 17, 20·8.

Corol. 3. A square beam cut out of the same cylinder, would have its side $= D\sqrt{\frac{1}{2}} = \frac{1}{2}D\sqrt{2}$. And its solidity would be to that of the strongest beam, as $\frac{1}{2}D^2$ to $\frac{1}{3}D^2\sqrt{2}$, or as 3 to $2\sqrt{2}$, or as 3 to 2·828; while its strength would be to that of the strongest beam, as $(D\sqrt{\frac{1}{2}})^3$ to $D\sqrt{\frac{1}{3}} \times \frac{2}{3}D^2$, or as $\frac{1}{4}\sqrt{2}$ to $\frac{2}{9}\sqrt{3}$, or as $9\sqrt{2}$ to $8\sqrt{3}$, or nearly as 101 to 110.

Corol. 4. Either of these beams will exert the greatest lateral strength, when the diagonal of its end is placed vertically.

Corol. 5. The strength of the whole cylinder will be to that of the square beam, when placed with its diagonal vertically, as the area of the circle to that of its inscribed square. For, the centre of the circle will be the centre of gravity of both beams, and is at the distance of the radius from the

lowest point in each of them; conseq. their strengths will be as their areas.

PROBLEM XXXII.

To determine the difference in the strength of a triangular beam, according as it lies with the edge or with the flat side upwards.

In the same beam, the area is the same, and therefore the strength can only vary with the distance of the centre of gravity from the highest or lowest point; but, in a triangle, the distance of the centre of gravity from an angle, is double of its distance from the opposite side; therefore the strength of the beam will be as 2 to 1 with the different sides upwards, under different circumstances, viz. when the centre of gravity is farthest from the place where the fracture ends; that is, with the angle upwards when the beam is supported at both ends; but with the side upwards, when it is supported only at one end, because in the former case the beam breaks first below, but the reverse in the latter case.

PROBLEM XXXIII.

Given the length and weight of a cylinder or prism, placed horizontally with one end firmly fixed, and will just support a given weight at the other end without breaking; it is required to find the length of a similar prism or cylinder which, when supported in like manner at one end, shall just bear without breaking another given weight at the unsupported end.

Let l denote the length of the given cylinder or prism, d the diameter or depth of its end, w its weight, and u the weight hanging at the unsupported end; also let the like capitals L, D, W, U, denote the corresponding particulars of the other prism or cylinder. Then, the weights of similar solids of the same matter being as the cubes of their lengths, as $l^3 : \mathrm{L}^3 :: w : \frac{\mathrm{L}^3}{l^3}w$, the weight of the prism whose length is L. Now $\frac{1}{2}wl$ will be the stress on the first beam by its own weight w acting at its centre of gravity, or at half its length; and lu the stress of the added weight u at its extremity, their sum $(\frac{1}{2}w + u)l$ will therefore be the whole stress on the given beam; in like manner the whole stress on the other beam,

whose weight is W or $\frac{L^3}{l^3}w$, will be $(\frac{1}{2}W+U)L$ or $(\frac{L^3}{2l^3}w+U)L$.

But the lateral strength of the first beam is to that of the second, as d^3 to D^3, or as l^3 to L^3; and the strengths and stresses of the two beams must be in the same ratio, to answer the conditions of the problem; therefore as $(\frac{1}{2}W+u)l : \frac{L^3}{2l^3}w + U)L :: l^3 : L^3$; this analogy, turned into an equation, gives $L^3 - \frac{w+2u}{w}lL^2 + \frac{2}{w}l^2U = 0$, a cubic equation, from which the numeral values of L may be easily determined, when those of the other letters are known.

Corol. 1. When U vanishes, the equation gives $L^3 = \frac{w+2u}{w}lL^2$, or $L = \frac{w+2u}{w}l$, whence $w : w + 2u :: l : L$, for the length of the beam, which will but just support its own weight.

Corol. 2. If a beam just only support its own weight when fixed at one end; then a beam of double its length, fixed at both ends, will also just sustain itself: or if the one just break, the other will do the same.

PROBLEM XXXIV.

Given the length and weight of a cylinder or prism, fixed horizontally as in the foregoing problem, and a weight which, when hung at a given point, breaks the prism: it is required to determine how much longer the prism, of equal diameter or of equal breadth and depth, may be extended before it break, either by its own weight, or by the addition of any other adventitious weight.

Let l denote the length of the given prism, w its weight, and u a weight attached to it at the distance d from the fixed end; also let L denote the required length of the other prism, and U the weight attached to it at the distance D. Now the strain occasioned by the weight of the first beam is $\frac{1}{2}wl$, and that by the weight u at the distance d, is du, their sum $\frac{1}{2}wl + du$ being the whole strain. In like mannnr $\frac{1}{2}WL + DU$ is the strain on the second beam; but $l : L :: w : \frac{Lw}{l} = W$ the weight of this beam, theref. $\frac{wL^2}{2l} + DU =$ its strain. But the

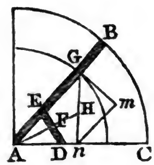
strength of the beam, which is just sufficient to resist these strains, is the same in both cases; therefore $\frac{w\text{L}^2}{2l} + \text{DU} = \frac{1}{2}wl + du$, and hence, by reduction, the required length $\text{L} = \sqrt{(l \times \frac{wl + 2du - 2\text{DU}}{w})}$.

Corol. 1. When the lengthened beam just breaks by its own weight, then $\text{U}=0$ or vanishes, and the required length becomes $\text{L} = \sqrt{(l \times \frac{wl + 2du}{w})}$.

Corol. 2. Also when U vanishes, if d become $= l$, then $\text{L} = l\sqrt{\frac{w+2u}{w}}$ is the required length.

PROBLEM XXXV.

Let AB *be a beam moveable about the end* A, *so as to make any angle* BAC *with the plane of the horizon* AC: *it is required to determine the position of a prop or supporter* DE *of a given length, which shall sustain it with the greatest ease in any given position; also to ascertain the angle* BAC *when the least force which can sustain* AB, *is greater than the least force in any other position.*



Let G be the centre of gravity of the beam; and draw Gm perp. to AB, Gn to AC, nm to Gm, and AFH to DE. Put $r = \text{AG}$, $p = \text{DE}$, $w =$ the weight of the beam AB, and An $= x$. Then, by the nature of the parallelogram of forces, Gn : Gm, or by sim. triangles, $\text{AG} = r : \text{A}n = x :: w : \frac{wx}{r}$, the force which acting at G in the direction mG, is sufficient to sustain the beam; and, by the nature of the lever, $\text{AE} : \text{AG} = r :: \frac{x\text{W}}{\text{AG}}$ the requisite force at G : $\frac{wx}{\text{AE}}$, the force capable of supporting it at E in a direction perp. to AB or parallel to mG; and again as $\text{AF} : \text{AE} :: \frac{wx}{\text{AE}} : \frac{wx}{\text{AF}}$, the force or pressure actually sustained by the given prop DE in a direction perp. to AF. And this latter force will manifestly be the least possible when the perp. AF upon DE is the greatest possible, whatever the angle BAC

may be, which is when the triangle ADE is isosceles, or has the side AD = AE.

Secondly, for a solution to the latter part of the problem, we have to find when $\frac{wx}{\text{AF}}$ is a maximum; the angles D and E being always eqaul to each other, while they vary in magnitude by the change in the position of AB. Let AF produced meet Gn in H: then, in the similar triangles ADF, AHn, it will be AF : An = x : : DF = $\frac{1}{2}p$: Hn, hence $\frac{x}{\text{AF}} = \frac{\text{H}n}{\frac{1}{2}p}$, and conseq. $\frac{x}{\text{F}} \times w = \frac{\text{H}n}{\frac{1}{2}p} \times w$. But, by theor. 83 Geom. and comp. AG + An = $r + x$: An = x : : Gn = $\sqrt{(r^2 - x^2)}$: Hn = $\frac{x}{r+x}\sqrt{(r^2-x^2)} = x\sqrt{\frac{r-x}{r+x}}$: consequently the force $\frac{\text{H}n}{\frac{1}{2}p} \times w$, acting on the prop, is also truly expressed by $\frac{wx}{\frac{1}{2}p}\sqrt{\frac{r-x}{r+x}}$. Then the fluxion of this made to vanish gives $x = \frac{\sqrt{5}-1}{2}r$ the cos. angle BAC = 51° 50′, the inclination required.

PROBLEM XXXVI.

Suppose the beam AB, *instead of being moveable about the centre* A, *as in the last problem, to be supported in a given position by means of the given prop* DE: *it is required to determine the position of that prop, so that the prismatic beam* AC, *on which it stands, may be the least liable to breaking, this latter beam being only supported at its two ends* A *and* C.

Put the base AC = b, the prop. DE = p, AG = r, the weight of AB = w, s and c the sine and cosine of $\angle$A, x = sin. $\angle$E, y = sin. $\angle$D, and z = AE. Then, by trigon. z : y : : p : s, or $\frac{y}{x} = \frac{s}{p}$, and AD = $\frac{px}{s}$; also cw = the force of the beam at G in direction Gm. Let F denote the force sustaining the beam at E in the direction ED: then, because action and reaction are equal and opposite, the same force will be exerted at D in the direction DE: therefore AG . cw = Fzx, and F = $\frac{rcw}{zx}$. Again, the vertical stress at D, will be as F × sine D × AD . DC = Fy . AD . DC = $\frac{rcwy}{zx}$ ×

$\frac{px}{s}(b - \frac{px}{s}) =$ (substituting $\frac{s}{p}$ for its equal $\frac{y}{z}$) $\frac{rcws}{px} \times \frac{px}{s} \times \frac{bs-px}{s} = rcw \times \frac{bs-px}{s} = \frac{rcwp}{s} \times (\frac{bs}{p} - x) =$ a minimum by the problem. Conseq. $\frac{bs}{p} - x$ is a minimum, or x a maximum, that is, $x = 1$, and the angle E is a right angle. Hence the point E is easily found by this proportion, sin. A : cos. A : : ED : EA.

This problem may also be solved *geometrically*, thus :—

The stress or pressure upon the prop ED, is as $\frac{1}{\text{AF}}$ (AF $\perp$ ED). The force upon D in perp. direction $\propto \frac{1}{\text{AD}}$ (it being to the absolute force in direction ED, as AF : AD).

But the force necessary to break the beam at D $\propto \frac{1}{\text{AD . DC}}$:

And $\frac{1}{\text{AD . DC}} : \frac{1}{\text{AD}} :: \frac{1}{\text{AD . DC}} : \frac{\text{DC}}{\text{AD . DC}} :: 1 : \text{DC}$.

Conseq. DC must be a min. and $\therefore$ AD a max.
But AD : sin E in the given ratio of DE : sin A.
Conseq. AD will be a max. when sin E is a max. which is evidently, when E is a right angle.
$\therefore$ make EA : ED : : cos A : sin A.

Place the given piece ED at the point E, so as to make right angles with AB ; and the point D, where it meets AC, is that in which it has the least tendency to break it.

PROBLEM XXXVII.

To explain the disposition of the parts of machines.

When several pieces of timber, iron, or any other materials, are employed in a machine or structure of any kind, all the parts, both of the same piece, and of the different pieces in the fabric, ought to be so adjusted with respect to magnitude, that the strength in every part may be, as near as possible, in a constant proportion to the stress or strain to which they will be subjected. Thus, in the construction of any engine, the weight and pressure on every part should be investigated, and the strength apportioned accordingly. All levers, for instance, should be made strongest where they are most strained : viz. levers of the first kind, at the fulcrum ; levers of the second kind, where the weight acts ; and those of the third kind, where the power is applied. The axles of wheels and pulleys, the teeth of wheels, also ropes, &c. must be made

stronger or weaker, as they are to be more or less acted on. The strength allotted should be more than fully competent to the stress to which the parts can ever be liable; but without allowing the surplus to be extravagant: for an over excess of strength in any part, instead of being serviceable, would be very injurious, by increasing the resistance the machine has to overcome, and thus encumbering, impeding, and even preventing the requisite motion; while, on the other hand, a defect of strength in any part will cause a failure there, and either render the whole useless, or demand very frequent repairs.

PROBLEM XXXVIII.

To ascertain the strength of various substances.

The proportions that we have given on the strength and stress of materials, however true, according to the principles assumed, are of little or no use in practice, till the comparative strength of different substances is ascertained: and even then they will apply more or less accurately to different substances. Hitherto they have been applied almost exclusively to the resisting force of beams of timber; though probably no materials whatever accord less with the theory than timber of all kinds. In the theory, the resisting body is supposed to be perfectly homogeneous, or composed of parallel fibres, equally distributed round an axis, and presenting uniform resistance to rupture. But this is not the case in a beam of timber: for, by tracing the process of vegetation, it is readily seen that the ligneous coats of a tree, formed by its annual growth, are almost concentric; being like so many hollow cylinders thrust into each other, and united by a kind of medullary substance, which offers but little resistance: these hollow cylinders therefore furnish the chief strength and resistance to the force which tends to break them.

Now, when the trunk of a tree is squared, in order that it may be converted into a beam, it is plain that all the ligneous cylinders greater than the circle inscribed in the square or rectangle, which is the transverse section of the beam, are cut off at the sides; and therefore almost the whole strength or resistance arises from the cylindric trunk inscribed in the solid part of the beam; the portions of the cylindric coats, situated towards the angles, adding but little comparatively to the strength and resistance of the beam. Hence it follows that we cannot, by legitimate comparison, accurately deduce the strength of a joist, cut from a small tree, by experiments on another which has been sawn from a much larger tree or block. As to the concentric cylinders above mentioned, they are evidently not all of equal strength; those nearest the

centre, being the oldest, are also the hardest and strongest; which again is contrary to the theory, in which they are supposed uniform throughout. But yet, after all, however, it is still found that, in some of the most important problems, the results of the theory and well-conducted experiments coincide, even with regard to timber; thus, for example, the experiments on rectangular beams afford results deviating but in a very slight degree from the theorem, that the strength is proportional to the product of the breadth and the square of the depth.

Experiments on the strength of different kinds of wood are by no means so numerous as might be wished: the most useful seem to be those made by Muschenbroek, Buffon, Emerson, Parent, Banks, Girard, Barlow, and Tredgold. But it will be at all times highly advantageous to make new experiments on the same subject; a labour especially reserved for engineers who possess skill and zeal for the advancement of their profession. It has been found by experiments, that the same kind of wood, and of the same shape and dimensions, will bear or break with very different weights: that one piece is much stronger than another, not only cut out of the same tree, but out of the same rod; and that even, if a piece of any length, planed equally thick throughout, be separated into three or four pieces of an equal length, it will often be found that these pieces require different weights to break them. Emerson observes that wood from the boughs and branches of trees is far weaker than that of the trunk or body; the wood of the large limbs stronger than that of the smaller ones; and the wood in the heart of a sound tree strongest of all; though some authors differ on this point. It is also observed that a piece of timber which has borne a great weight for a short time has broke with a far less weight, when left upon it for a much longer time. Wood is also weaker when green, and strongest when thoroughly dried, in the course of two or three years, at least. Wood is often very much weakened by knots in it; also when cross-grained, as often happens in sawing, it will be weakened in a greater or less degree, according as the cut runs more or less across the grain. From all which it follows, that a considerable allowance ought to be made for the various strength of wood, when applied to any use where strength and durability are required.

Iron is much more uniform in its strength than wood. Yet experiments show that there is some difference arising from different kinds of ore: a difference is also found not only in iron from different furnaces, but from the same furnace, and even from the same melting; which may arise in a great

measure from the different degrees of heat it has when poured into the mould.

Every beam or bar, whether of wood, iron, or stone, is more easily broken by any transverse strain, while it is also suffering any very great compression endways; so much so indeed that we have sometimes seen a rod, or a long slender beam, when used as a prop or shore, urged home to such a degree, that it has burst asunder with a violent spring. Several experiments have been made on this kind of strain: a piece of white marble, $\frac{1}{4}$ of an inch square, and 3 inches long, bore 38 lbs.; but when compressed endways with 300 lbs. it broke with $14\frac{1}{2}$ lbs. The effect is much more observable in timber, and more elastic bodies; but is considerable in all. This is a point therefore that must be attended to in all experiments; as well as the following, viz. that a beam supported at both ends will carry almost twice as much when the ends beyond the props are kept from rising, as when the beam rests loosely on the props.

The following list of the absolute strength of several materials is extracted from the collection made by professor Robison, from the experiments of Muschenbroek and other experimentalists. The specimens are supposed to be prisms or cylinders of one square inch transverse area, which are stretched or drawn lengthways by suspended weights, gradually increased till the bars parted or were torn asunder, by the number of avoirdupois pounds, on a medium of many trials, set opposite each name.

1st. Metals.

	lbs.		lbs.
Gold, cast	22,000	Tin, cast	5,000
Silver, cast	42,000	Lead, cast	860
Copper, cast . . .	34,000	Regulus of Antimony	1,000
Iron, cast	50,000	Zinc	2,600
Iron, bar	70,000	Bismuth	2,900
Steel, bar	135,000		

It is very remarkable that almost all the metallic mixtures are more tenacious than the metals themselves. The change of tenacity depends much on the proportion of the ingredients; and yet the proportion which produces the most tenacious mixture is different in the different metals. The proportion of ingredients here selected, is that which produces the greatest strength.

	lbs.		lbs.
2 parts gold with 1 silver . . .	28,000	5 pts gold, 1 copper .	50,000
4 silver, 1 tin .	41,000	5 silver, 1 copper .	48,500
		8 tin, 1 zinc . . .	10,000

6 copper, 1 tin . .	60,000	4 tin, 1 regul. antim.	12,000
Brass, of copper & tin	51,000	8 lead, 1 zinc. . .	4,500
3 tin, 1 lead . .	10,200	4 tin, 1 lead, 1 zinc	13,000

These numbers are of considerable use in the arts. The mixtures of copper and tin are particularly interesting in the fabric of great guns. By mixing copper, whose greatest strength does not exceed 37,000, with tin which does not exceed 6000, is produced a metal whose tenacity is almost double, at the same time that it is harder and more easily wrought: it is however more fusible. We see also that a very small addition of zinc almost doubles the tenacity of tin, and increases the tenacity of lead 5 times; and a small addition of lead doubles the tenacity of tin. These are economical mixtures; and afford valuable information to plumbers for augmenting the strength of water-pipes.

Brass being peculiarly liable to decomposition in the atmosphere of London, Captain Kater directed Mr. Bate, of the Poultry, the artist employed to conduct the new standard of linear measure, to make some experiments, in order to ascertain the proportions of tin and copper, which might produce a metal equal in hardness, and which might be worked with the same facility as hammered brass; and after some trials, it was found that a mixture of 576 parts of copper, 59 of tin, and 48 of brass, afforded a beautiful metal, which possessed all the qualities desired.

2d. Woods, &c.

	lbs.		lbs.
Locust tree . . .	20,100	Tamarind	8,750
Jujeb	18,500	Fir	8,330
Beech, Oak . .	17,300	Walnut	8,130
Orange	15,500	Pitch pine	7,650
Alder	13,900	Quince	6,750
Elm	13,200	Cypress	6,000
Mulberry . . .	12,500	Poplar	5,500
Willow . . .	12,500	Cedar	4,880
Ash	12,000	Ivory	16,270
Plum	11,800	Bone	5,250
Elder	10,000	Horn	8,750
Pomegranate .	9,750	Whalebone . . .	7,500
Lemon	9,250	Tooth of sea-calf . .	4,075

It may be said in general, that $\frac{2}{3}$ of these weights will sensibly impair the strength, after acting a considerable while, and that one-half is the utmost that can remain permanently suspended at the rods with safety; and it is this last allotment that the engineer should reckon upon in his

constructions. There is, however, considerable difference in this respect: woods of a very straight fibre, such as fir, will be less impaired by any load which is not sufficient to break them immediately. Perhaps, also, subsequent experiments have been more correctly conducted. According to Mr. Emerson, the load which may be safely suspended to an inch square of various materials, is as follows.

	lbs.		lbs.
Iron	76,400	Red fir, holly, elder, plane	5,000
Brass	35,600	Cherry, hazel	4,760
Hempen rope	19,600	Alder, asp, birch, willow	4,290
Ivory	15,700	Freestone	914
Oak, box, yew, plum	7,850	Lead	430
Elm, ash, beech	6,070		
Walnut, plum	5,360		

He gives farther the practical rule, that a cylinder whose diameter is d inches loaded to $\frac{1}{4}$ of its absolute strength, will carry permanently as here annexed.

	cwt.
Iron	$135d^2$
Good rope	$22d^2$
Oak	$14d^2$
Fir	$9d^2$

We may here also introduce a valuable table of the strength of cohesion of wood, from very cautious experiments by Mr. B. Bevan, Civil Engineer.

Species of Wood.	Spec. Grav.	Cohesion in lbs.	Species of Wood.	Spec. Grav.	Cohesion in lbs.
1. Acacia	.85	16,000+	27. Mahogany	.80	16,500
2. Ash	.84	16,700	28. Maple	.66	17,400
3. Ditto	.78	19,600	29. Mulberry	.66	10,600
4. Beech	.72	22,200	30. Oak, English	.70	19,800+
5. Birch	.64	15,000—	31. Ditto	.76	15,000
6. Box	.99	15,500—	32. Ditto, old	.76	14,000
7. Cane	.40	6,300	33. Oak pile out of the river Cam	.61	4,500
8. Cedar	.54	11,400	34. Oak, black Linc. log	.67	7,700—
9. Chestnut (horse)	.61	12,100—	35. Oak, Hamboro'	.66	16,300+
10. Ditto (sweet)	.61	10,500—	36. Ditto ditto	.6	14,000
11. Damson	.79	14,000	37. Pine, Petersburg	.49	13,300—
12. Deal, Norway spruce	.34	18,100+	38. Do. Norway.	.59	12,400—
13. Ditto ditto		17,600+	39. Ditto ditto	.66	14,300
14. Do. Christiana	.46	12,400	40. Do. Petersburg	.55	13,100+
15. Ditto ditto	.46	12,300	41. Poplar	.36	7,200—
16. Ditto ditto	.46	14,000	42. Sallow	.70	18,600
17. Do. English	.47	7,000	43. Sycamore	.69	13,000+
18. Elder	.73	15,000	44. Teak, old	.53	8,200
19. Hawthorn	.91	10,700—	45. Walnut	.59	7,800
20. Ditto		9,200	46. Willow	.39	14,000
21. Holly	.76	16,000	47. Yew	.79	8,000
22. Laburnum	.92	10,500			
23. Lance-wood	1.01	23,400+			
24. Lignum Vitæ	1.22	11,800			
25. Lime-tree	.76	23,500+			
26. Mahogany	.87	21,800+			

Supplementary table of various substances.

	Spec. gr.	Cohesion per square inch. lbs.
Apple	·71	19,500
Elm	·69	14,400
Hazel	·86	18,000+
Hornbeam	·82	20,240+
Larc	·57	8,900—
Plane	·64	11,700—

Nearly all the species of wood submitted to longitudinal strain, for obtaining the force of cohesion per square inch, were also subjected by Mr. Bevan to transverse fracture by a load applied to the middle of a bar placed horizontally, and supported at each end.

If l = length, b = breadth, d = depth of the prism, all in inches; w = the weight in lbs. applied to the middle of the beam; c = cohesion (as above) per square inch: then, if the resistance to compression were equal to that of extension, we should have $\frac{1 \cdot 5lw}{bd^2} = c$: the mean result of Mr. Bevan's experiments gives, for dry and seasoned wood,

$$\frac{2lw}{bd^2} = c, \text{ or } \frac{cbd^2}{2l} = w.$$

The reader may advantageously compare these formulæ and results, with those from *Mr. Barlow*, given at page 391, &c.

Experiments on the transverse strength of bodies are easily made, and accordingly are very numerous, especially those made on timber, being the most common and the most interesting. A very complete series is that given by Belidor, in his Science des Ingenieurs, and is exhibited in the following table. The first column simply indicates the number of experiments; the column b shows the breadth of the pieces, in inches; the column d contains their depths; the column l shows the lengths; and column *lbs* shows the weights in pounds which broke them, when suspended by their middle points, being the medium of 3 trials of each piece; the accompanying words, *fixed* and *loose*, denoting whether the ends were firmly fixed down, or simply lay loose on the supports.

No.	b	d	l	*lbs.*	
1	1	1	18	406	loose.
2	1	1	18	608	fixed.
3	2	1	18	805	loose.
4	1	2	18	1580	loose.
5	1	1	36	187	loose.
6	1	1	36	283	fixed.
7	2	2	36	1585	loose.
8	$1\frac{2}{3}$	$2\frac{1}{3}$	36	1660	loose.

By comparing experiments 1 and 3, the strength appears proportional to the breadth.

Experiments 3 and 4 show the strength to be as the breadth multiplied by the square of the depth.

Experiments 1 and 5 show the strength nearly in the inverse ratio of the lengths, but with a sensible deficiency in the longer pieces.

Experiments 5 and 7 show the strength to be proportional to the breadth and the square of the depth.

Experiments 1 and 7 show the same thing, compounded with the inverse ratio of the length; the deficiency of which is not so remarkable here.

Experiments 1 and 2, and experiments 5 and 6, show the increase or strength, by fastening down the ends, to be in the proportion of 2 to 3; which the theory states as 2 to 4, the difference being probably owing to the manner of fixing.

Mr. Buffon made numerous experiments, both on small bars, and on large ones, which are the best. The following is a specimen of one set, made on bars of sound oak, clear of knots.

Length feet.	Weight lbs.	Broke with lbs.	Bent. inch.	Time. min.
7	60	5350	3·5	29′
	56	5275	4·5	22
8	68	4600	3·75	15
	63	4500	4·7	13
9	77	4100	4·85	14
	71	3950	5·5	12
10	84	3625	5·83	15
	82	3600	6·5	15
12	100	3050	7	
	98	2925	8	

Column 1 shows the length of the bar, in feet, clear between the supports.—Column 2 is the weight of the bar in lbs., the second day after it was felled.—Column 3 shows the number of sounds necessary for breaking the tree in a few minutes.—Col. 4 is the number of inches it bent down before breaking.—Col. 5 is the time at which it broke.—The parts next the root were always the heaviest and strongest.

The following experiments on other sizes were made in the same way, two at least of each length being taken; and the table contains the mean results. The beams were all squared, and their sides in inches are placed at the top of the columns, their lengths in feet being in the first column. The numbers

in the other columns are the pounds weight which broke the pieces.

	4	5	6	7	8	A
7	5312	11525	18950	32200	47649	11525
8	4550	9787	15525	26050	39750	10085
9	4025	8308	13150	22350	32800	8964
10	3612	7125	11250	19475	27750	8068
12	2987	6075	9100	16175	23450	6723
14		5300	7475	13225	19775	5763
16		4350	6362	11000	16375	5042
18		3700	5562	9245	13200	4482
20		3225	4950	8375	11487	4034
22		2975				3667
24		2162				3362
28		1775				2881

Mr. Buffon had found, by many trials, that oak timber lost much of its strength in the course of seasoning or drying; and therefore, to secure uniformity, his trees were all felled in the same season of the year, were squared the day after, and the experiments tried the 3d day. Trying them in this green state gave him an opportunity of observing a very curious phenomenon. When the weights were laid quickly on, nearly sufficient to break the beam, a very sensible smoke was observed to issue from the two ends with a sharp hissing sound; which continued all the time the tree was bending and cracking. This shows the great effects of the compression, and that the beam is strained through its whole length, which is shown also by its bending through the whole length.

Mr. Buffon considers the experiments with the 5-inch bars as the standard of comparison, having both extended these to greater lengths, and also tried more pieces of each length. Now, the theory determines the relative strength of bars, of the same section, to be inversely as their lengths: but most of the trials show a great deviation from this rule, probably owing, in part at least, to the weights of the pieces themselves. Thus, the 5-inch bar of 28 feet long should have half the strength of that of 14 feet, or 2650, whereas it is only 1775; the bar of 14 feet should have half the strength of that of 7 feet, or 5762, but is only 5300; and so of others. The column A is added, to show the strength that each of the 5-inch bars ought to have by the theory.

Mr. Banks, an ingenious lecturer on natural philosophy, made many experiments on the strength of oak, deal, and

iron. He found that the worst or weakest piece of dry heart of oak, 1 inch square, and 1 foot long, broke with 602 lbs., and the strongest piece with 974 lbs.: the worst piece of deal broke with 464 lbs., and the best with 690 lbs. A like bar of the worst kind of cast iron 2190 lbs. Bars of iron set up in positions oblique to the horizon, showed strength nearly proportional to the sines of elevation of the pieces. Equal bars placed horizontally, on supports 3 feet distant, bore $6\frac{3}{4}$ cwt.; the same at $2\frac{1}{2}$ feet distance broke only with 9 cwt.—An arched rib of $29\frac{1}{2}$ feet span, and 11 inches high in the centre, supported $99\frac{1}{2}$ cwt.; it sunk in the middle of $3\frac{7}{8}$ inches, and rose again $\frac{3}{4}$ on removing the load. The same rib tried without abutments, broke with 55 cwt. Another rib, a segment of a circle, $29\frac{1}{4}$ feet span, and 3 feet high in the middle, bore $100\frac{1}{2}$ cwt., and sunk $1\frac{3}{16}$ in the middle. The same rib without abutments, broke with $64\frac{1}{2}$ cwt.

Mr. Banks made also experiments at another foundry, on like bars of 1 inch square, each yard in length weighing 9 lbs. the props at 3 feet asunder.

	lbs.
The 1st bar broke with	963
The 2d ditto	958
The 3d ditto	994
Bar made from the cupola, broke with . .	864
Bar equally thick in the middle, but the ends shaped into a parabola, and weighed $6\frac{3}{16}$ lbs., broke with	874

From these and many other experiments, Mr. Banks concludes, that cast iron is from $3\frac{1}{3}$ to $4\frac{1}{2}$ times stronger than oak of the same dimensions, and from 5 to $6\frac{1}{2}$ times stronger than deal*.

Some examples for practice.

The theory, as has been before mentioned, is, that the strength of a bar, or the weight it will bear, is directly as the breadth and square of the depth divided by the length. So that, if b denote the breadth of a bar, d the depth, l the length, and w the weight it will bear; and the capitals B, D, L, W denote the like quantities in another bar; then, by the rule $\frac{bd^2}{l} : w :: \frac{BD^2}{L} : W$, which gives this general equation

* See farther, on this subject, page 390, &c., *Barlow on the Strength of Timber, &c. Tredgold's Carpentry, and Gregory's Mathematics for Practical men.*

$bd^2\text{LW} = \text{BD}^2lw$, from which any one of the letters is easily found, when the rest are given.

Now, if we take, for a standard of comparison, this experiment of Mr. Banks, that a bar of oak an inch square and a foot in length, lying on a prop at each end, and its strength, or the utmost weight it can bear, on its middle, 660lbs. : here $b = 1$, $d = 1$, $l = 1$, $w = 660$; these substituted in the above equation, it becomes $\text{LW} = 660\text{BD}^2$, from which any one of the four quantities, L, W, B, D, may be found, when the other three are given, when the calculation respects oak timber. But for fir, the like rule will be $\text{LW} = 440\text{BD}^2$; and for iron, $\text{LW} = 2640\text{BD}^2$.

Exam. 1. Required the utmost strength of an oak beam, of 6 inches square and 8 feet long, supported at each end, or the weight to break it in the middle ?

Here are given $\text{B} = 6$, $\text{D} = 6$, $\text{L} = 8$, to find $\text{W} =$
$$\frac{660\text{BD}^2}{\text{L}} = \frac{660 \times 6 \times 36}{8} = 660 \times 3 \times 9 = 17820 \text{ lbs.}$$

Exam. 2. Required the depth of an oak beam, of the same length and strength as above, but only 6 inches breadth ?

Here, as $3 : 6 :: 36 : \text{D}^2 = 72$, theref. $\text{D} = \sqrt{72} = 8{\cdot}485$ the depth.

This last beam, though as strong as the former, is but little more than $\frac{2}{3}$ of its size or quantity. And thus, by making joists thinner, a great part of the expense is saved, as in the modern style of flooring, &c.

Exam. 3. To determine the utmost strength of a deal joist of 2 inches thick and 8 inches deep, the bearing or breadth of the room being 12 feet ? Here $\text{B} = 2$, $\text{D} = 8$, $\text{L} = 12$; then the rule $\text{LW} = 440\text{BD}^2$ gives $\text{W} =$
$$\frac{440 \times \text{B} \times \text{D}^2}{\text{L}} = \frac{440 \times 2 \times 64}{12} = \frac{440 \times 32}{3} = 4693 \text{ lbs.}$$

Exam. 4. Required the depth of a bar of iron 2 inches broad and 8 feet long, to sustain a load of 20,000 lbs. ? Here $\text{B} = 2$, $\text{L} = 8$, and $\text{W} = 20{,}000$, to find D from the equation $\text{LW} = 2640\text{BD}^2$, viz. $\text{D}^2 = \frac{\text{LW}}{2640\text{B}} = \frac{8 \times 20000}{2640 \times 2} = \frac{1000}{33} = 30{\cdot}3$, and $\text{D} = \sqrt{30{\cdot}3} = 5\frac{1}{2}$ inches, the depth.

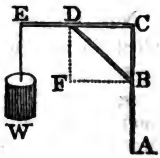
Exam. 5. To find the length of a bar of oak, an inch square, so that when supported at both ends it may just break by its own weight ?—Here, according to the notation and

calculation in prob. 36, $l = 1$, $w = \frac{2}{5}$ of a lb., the weight of 1 foot in length, and $u = 660$ lbs. Then $L =$
$l\sqrt{\dfrac{w+2u}{w}} = \sqrt{3301} = 57{\cdot}45$ feet, nearly.

Exam. 6. To find the length of an iron bar an inch square, that it may break by its own weight, when it is supported at both ends.—Here, as before, $l = 1$, $w = 3$ lbs. nearly the weight of 1 foot in length, also $u = 2640$. Therefore $L =$
$l\sqrt{\dfrac{w+2u}{w}} = 41{\cdot}97$ feet nearly.

Note. It might perhaps have been supposed that this last result should exceed the preceding one: but it must be considered that while iron is only about 4 times stronger than oak, it is at least 8 times heavier.

Exam. 7. When a weight W is suspended from E on the arm of a crane ABCDE, it is required to find the pressure at the end D of the spur, and that at B against the upright post AC.

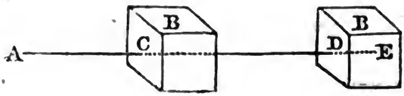


Here, by the nature of the lever, $\frac{CE}{CD}W$ = the pressure at D in the vertical direction DF: but this pressure in DF is to that in DB as DF to DB, viz. DF : DB :: $\frac{CE}{CD}W : \frac{CE \,.\, DB}{DF \,.\, CD}W$ the pressure in DB; and again, DB : FB or CD :: $\frac{CE \,.\, DB}{DF \,.\, CD}W : \frac{CE}{DF}W = \frac{CE}{BC}W$ the pressure against B in direction FB.

Thus, for example, if $CE = 16$ feet, $BC = 6$, $CD = 8$, $BD = 10$, and $W = 3$ tons: then $\frac{CE \,.\, BD}{BC \,.\, CD}W = \frac{16 \,.\, 10}{6 \,.\, 8} \times 3 = 10$ tons, for the pressure on the spur DB. Also $\frac{CE}{CB}W = \frac{16}{6} \times 3 =$ 8 tons, the force tending to break the bar AC at B.

PROBLEM XXXIX.

To determine the circumstances of space, penetration, velocity, and time, arising from a ball moving with a given velocity, and striking a moveable block of wood, or other substance.



Let the ball move in the direction AE passing through the centre of gravity of the block E, impinging on the point C; and when the block has moved through the space CD in consequence of the blow, let the ball have penetrated to the depth DE.

Let B = the mass or matter in the block,
b = the same in the ball,
s = CD the space moved by the block,
x = DE the penetration of the ball, and theref.
$s + x$ = CE the space described by the ball,
a = the first velocity of the ball,
v = the velocity of the ball at E,
u = velocity of the block at the same instant,
t = the time of penetration, or of the motion,
r = the resisting force of the wood.

Then shall $\frac{r}{B}$ be the accelerating force of the block,

and $\frac{r}{b}$ the retarding force of the ball.

Now because the momentum $B\dot{u}$, communicated to the block in the time $\dot{t}$, is that which is lost by the ball, namely, $-b\dot{v}$, therefore $B\dot{u} = -b\dot{v}$, and $Bu = -bv$. But when $v = a$, $u = 0$; therefore, by correcting, $Bu = b(a - v)$; or the momentum of the block is every where equal to the momentum lost by the ball. And when the ball has penetrated to the utmost depth, or when $u = v$, this becomes $Bu = b(a - u)$, or $ab = (B + b)u$; that is, the momentum before the stroke, is equal to the momentum after it. And the velocity communicated will be the same, whatever be the resisting force of the block, the weight being the same.

Again, by theor. 6, Forces, this vol. it is $u^2 = \frac{2grs}{B}$, and $-v^2 = \frac{2gr}{b} \times (s + x)$ or rather, by correction, $a^2 - v^2 = \frac{2gr}{b}(s + x)$. Hence the penetration or $x = \frac{b(a^2 - v^2) - 2grs}{2gr}$.

And when $v = u$, by substituting u for v, and Bu^2 for $2grs$, the greatest penetration becomes $\frac{ba^2 - (B + b)u^2}{2gr}$; and this again, by writing ab for its value $(B + b)u$, gives the greatest penetration $x = \frac{Bba^2}{2gr(B + b)} = \frac{ba^2}{2gr} \times (1 - \frac{b}{B + b})$. Which is barely equal to $\frac{ba^2}{2gr}$ when the block is fixed, or infinitely great; and

is always very nearly equal to the same $\frac{ba^2}{2gr}$ when B is very great in respect of b.

Hence $s+x=\frac{a^2-u^2}{2gr}b=\frac{a^2-\frac{a^2b^2}{(B+b)^2}}{2gr}b=\frac{B^2+2Bb}{(B+b)^2}\times\frac{a^2b}{2gr}$.

And theref. $B+b : B+2b :: x : s+x$, or $B+b : b :: x : s$,

and $s=\frac{bx}{B+b}=\frac{Bb^2a^2}{2gr(B+b)^4}$.

Exam. When the ball is iron, and weighs 1 pound, it penetrates elm about 13 inches when it moves with a velocity of 1500 feet per second: in which case,

$$\frac{r}{b}=\frac{a^2}{2gx}=\frac{1500^2}{4\times16\frac{1}{12}\times\frac{13}{12}}=\frac{9000^2}{193\times13}=32284 \text{ nearly.}$$

When $B=500$lb., and $b=1$; then $u=\frac{ab}{B+b}=\frac{1500}{501}=$ 3 feet nearly per second, the velocity of the block.

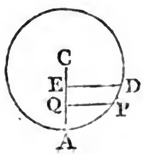
Also $s=\frac{Bu^2}{2gr}=\frac{500\times9}{64\frac{1}{3}\times32284}=\frac{1}{461\frac{1}{4}}$ part of a foot, or $\frac{2}{77}$ of an inch, which is the space moved by the block when the ball has completed its penetration.

And $t=\frac{2s}{u}=\frac{2}{461\frac{1}{4}\times3}=\frac{1}{692}$ part of a second, or

$$t=\frac{2s+2x}{v}=\frac{\frac{2}{461\frac{1}{4}}+\frac{26}{12}}{1500}=\frac{6+13\,.\,231}{6.231\,.\,1500}=\frac{1}{692}$$ part of a second, the time of penetration.

PROBLEM XL.

To find the velocity and time of a heavy body descending down the arc of a circle, or vibrating in the arc by a line fixed in the centre.



Let D be the beginning of the descent, C the centre, and A the lowest point of the circle; draw DE and PQ perpendicular to AC. Then the velocity in P being the same as in Q by falling through EQ, it will be $v=2\sqrt{(\frac{1}{2}g\times EQ)}=8\sqrt{(a-x)}$ nearly, when $a=AE$, $x=AQ$; since $AQ\,.\,2r=AP^2$, v . in A $\propto$ chord AP.

But the fluxion of the time $\dot{t}$ is $=\frac{-\dot{\text{AP}}}{v}$, and, $\dot{\text{AP}}=\frac{r\dot{x}}{\sqrt{(2rx-x^2)}}$ where r = the radius AC. Theref. $\dot{t}=\frac{r}{8}\times\frac{-\dot{x}}{\sqrt{(2rx-x^2}\times\sqrt{(d-x)}}$ $=\frac{d}{16}\times\frac{-\dot{x}}{\sqrt{(ax-x^2)}\times\sqrt{(d-x)}}$, because $(2rx-x^2)(a-x)=(dx-x^2)(a-x)=(ax-x^2)(d-x)=\frac{-\sqrt{d}}{16}\times$ $\frac{\dot{x}}{\sqrt{(ax-x^2)}\times\sqrt{(1-\frac{x}{d})}}$, where $d=2r$ the diameter.

Or $\dot{t}=\frac{-\sqrt{d}}{16}\times\frac{\dot{x}}{\sqrt{(ax-x^2)}}(1+\frac{x}{2d}+\frac{1.3x^2}{2.4d^2}+\frac{1.3.5x^3}{2.4.6d^3}\text{\&c.})$, by developing $1\div\sqrt{(1-\frac{x}{d})}$, or $(1-\frac{x}{d})^{-\frac{1}{2}}$, in a series.

But the fluent of $\frac{\dot{x}}{\sqrt{(ax-x^2)}}$ is $\frac{2}{a}\times$ arc. to radius $\frac{1}{2}a$ and vers. x, or it is the arc whose rad. is 1 and vers. $\frac{2x}{a}$: which call A. And let the fluents of the succeeding terms, without the coefficients, be B, C, D, E, &c. Then will the flux. of any one as $\dot{\text{Q}}$, at n distance from A, be $\dot{\text{Q}}=x^n\dot{\text{A}}=x\dot{\text{P}}$, which suppose also = the flux. of $b\text{P}-dx^{n-1}\sqrt{(ax-x^2)}=b\dot{\text{P}}-$ $d(n-1)\dot{x}x^{n-2}\sqrt{(ax-x^2)}-d\dot{x}x^{n-2}\times\frac{\frac{1}{2}ax-x^2}{\sqrt{(ax-x^2)}}=b\dot{\text{P}}-d\dot{x}\times$ $\frac{(n-\frac{1}{2})ax^{n-1}-nx^n}{\sqrt{(ax-x^2)}}=b\dot{\text{P}}-d(n-\frac{1}{2})a\dot{\text{P}}+dnx\dot{\text{P}}$.

Hence, by equating the coefficients of the like terms, $d=\frac{1}{n}$; $b=\frac{2n-1}{2n}a$; and $\text{Q}=\frac{(2n-1)a\text{P}-2x^{n-1}\sqrt{(ax-x^2)}}{2n}$.

Which being substituted, the fluential terms become $\frac{\sqrt{d}}{16}\times$ $(-\text{A}-\frac{1}{2d}\cdot\frac{a\text{A}-2\sqrt{(ax-x^2)}}{2}-\frac{1.3}{2.4d^2}\cdot\frac{3a\text{B}-2x\sqrt{(ax-x^2)}}{4}-$ $\frac{1.3.5}{2.4.6a^3}\cdot\frac{5a\text{C}-2x^2\sqrt{(ax-x^2)}}{6}-\text{\&c.})$. Or the same fluents will be found by art. 85, p. 353.

But when $x=a$, those terms become barely $\frac{3\cdot1416\sqrt{d}}{16}\times$

$(-1 + \frac{1^2 a}{2^2 d} - \frac{1^2 \,.\, 3^2 a^2}{2^2 \,.\, 4^2 d^2} - \frac{1^2 \,.\, 3^2 \,.\, 5^2 a^3}{2^2 \,.\, 4^2 \,.\, 6^2 d^3} - \text{\&c.})$; which being subtracted, and x taken $= 0$, there arises for the whole time of descending down DA, or the corrected value of $t =$ $\frac{3{\cdot}1416\sqrt{d}}{16} \times (1 + \frac{1^2 a}{2^2 d} + \frac{1^2 \,.\, 3^2 a^2}{2^2 \,.\, 4^2 d^2} + \frac{1^2 \,.\, 3^2 \,.\, 5^2 a^3}{2^2 \,.\, 4^2 \,.\, 6^2 d^3} + \text{\&c.})$.

When the arc is small, as in the vibration of the pendulum of a clock, all the terms of the series may be omitted after the second, and then the time of a semi-vibration t is nearly $= \frac{1{\cdot}5708}{4} \sqrt{\frac{r}{2}} \times (1 + \frac{a}{8r})$. And theref. the times of vibration of a pendulum, in different arcs, are as $8r + a$, or 8 times the radius added to the versed sine of the arc.

If D be the degrees of the pendulum's vibration, on each side of the lowest point of the small arc, the radius being r, the diameter d, and $3{\cdot}1416 = p$; then is the length of that arc $\text{A} = \frac{pr\text{D}}{180} = \frac{pd\text{D}}{360}$. But the versed sine in terms of the arc is $a = \frac{\text{A}^2}{2r} - \frac{\text{A}^4}{24r^3} + \text{\&c.} = \frac{\text{A}^2}{d} - \frac{\text{A}^4}{3d^3} + \text{\&c.}$ Therefore $\frac{a}{d} = \frac{\text{A}^2}{d^2} - \frac{\text{A}^4}{3d^4} + \text{\&c.} = \frac{p^2\text{D}^2}{360^2} - \frac{p^4\text{D}^4}{3{\cdot}360^4} + \text{\&c.}$, or only $= \frac{p^2\text{D}^2}{360^2}$ the first term, by rejecting all the rest of the terms on account of their smallness, or $\frac{a}{d} = \frac{a}{2r}$ nearly $= \frac{\text{D}^2}{13131}$. This value then being substituted for $\frac{a}{d}$ or $\frac{a}{2r}$ in the last near value of the time, it becomes $t = \frac{1{\cdot}5708}{4} \sqrt{\frac{r}{2}} \times (1 + \frac{\text{D}^2}{52524})$ nearly. And therefore the times of vibration in different small arcs, are as $52524 + \text{D}^2$, or as 52524 added to the square of the number of degrees in the arc.

Hence it follows that the time lost in each second, by vibrating in a circle, instead of the cycloid is $\frac{\text{D}^2}{52524}$; and consequently the time lost in a whole day of 24 hours, or $24 \times 60 \times 60$ seconds, is $\frac{5}{3}\text{D}^2$ nearly. In like manner, the seconds lost per day by vibrating in the arc of Δ degrees, is $\frac{5}{3}\Delta^2$. Therefore, if the pendulum keep true time in one of these arcs, the seconds lost or gained per day, by vibrating in the other, will be $\frac{5}{3}(\text{D}^2 - \Delta^2)$. So, for example, if a pendulum

measure true time in an arc of 3 degrees, it will lose $11\frac{3}{4}$ seconds a day by vibrating 4 degrees; and $26\frac{1}{4}$ seconds a day by vibrating 5 degrees; and so on.

And in like manner we might proceed for any other curve, as the ellipse, hyperbola, parabola, &c.

Scholium. By comparing this with the results of the problems 13 and 14, page 410, &c. it will appear that the times in the cycloid, and in the arc of a circle, and in any chord of the circle, are respectively as the three quantities,

$$1,\ 1 + \frac{a}{8r}\ \text{\&c., and } \frac{1}{\cdot 7854},$$

or nearly as the three quantities 1, $1 + \frac{a}{8r}$, 1·27324; the first and last being constant, but the middle one, or the time in the circle, varying with the extent of the arc of vibration. Also the time in the cycloid is the least, but in the chord the greatest; for the greatest value of the series, in this prob. when $a = r$, or the arc AD is a quadrant, is 1·18014; and in that case the proportion of the three times is as the numbers 1, 1·18014, 1·27324. Moreover the time in the circle approaches to that in the cycloid, as the arc decreases, and they are very nearly equal when that arc is very small.

PROBLEM XLI.

To find the time and velocity of a chain consisting of very small links, descending from a smooth horizontal plane; the chain being 100 *inches long, and* 1 *inch of it hanging off the plane at the commencement of motion.*

Put $a = 1$ inch, the length at the beginning;
$l = 100$ the whole length of the chain:
$x =$ any variable length off the plane.
Then x is the motive force to move the body,

and $\frac{x}{l} = f$ the accelerative force.

Hence $v\dot{v} = gf\dot{s} = g \times \frac{x}{l} \times \dot{x} = \frac{gx\dot{x}}{l}$.

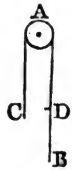
The fluents give $v^2 = \frac{gx^2}{l}$. But $v = 0$ when $x = a$, theref. by correction, $v^2 = g \times \frac{x^2 - a^2}{l}$, and $v = \sqrt{(g \times \frac{x^2 - a^2}{l})}$ the velocity for any length x. And when the chain just

quits the plane, $x = l$, and then the greatest velocity is $\sqrt{(g \times \frac{l^2-a^2}{l})} = \sqrt{(2 \times 193 \times \frac{100^2-1^2}{100})} = \sqrt{\frac{386 \times 9999}{100}}$ $= 196\cdot45902$ inches, or $16\cdot371585$ feet, per second.

Again $\dot{t}$ or $\frac{\dot{s}}{v} = \sqrt{\frac{l}{g}} \times \frac{\dot{x}}{\sqrt{(x^2-a^2)}}$; the correct fluent of which is $t = \sqrt{\frac{l}{g}} \times \log. \frac{x+\sqrt{(x^2-a^2)}}{a}$, the time for any length x. And when $x = l = 100$, it is $t = \sqrt{\frac{100}{386}} \times \log. \frac{100 \times \sqrt{9999}}{1} = 2\cdot69676$ seconds, the time when the last of the chain just quits the plane.

PROBLEM XLII.

To find the time and velocity of a chain, of very small links, quitting a pulley, by passing freely over it: the whole length being 200 inches, and the one end hanging 2 inches below the other at the beginning.



Put $a = 2$, $l = 200$, and $x = $ BD any variable difference of the two parts AB, AC. Then $\frac{x}{l} = f$, and $v\dot{v}$ or $gf\dot{s} = g \cdot \frac{x}{l} \cdot \frac{1}{2}\dot{x} = \frac{gx\dot{x}}{2l}$.

Hence the correct fluent is $v^2 = g \times \frac{x^2-a^2}{2l}$ and $v = \sqrt{(\frac{1}{2}g \times \frac{x^2-a^2}{l})}$, the general expression for the veloc. And when $x = l$, or when C arrives at A, it is $v = \sqrt{(\frac{1}{2}g \times \frac{l^2-a^2}{l})} = \sqrt{(193 \times \frac{200^2-2^2}{200})} = \sqrt{(386 \times \frac{100^2-1^2}{100})}$ $= \sqrt{\frac{386 \times 9999}{100}} = 196\cdot45902$ inches, or $16\cdot371585$ feet for the greatest velocity when the chain just quits the pulley.

Again, $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{\dot{x}}{2v} = \sqrt{\frac{l}{2g}} \times \frac{\dot{x}}{\sqrt{(x^2-a^2)}}$. And the correct fluent is $t = \sqrt{\frac{l}{2g}} \times \log. \frac{x+\sqrt{(x^2-a^2)}}{a}$, the general expression for the time. And when $x = l$, it becomes $t =$

$$\sqrt{\frac{l}{2g}} \times \text{log.}\ \frac{l+\sqrt{(l^2-a^2)}}{a} = \sqrt{\frac{200}{772}} \times \text{log.}\ \frac{200\times\sqrt{(200^2-2^2)}}{2}$$

$$= \sqrt{\frac{100}{386}} \times \text{log.}\ \frac{100+\sqrt{9999}}{1} = 2{\cdot}69676 \text{ seconds, the whole}$$

time when the chain just quits the pulley.

So that the velocity and time at quitting the pulley in this prob. and the plane in the last prob. are the same; the distance descended 99 being the same in both. For, though the weight l moved in this latter case, be double of what it was in the former, the moving force x is also double, because here the one end of the chain shortens as much as the other end lengthens, so that the space descended $\frac{1}{2}x$ is doubled, and becomes x; and hence the accelerative force $\frac{x}{l}$ or f is the same in both; and of course the velocity and time the same for the same distance descended.

PROBLEM XLIII.

If a fine chain 30 inches long be laid straight on a horizontal plane, and 6 inches below this plane there be a parallel plane to receive the chain falling from the first: it is required to determine the velocity and time of the chain's quitting the first plane, 6 inches of the chain hanging off at the commencement of the motion.

Solution.

Let l the whole length of the chain, p = the part hanging down, x = the variable part or that above the nether plane, $g = 32\frac{1}{6}$, v = the velocity per second of the end of the chain along the upper plane, and t = the time of motion.

Then the moving force is as p, and the matter moved as x, whence f the accelerating force urging the chain in the direction of its length is $\frac{p}{x}$ or px^{-1}: therefore (page 400) $v\dot{v} = g f\dot{s} = -\frac{pg\dot{x}}{x}$. The correct fluent of which gives $v^2 = 2pg \times \text{hyp. log. of } \frac{l}{x}$, or $v = \sqrt{(pg \text{ hyp. of } \frac{l}{x})}$. Hence $\dot{t} = \frac{-\dot{x}}{v} = \frac{x\dot{v}}{pg}$, and t = fluent $\frac{x\dot{v}}{pg} = \frac{xv}{pg}$ — flu. $\frac{v\dot{x}}{pg} = \frac{xv}{pg}$ + flu. $\frac{xv^2\dot{v}}{(pg)^2} = \frac{xv}{pg} + \frac{xv^3}{3(pg)^2}$ — flu. $\frac{v^3\dot{x}}{3(pg)^2} = \frac{xv}{pg} + \frac{xv^3}{3(pg)^2}$ + flu. $\frac{xv^4\dot{v}}{3(pg)^2}$ =

$\frac{x\dot{v}}{pg}+\frac{xv^3}{3(pg)^2}+\frac{xv^5}{3\cdot5(pg)^3}-$ flu. $\frac{v^5\dot{x}}{3\cdot5(pg)^3}$, whence the law of continuing the series is manifest, and $t=\frac{xv}{pg}(1+\frac{v^2}{3(pg)}+\frac{v^4}{3\cdot5(pg)^2}+\frac{v^6}{3\cdot5\cdot7(pg)^3}+\frac{v^8}{3\cdot5\cdot7\cdot9(pg)^4}+$&c.), which is a general expression for the time of describing any variable length of the chain. But $v^2=pg$ hyp. log. of $\frac{l}{x}=pgc$, and $t=\frac{x\sqrt{c}}{\sqrt{\frac{1}{2}pg}}$ $(1+\frac{2c}{3}+\frac{4c^2}{3\cdot5}+\frac{8c^3}{3\cdot5\cdot7}+\frac{16c^4}{3\cdot5\cdot7\cdot9}+$ &c.). And when $x=\frac{1}{2}$, $l=\frac{5}{2}$ feet, $p=\frac{1}{2}$, and $c=$ hyp. log. of 5, we have $t=$ ·7244153 of a second. Also $v=7\cdot19515$ feet, the velocity per second with which the end of the chain quits the upper plane.

PROBLEM XLIV.

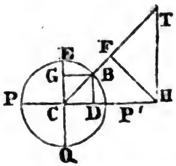
To find the weight of a column infinitely high, whose base is B.

Let $r=$ radius of the earth, $x=$ any height above it, and x' any indefinitely small height; also, let s the specific gravity of the matter of which the pillar is constituted. Then $\text{B}sx'=$ weight of an indefinitely small portion of the column at the earth's surface. And by the laws of gravity $\frac{1}{r^2}:\text{B}sx'::\frac{1}{(r+x)^2}:\frac{r^2\text{B}sx'}{(r+x)^2}$, the weight of an equal part $\text{B}x'$ at the height x. Consequently the fluxion of the weight is $\frac{r^2\text{B}sx'}{(r+x)^2}=r^2\text{B}s\dot{x}\,(r+x)^{-2}$: the fluent of this is $-r^2\text{B}s\times$ $r+x)^{-1}$, or $\frac{-r^2\text{B}s}{r+x}$; or corrected, $\frac{r^2\text{B}s}{r}-\frac{r^2\text{B}s}{r+x}$, the weight of the pillar whose height is x. This reduces to $r^2\text{B}s(\frac{1}{r}-\frac{1}{r+x})=r^2\text{B}s(\frac{x}{r(r+x)})=r\text{B}s\,(\frac{x}{r+x})$. When x is infinite the weight of the infinite pillar becomes $r\text{B}s\,(\frac{x}{x})$, or simply $r\text{B}s$. That is, the weight would be equal to that of a bent pillar of the same matter, and the same base which should be in horizontal contact with an arc of 57·2957795 degrees of a great circle of the earth.

Note. This is independent of the consideration of centrifugal force, which would take off the stones from the top of the column, when its height was little more than $5 \cdot 6r$, at the equator, &c. See the next problem.

PROBLEM XLV.

Required the altitude of the highest edifices that could possibly be raised on any part of the earth's surface.



Let C be the centre of the earth, PP its poles, EQ the equator, BT the altitude of the edifice erected perpendicularly at B, whose latitude ECB is l. Draw BD, TH, each perpendicular to PP, or to PP produced, and from H, HF perpendicular to CT. Now, it is evident that the edifice cannot be raised any higher than the point T, where the centrifugal force reduced to the direction CT is equal to the force of gravity; for at a greater altitude the centrifugal force would exceed the gravitating force, and the materials would fly off.

Let CE = CB = CP = 1, CD = BG = sin l, BD = cos l, $c = \frac{1}{289}$ = centrifugal force at E, that of gravity being expressed by unity, and CT = x. Then, as CB : BD :: CT : TH, or 1 : cos l :: x : $x \cos l$ = TH : hence, as CE : TH, or 1 : $x \cos l$ :: c : $cx \cos l$ = centrifugal force at T in direction HT, T revolving about H. And, as HT : FT :: CB : BD :: rad : cos l :: $cx \cos l$: $cx \cos^2 l$, the said centrifugal force reduced to the direction TC, opposite that of gravity. Again, as $\frac{1}{\text{CB}^2} : \frac{1}{\text{CT}^2}$ :: grav. at B : grav. at T $= \frac{1}{x^2}$. Hence, by the above, $cx \cos^2 l = \frac{1}{x^2}$, and $x^3 = \frac{1}{c \cos^2 l} = \frac{\sec^2 l}{c}$, and $x = \sqrt[3]{(289 \sec.^2 l)}$. Consequently, BT = CT − CB = $\sqrt[3]{(289 \sec.^2 l)} - 1$.

1. Suppose the place to be the equator, then sec. $l = 1$, and BT $= \sqrt[3]{289} - 1 = 6 \cdot 611489 - 1 = 5 \cdot 611489$ radii of the earth.

2. Suppose the latitude to be 45°, then BT $= \sqrt[3]{(289 \sec^2 45^\circ)} - 1 = \sqrt[3]{(289 \times 2)} - 1 = 8 \cdot 320335 - 1 = 7 \cdot 320335$ radii.

3. If the latitude be 60°, then BT $= \sqrt[3]{(289 \sec^2 60^\circ)} - 1 = \sqrt[3]{(289 \times 4)} - 1 = 10 \cdot 49508 - 1 = 9 \cdot 49508$ radii.

4. At either pole sec l is infinite, and the height of the edifice would have no limit.

PROBLEM XLVI.

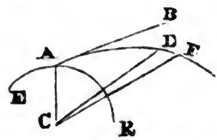
Supposing the earth's rotation about its axis were entirely to cease, how much would pendulum clocks gain in 24 hours in the latitude 51° 30′?

It has been fully ascertained, that when the attraction of gravitation at the equator is 1, the centrifugal force there is $\frac{1}{289}$; and it is easily demonstrated (vide preceding problem, or, *Gregory's Mechanics*, vol. i. p. 259,) that the central force under the equator is to the central force in any proposed latitude, as the square of the radius to the square of the cosine of the said latitude. Hence we shall have $\frac{1}{289}\cos^2 51° 30' = \cdot 0013492$, the measure of the central force in latitude 51° 30′; and this taken from the measure of the attraction gives ·9986508 for the force with which the pendulum is actuated in the proposed latitude. But the number of vibrations performed by equal pendulums in the same time, is in the subduplicate ratio of the forces: therefore $\sqrt{(\cdot 9986508)} : \sqrt{1} :: 24 \times 3600\ (=86400) : 86458\cdot 34$, the number of vibrations of a second pendulum that would be performed in 24 hours, if the earth's rotation were to cease. Consequently, the number of vibrations, or nearly of seconds gained would be $86458\cdot 34 - 86400 = 58\cdot 34 = 58'' \, 20'''$.

N. B. Computations of the kind required in the preceding solution are best performed by logarithms.

PROBLEM XLVII.

Required the least velocity with which a cannon ball must be projected from the surface of the earth (suppose at an angle of 40° *elevation) so that it shall never return.*



Let C be earth's centre, EAR a portion of the surface, AB the projectile's direction, and ADF its trajectory. Suppose CD, CF indefinitely near each other, and call CA, (the earth's radius = 21000000 feet) r, CD, x; 32·2 feet, the velocity generated in a second at the earth's surface g; v the velocity in D; V the required velocity. Then

the centripetal force in D will be $\frac{r^2g}{x^2}$ (being reciprocally as the square of the distance from the earth's centre), and the force to retard the motion in the direction DF, $\frac{r^2gx}{x^2 \times DF}$; this retarding force drawn into the fluxion of the time, being equal to the fluxion of the velocity, $\frac{r^2g\dot{x}}{vx^2}$ will be $= -\dot{v}$; therefore $v\dot{v} = -\frac{r^2g\dot{x}}{xx}$, and the fluent $\frac{vv}{2} = \frac{r^2g}{x}$. But in A ($v$ being $= \text{v}$, and $x = r$) the correct fluent gives $v = \sqrt{(\text{vv} - 2rg + 2r^2gx^{-1})}$. After an infinite time, x will be infinitely great, and $\sqrt{(\text{vv} - 2rg + 2r^2gx^{-1})}$ infinitely small, and therefore may be put $= 0$, in which equation r is nothing in respect of the value of x; and therefore $\text{v} = \sqrt{2rg} = 36775$ feet $= 6{\cdot}9655$ miles. Hence, there is no limit with regard to the angle of direction; but if a body be projected from the earth's surface, in *any* direction whatever above the horizon, with such a velocity as will carry it about or above 7 miles per second, it will never return.

PROBLEM XLVIII.

To determine the ratio of the densities of the sun and the earth.

Let R and r be the radii or semidiameters of the orbits of the earth and moon, P and p the periodic times in those orbits, S and s the sun's mean apparent semidiameter and moon's mean horizontal parallax, and N and n any two numbers in the required ratio of the densities of the sun and earth respectively. Then the real semidiameters of the sun and earth being in the ratio of R S to $r\ s$, their masses will be as $R^3 S^3 \times N : r^3 s^3 \times n$; and consequently their forces, at the distances R and r, as $\frac{R^3S^3N}{R^2} : \frac{r^3s^3n}{r^2}$, or as $RS^3N : rs^3n$.

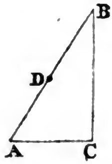
But these are (by the laws of central forces) also as $\frac{R}{PP} : \frac{r}{pp}$; therefore by dividing the antecedents of these equal ratios by RS^3, and the consequents by rs^3, we have as $N : n :: \frac{1}{P^2S^3} : \frac{1}{p^2s^3} :: 1 : \frac{P^2}{p^2} \times \frac{S^3}{s^3}$; which in numbers, taking $P = 365d.\ 5h.\ 49m.$ $p = 27d.\ 7h\ 43m.$ $S = 16'\ 5\frac{1}{4}''$. and $s = 57'.\ 17\frac{1}{2}''$. will come out as 1 to 3·957 for the ratio of the

density of the sun to that of the earth. *Laplace*, taking the values of the given quantities rather differently, makes the ratio 1 to 3·9393.

PROBLEM XLIX.

Suppose a given right-angled plane triangle, whose hypothenuse is a uniform slender rod, and its base parallel to the horizon, to revolve about its perpendicular as an axis, while a ring slides freely along its hypothenuse: it is required to determine the time of the ring's descent along the said hypothenuse, its length being = 50 feet, the perpendicular = 40 feet, and the time of one revolution round the axis = 3 seconds?

Let BA represent the given rod, revolving about the axis BC perpendicular to the horizontal line AC, D the place of the ring at the end of any time t, and v its velocity, there, in the direction DA.



Put AB (= 50 feet) = r, BC (= 40) = p, AC (= 30) = b, $32\frac{1}{6}$ as usual = g, $3''$ (the time of one revolution) = s, 3·141593, &c. = $\frac{1}{2}c$, and BD = x; then will $bccx \div rss$ = the centrifugal force in a direction parallel to the horizon, and its effect in the direction DA (by the resolution of forces) = $bbccx \div rrss$; which added to $pg \div r$, the true measure of the force of gravity in the same direction, gives $\frac{bbccx}{rrss}+\frac{pg}{r}=$ the whole force accelerating the ring along the rod, at D. Whence by the principles of motion, $\frac{bbccx\dot{x}}{rrss}+\frac{pg\dot{x}}{r} = v\dot{v}$, and (putting $\frac{bbcc}{rrss}= m$, and $\frac{pg}{r} = n$) $v = \surd(mxx + 2nx)$; consequently $\dot{t} = \frac{1}{\surd m} \times \frac{\dot{x}}{\surd(2qx+xx)}$, putting $q = \frac{n}{m}$; and, taking the correct fluent $t = \frac{1}{\surd m} \times$ hyp. log. of $\frac{q+x+\surd(2qx+xx)}{q}$ =(when x becomes = BA = 50) 1·6559″, &c. = $1''\,39'''\,21''''$, &c. the time required.

Cor. 1. The time of descent along the rod in motion, is less than the time of descent along it when at rest, by $18''\,51''''$, &c.

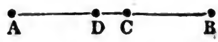
Cor. 2. Making $\frac{bg}{p}$ (the gravity of the ring on the rod) = $\frac{bc^2x}{rs^2}$, (the horizontal centrifugal force above found), it appears that when $x = \frac{grs^2}{pc^2}$, the ring, if at entire liberty, would no longer slide along the rod, but fly off.

Note. For an investigation of the actual length of the track described in the circumstances of this problem, see *Gentleman's Diary*, for 1824, p. 44, &c.

PROBLEM L.

Suppose a uniform slender rod AB. *parallel to the horizon (considered without regard to its weight) to have two equal bodies fixed to it, one at each end, and to revolve round the point* C *in that rod, as a centre, in any given time; the length of the rod being given, as also* CD *the distance of the middle of the rod from the centre of motion* C: *it is required to determine the velocity with which the rod passes through the point* C, *as also the time of description. Supposing* AB *to be* 100 *feet,* CD = 1 *inch at first, and the time of each revolution* 20 *seconds.*

Solution.



Let the line AB represent the rod, D its middle point, C the point or centre about which it revolves, and through which it slides. Put AD = DB = a = 50 feet, the variable distance CD = x; t the time the rod has been in motion, and v the velocity of the motion (of the rod in its own direction) through the point C; also, let $c = \frac{1}{12}$ of a foot, the distance CD at the beginning of the motion, b the weight or quantity of matter in each of the equal bodies, d = 10 seconds, half the time of revolution, and p = 3·1416. Then, by the nature of centrifugal forces $\frac{p^2}{d^2} \cdot b\ (a + x)$ = the motive centrifugal force in the direction CA, and $\frac{p^2}{d^2}\ b\ (a - x)$ = that in the direction CB. Consequently the differences of these two forces being divided by the quantity of matter, $2b$, in both

the bodies, gives $\frac{p^2x}{d^2}$, for the accelerating force of the rods in the direction CA, or $= n^2x$, putting $n^2 = \frac{p^2}{d^2}$. Then, by the laws of motion, $\text{F}\dot{x} = v\dot{v}$, or $n^2x\dot{x} = v\dot{v}$, the fluents of which give $n^2x^2 = v^2$. But when $x = c$, $v = 0$; therefore, the fluents corrected are $v^2 = n^2(x^2 - c^2)$. Again, by the fundamental equations for motion (pa. 400, 401,) $\dot{t} = \frac{\dot{x}}{v} = \frac{\dot{x}}{n\sqrt{x^2-c^2}}$ the fluxion of the time; the fluent of which is $\frac{1}{n} \times$ hyp. log. $[x + \sqrt{(x^2-c^2)}]$; or corrected as above $t = \frac{1}{n} \times$ hyp. log. $\frac{x+\sqrt{(x^2-c^2)}}{c}$, for the true time of describing the distance $x-c$. And this when $x = 50$, gives $t = 22{\cdot}56$ seconds, the time which will elapse before the end B passes through C.

From this solution it is evident that the magnitude of the equal weights appended to A and B, has no effect upon the result.

PROBLEM LI.

To find the number of vibrations made by two weights, connected by a very fine thread, passing freely over a tack or a pulley, while the less weight is drawn up to it by the descent of the heavier weight at the other end; the extent of the vibrations being indefinitely small.

Suppose the motion to commence at equal distances below the pulley at B; and that the weights are 1 and 2 pounds.

Put a = AB, half the length of the thread;

$b = 39\frac{1}{8}$ inc. or $3\frac{2}{9}\frac{3}{6}$ feet, length of second's pend.

x = Bw = BW, any space passed over;

z = the number of vibrations.

Then $\frac{\text{W}-w}{\text{W}+w} = f = \frac{1}{3}$ is the accelerating force.

And hence v or $\sqrt{2gfs} = \sqrt{2gfx}$, and $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{2gfx}}$.

But, by the nature of pendulums, $\sqrt{(a \pm x)} : \sqrt{b} :: 1$ vibr. : $\sqrt{\frac{b}{a \pm x}}$ the vibrations per second made by either weight,

namely, the longer or shorter, according as the upper or under sign is used,, if the threads were to continue of that length for 1 second. Hence, then, as

$$1'' \; \dot{t} :: \sqrt{\frac{b}{a \pm x}} : \dot{z} = \dot{t}\sqrt{\frac{b}{a \pm x}} = \sqrt{\frac{b}{2gf}} \times \frac{\dot{x}}{\sqrt{(ax \pm x^2)}},$$

the fluxion of the number of vibrations.

Now when the upper sign $+$ takes place, the fluent is

$$z = 2\sqrt{\frac{b}{2gf}} \times \mathrm{l.}\frac{\sqrt{x} + \sqrt{(a+x)}}{\sqrt{a}} = \sqrt{\frac{2b}{gf}} \times \mathrm{l.}\frac{\sqrt{ax} + \sqrt{(ax+x^2)}}{a}.$$

And when $x = a$, the same then becomes $z = \sqrt{\frac{2b}{2g}} \times \log.$ $1 + \sqrt{2} = \sqrt{\frac{3b}{2g}} \times \log.\ 1 + \sqrt{2} = \sqrt{\frac{117\frac{3}{8}}{193}} \times \log.\ 1 + \sqrt{2}$ $= \cdot 688511$, the whole number of vibrations made by the descending weight.

But when the lower sign, or —, takes place, the fluent is $\sqrt{\frac{b}{2gf}} \times$ arc to rad. 1 and vers. $\frac{2x}{a}$. Which, when $x = a$, gives $\frac{1}{2}p\sqrt{\frac{2b}{gf}} = 3\cdot1416 \times \sqrt{\frac{3 \times 39\frac{1}{8}}{4 \times 193}} = \frac{3\cdot1416}{2} \times \sqrt{\frac{117\frac{3}{8}}{193}}$ $= 1\cdot227091$, the whole number of vibrations made by the lesser or ascending weight.

Schol. It is evident that the whole number of vibrations, in each case, is the same, whatever the length of the thread is. And that the greater number is to the less, as $1\cdot5708$ to the hyp. log. of $1 + \sqrt{2}$.

Further, the number of vibrations performed in the same time t, by an invariable pendulum, constantly of the same length a, is $\sqrt{\frac{2b}{gf}} = \cdot781190$. For, the time of descending the space a, or the fluent of $\dot{t} = \frac{\dot{x}}{\sqrt{2gfx}}$, when $x = a$, is $t = \sqrt{\frac{2a}{gf}}$. And, by the nature of pendulums, $\sqrt{a} : \sqrt{b} :: 1$ vibr. : $\sqrt{\frac{b}{a}}$ the number of vibrations performed in 1 second; hence $1'' : t :: \sqrt{\frac{b}{a}} : t\sqrt{\frac{b}{a}} = \sqrt{\frac{b}{\frac{1}{2}gf}}$, the constant number of vibrations.

So that the three numbers of vibrations, namely of the

ascending, constant, and descending pendulums, are proportional to the numbers 1·5708, 1, and hyp. log. $1 + \sqrt{2}$, or as 1·5708, 1, and ·88137 ; whatever be the length of the thread.

PROBLEM LII.

To determine the circumstances of the ascent and descent of two unequal weights, suspended at the two ends of a thread, passing over a pulley : the weight of the thread and of the pulley being considered in the solution.

Let $l =$ the whole length of the thread ;
$a =$ the weight of the same ;
$b =$ Aw the dif. of lengths at first ;
$d =$ W—w the dif. of the two weights ;
$e =$ a weight applied to the circumference, such as to be equal to its whole wt. and friction reduced to the circumference ;
$s =$ W $+ w + a + c$ the sum of the weights moved.

Then the weight of b is $\frac{ab}{l}$, and $d - \frac{ab}{l}$ is the moving force at first. But if x denote any variable space descended by W, or ascended by w, the difference of the lengths of the thread will be altered $2x$; so that the difference will then be $b—2x$, and its weight $\frac{b—2x}{l}a$; conseq. the motive force there will be $d - \frac{b—2x}{l}a = \frac{dl-ab+2ax}{l}$, and theref. $\frac{dl-ab+2ax}{sl} = f$ the accelerating force there. Hence then $v\dot{v} = gf\dot{x} = g\dot{x} \times \frac{dl—ab+2ax}{sl}$; the fluents of which give $v^2=2gx \times \frac{dl—ab+ax}{sl}$, or $v = 2\sqrt{\frac{ag}{2sl}} \times \sqrt{(cx + x^2)}$ the general expression for the velocity, putting $e = \frac{dl—ab}{a}$. And when $x = b$, or W becomes as far below w as it was above it at the beginning, it is barely $v = 2\sqrt{\frac{bdg}{2s}}$ for the velocity at that time. Also, when a, the weight of the thread, is nothing, the velocity is only $2\sqrt{\frac{dgx}{2s}}$, as it ought.

Again, for the time, $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{sl}{\frac{1}{2}ag}} \times \frac{\dot{x}}{\sqrt{(ex + x^2)}}$; the

fluents of which give $t = \sqrt{\frac{sl}{\frac{1}{2}ag}} \times \log. \frac{\sqrt{x}+\sqrt{(e+x)}}{\sqrt{e}}$ the general expression for the time of descending any space x.

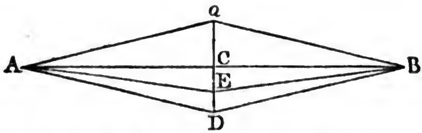
And if the radicals be expanded in a series, and the log. of it be taken, the same will become

$$t = \sqrt{\frac{sx}{\frac{1}{2}dg}} \times \sqrt{\frac{dl}{dl-ab}} \times (1 - \frac{x}{6e} + \frac{3x^2}{40e^2}, \&c.)$$

Which therefore becomes barely $\sqrt{\frac{sx}{\frac{1}{2}dg}}$ when a, the weight of the thread, is nothing; as it ought.

PROBLEM LIII.

To find the velocity and time of vibration of a small weight, fixed to the middle of a line, or fine thread void of gravity, and stretched by a given tension; the extent of the vibration being very small.



Let l = AC half the length of the thread;
a = CD the extent of the vibration;
x = CE any variable distance from C;
w = wt. of the small body fixed to the middle;
W = a wt. which, hung at each end of the thread, will be equal to the constant tension at each end, acting in the direction of the thread.

Now, by the nature of forces, AE : CE :: W the force in direction EA : the force in direction EC. Or, because AC is nearly = AE, the vibration being very small, taking AC instead of AE, it is AC : CE :: W : $\frac{Wx}{l}$ the force in EC arising from the tension in EA. Which will be also the same for that in EB. Therefore the sum is $\frac{2Wx}{l}$ = the whole motive force in EC arising from the tensions on both sides. Consequently $\frac{2Wx}{lw} = f$ the accelerative force there. Hence the equation of the fluxions $v\dot{v}$ or $gf\dot{s} = \frac{-2gWx\dot{x}}{lw}$; and the fluents

$v^2 = -\frac{2gwx^2}{lw}$. But when $x=a$, this is $-\frac{2gwa^2}{lw}$, and should be $= 0$; theref. the correct fluents are $v^2 + 2gw \times \frac{a^2-x^2}{lw}$, and hence $v = \sqrt{(2gw \times \frac{a^2-x^2}{lw})}$ the velocity of the little body w at any point E. And when $x = 0$, it is $v = 2a\sqrt{\frac{gw}{\frac{1}{2}lw}}$ for the greatest velocity at the point C.

Now if we suppose $w = 1$ grain, $w = 5$lb. troy, or 28800 grains, and $2l$ = AB = 3 feet; the velocity at C becomes $a\sqrt{\frac{8 \times 16\frac{1}{12} \times 28800}{3}} = 1111\frac{2}{3}a$. So that,

if $a = \frac{1}{12}$ inc. the greatest veloc. is $9\frac{7}{16}$ ft. per sec.
if $a = 1$ inc. the greatest veloc. is $92\frac{7}{36}$ ft. per sec.
if $a = 6$ inc. the greatest veloc. is $555\frac{7}{16}$ ft. per sec.

To find the time t, it is $\dot{t}$ or $\frac{-\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{lw}{\frac{1}{2}wg}} \times \frac{-\dot{x}}{\sqrt{(a^2-x^2)}}$.
Hence the correct fluent is $t = \frac{1}{2}\sqrt{\frac{wl}{\frac{1}{2}wg}} \times$ arc to cosine $\frac{x}{a}$ and radius 1, for the time in DE. And when $x = 0$, the whole time in DC, or of half a vibration; is $\cdot 7854\sqrt{\frac{wl}{\frac{1}{2}wg}}$; and conseq. the time of a whole vibration through Dd is $1\cdot 5708\sqrt{\frac{wl}{\frac{1}{2}wg}}$.

Using the foregoing numbers, namely $w = 1$, $w = 28800$, and $2l = 3$ feet; this expression for the time gives $\frac{1111\frac{2}{3}}{3\cdot 1416} = 353\frac{5}{8}$, the number of vibrations per second. But if $w = 2$, there would be 250 vibrations per second; and if $w = 100$, there would be $35\frac{1}{4}\frac{3}{8}$ vibrations per second.

PROBLEM LIV.

To determine the same as in the last problem, when the distance CD *bears some sensible proportion to the length* AB; *the tension of the thread however being still supposed a constant quantity.*

Using here the same notation as in the last problem, and taking the true variable length AE for AC, it is AE or EB : CE :: $2w : \frac{2wx}{AE} = \frac{2wx}{\sqrt{(l^2+x^2)}}$ the whole motive force from the two

equal tensions W in AE and EB; and theref. $\frac{2W}{w} \times \frac{x}{\sqrt{(l^2+x^2)}} = f$ is the accelerative force at E. Theref. the fluxional equation is $v\dot{v}$ or $gf\dot{s} = \frac{2Wg}{w} \times \frac{-x\dot{x}}{\sqrt{(l^2+x^2)}}$; and the fluents $v^2 = \frac{4Wg}{w} \times - \sqrt{(l^2+x^2)}$. But when $x = a$, these are $0 = \frac{4Wg}{w} \times - \sqrt{(l^2+a^2)}$; therefore the correct fluents are $v^2 = \frac{4Wg}{w} \times [\sqrt{(l^2+a^2)} - \sqrt{(l^2+x^2)}] = \frac{4Wg}{w} \times (\text{AD} - \text{AE})$. And hence $v = \sqrt{[\frac{4Wg}{w} \times (\text{AD} - \text{AE})]}$ the general expression for the velocity at E. And when E arrives at C, it gives the greatest velocity there $= \sqrt{[\frac{4Wg}{w} \times (\text{AD} - \text{AC})]}$. Which, when $W = 28800$, $w = 1$, $2l = 3$ feet, and CD = 6 inches or $\frac{1}{2}$ a foot, is $\sqrt{(8 \times 28800 \times 16\frac{1}{12} \times \frac{\sqrt{10}-3}{2})} = 548\frac{1}{3}$ feet per second. Which came out $555\frac{7}{12}$ in the last problem, by using always AC for AE in the value of f. But when the extent of the vibrations is very small, as $\frac{1}{16}$ of an inch, as it commonly is, this greatest velocity here will be $\sqrt{8 \times 28800 \times 16\frac{1}{12} \times \frac{1}{4320}} = 9\frac{1}{4}$ nearly, which in the last problem was $9\frac{5}{12}$ nearly.

To find the time, it is $\dot{t}$ or $\frac{-\dot{x}}{v} = \sqrt{\frac{w}{4Wg}} \times \frac{-\dot{x}}{\sqrt{[c-\sqrt{(l^2+x^2)}]}}$, making $c = \text{AD} = \sqrt{(l^2+a^2)}$. To find the fluent the easier, multiply the numer. and denom. both by $\sqrt{[c+\sqrt{(l^2+x^2)}]}$, so shall $\dot{t} = \sqrt{\frac{w}{4Wg}} \times \frac{-\dot{x}}{\sqrt{(a^2-x^2)}} \times \sqrt{[c+\sqrt{(l^2+x^2)}]}$. Expand now the quantity $\sqrt{[c+\sqrt{(l^2+x^2)}]}$ in a series, and put $d = c+l$, so shall $\dot{t} = \sqrt{\frac{wd}{4g}} \times \frac{-\dot{x}}{\sqrt{(a^2-x^2)}} (1 + \frac{x^2}{4dl} - \frac{2d+l}{32d^2l^3}x^4 + \frac{4d^2+2dl+l^2}{128d^3l^5}x^6 - \frac{40d^3+8d^2l+12dl^2+5l^3}{2048d^4l^7}x^8 \text{ \&c})$.

Now the fluent of the first term $\frac{\dot{x}}{\sqrt{(a^2-x^2)}}$ is = the arc to sine $\frac{x}{a}$ and radius 1, which arc call A; and let P, Q be the fluents of

any other two successive terms, without the coefficients, the distance of Q from the first term A being n; then it is evident that $\dot{Q} = x^2\dot{P} = x^{2n}\dot{A}$, and $\dot{P} = x^{2n-2}\dot{A}$. Assume theref. $Q = bP - ex^{2n-1}\sqrt{(a^2-x^2)}$; then is $\dot{Q}$ or $x^2\dot{P} = b\dot{P} - (2n-1)ex^{2n-2}\dot{x}\sqrt{(a^2-x^2)} + \frac{ex^{2n}\dot{x}}{\sqrt{(a^2-x^2)}} = b\dot{P} - \frac{(2n-1)ea^2x^{2n-2}\dot{x}}{\sqrt{(a^2-x^2)}} + \frac{(2n-1)ex^{2n}\dot{x}}{\sqrt{(a^2-x^2)}} + \frac{ex^{2n}\dot{x}}{\sqrt{(a^2-x^2)}} = b\dot{P} - (2n-1)ea^2\dot{P} + (2n-1)ex^2\dot{P} + ex^2\dot{P} = b\dot{P} - (2n-1)ea^2\dot{P} + 2nex^2\dot{P}$. Then comparing the coefficients of the like terms, we find $1 = 2en$, and $b = (2n-1)ea^2$; from which are obtained $e = \frac{1}{2n}$, and $b = \frac{2n-1}{2n}a^2$.

Consequently $Q = \frac{(2n-1)a^2P - x^{2n-1}\sqrt{(a^2-x^2)}}{2n}$, the general equation between any two successive terms, and by means of which the series may be continued as far as we please. And hence neglecting the coefficients, putting A = the first term, namely the arc whose sine is $\frac{x}{a}$, and B, C, D, &c, the following terms, the series is as follows, $A + \frac{a^2A - x\sqrt{(a^2-x^2)}}{2} + \frac{3a^2B - x^3\sqrt{(a^2-x^2)}}{4} + \frac{5a^2C - x^5\sqrt{(a^2-x^2)}}{6}$, &c. Now when $x = 0$, this series $= 0$; and when $x = a$, the series becomes $\frac{1}{2}p + \frac{a^2A}{2} + \frac{3a^2B}{4} + \frac{5a^2C}{6}$, &c. where $p = 3{\cdot}1416$, or the series is

$\frac{1}{2}p(1 + \frac{1}{2}a^2 + \frac{1\,.\,3}{2\,.\,4}a^4 + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}a^6$, &c.)

So that, by taking in the coefficients, the general time of passing over any distance DE will be

$$\sqrt{\frac{w(c+l)}{4wg}} \times \tfrac{1}{2}p \times \left(1 + \frac{1}{4dl} \cdot \tfrac{1}{2}a^2 - \frac{2d+l}{32d^2l^3} \cdot \frac{1\,.\,3}{2\,.\,4}a^4, \text{\&c.} - \text{arc sin.}\ \frac{x}{a} - \frac{1}{4dl} \cdot \frac{a^2A - x\sqrt{(a^2-x^2)}}{2} + \frac{2d+l}{32d^2l^3} \cdot \frac{3a^2B - x^3\sqrt{(a^2-x^2)}}{4}, \text{\&c.}\right)$$

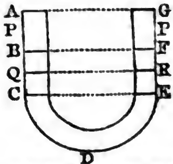
And hence, taking $x = 0$, and doubling, the time of a whole vibration, or double the time of passing over CD will be equal to $\frac{1}{2}p\sqrt{\frac{w(c+l)}{wg}} \times (1 + \frac{1}{4dl} \cdot \frac{1}{2}a^2 - \frac{2d+l}{32d^2l^3} \cdot \frac{1\,.\,3}{2\,.\,4}a^4 + \frac{4d^2+2dl+l^2}{128dl^5} \cdot \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}a^6 - \frac{40d^3+8d^2l+12dl^2+5l^3}{2048dl^7} \cdot \frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8}$

a^8, &c.) Which, when $a = 0$, or $c = l$, becomes only $\frac{1}{2}p$ $\sqrt{\frac{wl}{\frac{1}{2}\text{w}g}}$ the same as in the last problem, as it ought.

Taking here the same numbers as in the last problem, viz. $l = \frac{1}{3}$, $a = \frac{1}{2}$, $w = 2$, $\text{w} = 28800$, $\frac{1}{2}g = 16\frac{1}{12}$; then $\frac{1}{2}p\sqrt{\frac{w(c+l)}{\text{w}g}} = \cdot0040514$, and the series is $1 + \cdot006762 - \cdot000175 + \cdot000003$, &c. $= 1\cdot006590$; therefore $\cdot0040514 \times 1\cdot006590 = \cdot0040965 = \frac{1}{245\frac{1}{5}}$ is the time of one whole vibration, and consequently $245\frac{1}{5}$ vibrations are performed in a second; which were 250 in the last problem.

PROBLEM LV.

It is proposed to determine the velocity, and the time of vibration, of a fluid in the arms of a canal or bent tube.



Let the tube ABCDEF have its two branches AC, GE vertical, and the lower part CDE in any position whatever, the whole being of a uniform diameter or width throughout. Let water, or quicksilver, or any other fluid, be poured in, till it stand in equilibrio, at any horizontal line BF. Then let one surface be pressed or pushed down by shaking, from B to C, and the other will ascend through the equal space FG; after which let them be permitted freely to return. The surfaces will then continually vibrate in equal times between AC and EG. The velocity and times of which oscillations are therefore required.

When the surfaces are any where out of a horizontal line, as at P and Q, the parts of the fluid in QDR, on each side, below QR, will balance each other; and the weight of the part in PR, which is equal to 2PF, gives motion to the whole. So that the weight of the part 2PF is the motive force by which the whole fluid is urged, and therefore $\frac{\text{wt. of } 2\text{PF}}{\text{whole wt.}}$ is the accelerative force. Which weights being proportional to their lengths, if l be the length of the whole fluid, or axis of the tube filled, and $a = \text{FG}$ or BC; then is $\frac{2a}{l}$ the accelerative force. Putting theref. $x = \text{GP}$ any variable distance, v the

velocity, and t the time; then PF $= a - x$, and $\frac{2a-2x}{l} = f$ the accelerative force; hence $v\dot{v}$ or $gf\dot{s} = \frac{2g}{l}(a\dot{x} - x\dot{x})$; the fluents of which give $v^2 = \frac{2g}{l}(2ax - x^2)$, and $v = \sqrt{(2g \times \frac{2ax-x^2}{l})}$ is the general expression for the velocity at any term. And when $x=a$, it becomes $v=2a\sqrt{\frac{g}{2}}$ for the greatest velocity at B and F.

Again, for the time, we have $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{1}{2}\sqrt{\frac{l}{\frac{1}{2}g}} \times \frac{\dot{x}}{\sqrt{(2ax-x^2)}}$; the fluents of which give $t = \frac{1}{2}\sqrt{\frac{l}{\frac{1}{2}g}} \times$ arc to versed sine $\frac{x}{a}$ and radius 1, the general expression for the time. And when $x = a$, it becomes $t = \frac{1}{2}p\sqrt{\frac{l}{\frac{1}{2}g}}$ for the time of moving from G to F, p being $= 3\cdot1416$; and consequently $\frac{1}{2}p\sqrt{\frac{l}{\frac{1}{2}g}}$ the time of a whole vibration from G to E, or from C to A. And which therefore is the same, whatever AB is, the whole length l remaining the same.

And the time of vibration is also equal to the time of the vibration of a pendulum whose length is $\frac{1}{2}l$, or half the length of the axis of the fluid. So that, if the length l be $78\frac{1}{4}$ inches, it will oscillate in 1 second.

Scholium. This reciprocation of the water in the canal, according to Newton, is nearly similar to the motion of the waves of the sea. For the time of vibration is the same, however short the branches are, provided the whole length be the same. So that when the height is small, in proportion to the length of the canal, the motion is similar to that of a wave, from the top to the bottom or hollow, and from the bottom to the top of the next wave; being equal to two vibrations of the canal; the whole length of a wave, from top to top, being double the length of the canal. Hence the wave will move forward by a space nearly equal to its breadth, in the time of two vibrations of a pendulum whose length is ($\frac{1}{2}l$) half the length of the canal, or one-fourth the breadth of a wave, or in the time of one vibration of a pendulum whose length is the whole breadth of the wave, since the times of vibration are as the square roots of their lengths.

Consequently, waves whose breadth is equal to $39\frac{1}{8}$ inches, or $3\frac{25}{96}$ feet, will move over $3\frac{25}{96}$ feet in a second, or $195\frac{5}{8}$ feet in a minute, or nearly 2 miles and a quarter in an hour. And the velocity of greater or less waves will be increased or diminished in the subduplicate ratio of their breadths.

Thus, for instance, for a wave of 18 inches breadth, as $\sqrt{39\frac{1}{8}} : 39\frac{1}{8} :: \sqrt{18} : \sqrt{(39\frac{1}{8} \times 18)} = \frac{3}{2}\sqrt{313} = 26{\cdot}5377$ the velocity of the wave of 18 inches breadth.

But the motion of waves has been otherwise considered by Lagrange, and other philosophers.

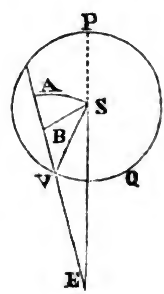
PROBLEM LVI.

To a pendulum SA *of a given length, suspended at* S, *a given weight* n *is affixed at* A; *to find where another weight* m *must be fixed, so that it may vibrate in the least time possible.*

Let F be the point required, and O the centre of oscillation; then the pendulum itself being considered as of no weight, if $\text{SA} = a$, and $\text{SF} = x$, $\text{SO} = \dfrac{na^2+mx^2}{na+mx}$. *Dynamics*, art. 229, p. 239. Now the time of oscillation $\propto \sqrt{\text{SO}}$; therefore, since the time is a minimum, $\dfrac{na^2+mx^2}{na+mx}$ is a minimum; and its fluxion, that is, $2mx\dot{x} \times (na+mx) - m\dot{x} \times (na^2+mx^2) = 0$, or $2nax + 2mx^2 = na^2 + mx^2$; $\therefore x^2 + \dfrac{2nax}{m} = \dfrac{na^2}{m}$; and from this quadratic $x = \dfrac{a}{m} \times \sqrt{(n^2+mn)} - \dfrac{na}{m} = \text{SF}$.

PROBLEM LVII.

To find the position of Venus when she shines with the greatest lustre.



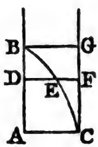
Let E be the earth, S the sun, and V Venus; join SV, SE, EV, and produce EV to A, making $\text{VA} = \text{VS}$; with V as a centre and VS radius describe the circular arc SA; and draw SB perpendicular to EA. Then SEV is the angle of elongation, SVA the exterior angle, VB its cosine, and BA its versed sine to the radius SV. Take $\text{SE} = a$, $\text{EV} = x$, $\text{VE} = y$, $\text{SV} = b$. Then since it is demonstrable that the area of the whole disc of the planet, is to the area of the enlight-

ened part, as the diameter of the circle to the versed sine of the exterior angle, it follows that the visible illumined part $\propto$ BA $\propto b - y$; while the brightness $\propto \frac{b-y}{x^2}, \propto \frac{b}{x^2} - \frac{y}{x^2}$, which by the problem is a maximum. Now, Geom. th. 36, $SE^2 = EV^2 + SV^2 + 2EV.VB$, or $a^2 = x^2 + b^2 + 2xy$; $\therefore y = \frac{a^2-b^2-x^2}{2x} =$ (if $m^2 = a^2 - b^2$) $\frac{m^2-x^2}{2x}$; $\therefore$ by substituting for y this value, we have $\frac{b}{x^2} - \frac{m^2-x^2}{2x^3}$ a maximum; that is, $\frac{3bx-m^2+x^2}{2x^3}$ is a maximum; hence $(2b\dot{x} + 2x\dot{x})\,2x^3 - 6x^2\dot{x}\,(2bx-m^2+x^2) = 0$: where, if we divide by $2x^2\dot{x}$, it becomes $-x^2 - 4bx + 3m^2 = 0$; $\therefore x^2+4bx=3m^2$, from which equation $x = -2b+\sqrt{(4b^2+3m^2)} = -2b+\sqrt{(3a^2+b^2)}$. Hence the three sides of the triangle SEV are known, to find the angle of elongation; which, if $a = 1$, $b = \cdot72333$, gives $x = \cdot43036$, and SEV $= 39^\circ\ 44'$, between the inferior conjunction and the greatest elongation.

Note. It may not be amiss to remark, that the equation $x = \sqrt{(3a^2 + b^2)} - 2b$ has a limit: for if b were equal to $\frac{1}{4}a$, the point V would fall on P, and the whole disc on the planet would be the *maximum* when in its superior conjunction with the sun. So again, if b were less than $\frac{1}{4}a$, the arch described from the centre E with the radius EV would not intersect the circle PVQ.

PROBLEM LVIII.

To determine the time of emptying any ditch, or inundation, &c. by a cut or notch, from the top to the bottom of it.



Let $x =$ AB the variable height of water at any time;
$b =$ AC the breadth of the cut;
$d =$ the whole or first depth of water;
A $=$ the area of the surface of the water in the ditch;
$\frac{1}{2}g = 16\frac{1}{12}$ feet, or $g = 32\frac{1}{6}$, as is usual.

The velocity at any point D, is as $\sqrt{BD}$, that is, as the ordinate DE of a parabola BEC, whose base is AC, and altitude AB. Therefore the velocities at all the points in AB, are as all the ordinates of the parabola. Consequently the quantity of water running through the cut ABGC, in any time, is to the quantity which would run through an equal aperture placed

all at the bottom in the same time, as the area of the parabola ABC, to the area of the parallelogram ABGC, that is, as 2 to 3.

But $\sqrt{\frac{1}{2}g} : \sqrt{x} :: g : \sqrt{\frac{1}{2}gx}$ the velocity at AC; therefore $\frac{2}{3} \times \sqrt{\frac{1}{2}gx} \times bx = \frac{2}{3}bx\sqrt{\frac{1}{2}gx}$ is the quantity discharged per second through ABGC; and consequently $\frac{2bx\sqrt{\frac{1}{2}gx}}{3A}$ is the velocity per second of the descending surface. Hence then $\frac{2bx\sqrt{\frac{1}{2}gx}}{3A} : -\dot{x} :: 1'' : \frac{-3A\dot{x}}{2bx\sqrt{\frac{1}{2}gx}} = \dot{t}$ the fluxion of the time of descending.

Now when A the surface of the water is constant, or the ditch is equally broad throughout, the correct fluent of this fluxion gives $t = \frac{3A}{2b\sqrt{\frac{1}{2}g}} \times \frac{\sqrt{d}-\sqrt{x}}{\sqrt{dx}}$ for the general time of sinking the surface to any depth x. And when $x = 0$, this expression is infinite; which shows that the time of a complete exhaustion is infinite.

But if $d = 9$ feet, $b = 2$ feet, $A = 21 \times 1000 = 21000$, and it be required to exhaust the water down to $\frac{1}{16}$ of a foot deep; then $x = \frac{1}{16}$, and the above expression becomes $\frac{3 \times 21000}{4 \times 4\frac{1}{16}} \times \frac{3-\frac{1}{4}}{\frac{3}{4}} = 14400''$, or just 4 hours for that time. And if it be required to depress it 8 feet, or till 1 foot depth of water remain in the ditch, the time of sinking the water to that point will be 43′ 38″.

Again, if the ditch be the same depth and length as before, but 20 feet broad at bottom, and 22 at top; then the descending surface will be a variable quantity, and, by prob. 16, p. 413, it will be $\frac{90+x}{90} \times 20000$; hence in this case the flux. of the time, or $\frac{-3A\dot{x}}{2bx\sqrt{\frac{1}{2}gx}}$, becomes $\frac{-500}{3b\sqrt{\frac{1}{2}g}} \times \frac{90+x}{x\sqrt{x}}$ $\dot{x}$; the correct fluent of which is $t = \frac{1000}{3b\sqrt{\frac{1}{2}g}} \times (\frac{90-x}{\sqrt{x}} - \frac{90-d}{\sqrt{d}})$ for the time of sinking the water to any depth x.

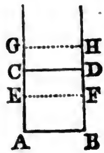
Now when $x = 0$, this expression for the complete exhaustion becomes infinite.

But if . . $x = 1$ foot, the time t is 42′ $56\frac{1}{2}$″.
And when $x = \frac{1}{16}$ foot, the time is 3h 50′ $28\frac{1}{2}$″.

PROBLEM LIX.

To determine the time of filling the ditches of a fortification 6 feet deep with water, through the sluice of a trunk of 3 feet square, the bottom of which is level with the bottom of the ditch, and the height of the supplying water is 9 feet above the bottom of the ditch.

Let ACDB represent the area of the vertical sluice, being a square of 9 square feet, and AB level with the bottom of the ditch. And suppose the ditch filled to any height AE, the surface being then at EF.



Put $a = 9$ the height of the head or supply ;

$b = 3 =$ AB $=$ AC ;

$\frac{1}{2}g = 16\frac{1}{12}$;

A $=$ the area of a horizontal section of the ditches ;

$x = a -$ AE, the height of the head above EF.

Then $\surd\frac{1}{2}g : \surd x :: g : \surd\frac{1}{2}gx$ the velocity with which the water presses through the part AEFB ; and theref. $\surd\frac{1}{2}gx \times$ AEFB $= b\surd\frac{1}{2}gx(a-x)$ is the quantity per second running through AEFB. Also, the quantity running per second through ECDF is $\surd\frac{1}{2}gx \times \frac{11}{12}$ ECDF $= \frac{11}{12}b\surd\frac{1}{2}gx(b-a+x)$ nearly. For the real quantity is, by proceeding as in the last prob. the difference between two parab. segs, the alt. of the one being x, its base b, and the alt. of the other $a-b$; and the medium of that dif. between its greatest state at AB, where it is $\frac{8}{15}$AD, and its least state at CD, where it is 0 is nearly $\frac{11}{12}$ED. Consequently the sum of the two, or $\frac{1}{6}b\surd gx$ $(a+11b-x)$ is the quantity per second running in by the whole sluice ACDB. Hence then $\frac{1}{6}b\surd gx \times \frac{a+11b-x}{A} = v$ is the rate or velocity per second with which the water rises in the ditches; and so $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-6A}{b\surd\frac{1}{2}g} \times \frac{x^{-\frac{1}{2}}\dot{x}}{c-x}$ the fluxion of the time of filling to any height AE, putting $c = a + 11b$.

Now when the ditches are of equal width throughout, A is a constant quantity, and in that case the correct fluent of this fluxion is $t = \frac{6A}{b\surd\frac{1}{2}gc} \times \log.\left(\frac{\surd c+\surd a}{\surd c-\surd a} \times \frac{\surd c-\surd x}{\surd c+\surd x}\right)$ the general expression for the time of filling to any height AE, or

$a - x$, not exceeding the height AC of the sluice. And when $x = \text{AC} = a - b = d$ suppose, then $t = \frac{6\text{A}}{b\sqrt{\frac{1}{2}gc}} \times$ log. $\sqrt{(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} \cdot \frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}+\sqrt{d}})}$ is the time of filling to CD the top of the sluice.

Again, for filling to any height GH above the sluice, x denoting as before a − AG the height of the head above GH, $2\sqrt{\frac{1}{2}gx}$ will be the velocity of the water through the whole sluice AD; and therefore $2b^2\sqrt{\frac{1}{2}gx}$ the quantity per second, and $\frac{2b^2\sqrt{\frac{1}{2}gx}}{\text{A}} = v$ the rise per second of the water in the ditches; consequently $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-\text{A}}{2b^2\sqrt{\frac{1}{2}g}} \times \frac{\dot{x}}{\sqrt{x}}$ the general fluxion of the time; the correct fluent of which, being 0 when $x = a - b = d$, is $t = \frac{\text{A}}{b^2\sqrt{\frac{1}{2}g}} (\sqrt{d} - \sqrt{x})$ the time of filling from CD to GH.

Then the sum of the two times, namely, that of filling from AB to CD, and that of filling from CD to GH, is $\frac{\text{A}}{b\sqrt{\frac{1}{2}g}}$ $[\frac{\sqrt{d}-\sqrt{x}}{d} + \frac{6}{\sqrt{c}} \log. (\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} \cdot \frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}+\sqrt{d}})]$ for the whole time required. And using the numbers in the prob. this becomes $\frac{\text{A}}{3\sqrt{g}} [\frac{\sqrt{6}-\sqrt{3}}{3} + \frac{6}{\sqrt{42}} \times \text{l.} (\frac{\sqrt{42}+\sqrt{9}}{\sqrt{42}-\sqrt{9}} \cdot \frac{\sqrt{42}-\sqrt{6}}{\sqrt{42}+\sqrt{6}})]$ $= 0{\cdot}03577277\text{A}$, the time in terms of A the area of the length and breadth, or horizontal section of the ditches. And if we suppose that area to be 200000 square feet, the time required will be 7154″, or $1^{\text{h}}\ 59'\ 14''$.

And if the sides of the ditch slope a little, so as to be a little narrower at the bottom than at top, the process will be nearly the same, substituting for A its variable value, as in the preceding problem. And the time of filling will be very nearly the same as that above determined.

PROBLEM LX.

But if the water, from which the ditches are to be filled, be the tide, which at low water is below the bottom of the trunk, and rises to 9 feet above the bottom of it by a regular rise of

one foot in half an hour; it is required to ascertain the time of filling it to 6 feet high, as before in the last problem.

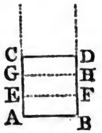
Let ACDB represent the sluice; and when the tide has risen to any height GH, below CD the top of the sluice, without the ditches, let EF be the mean height of the water within. And put $b = 3 = $ AB $=$ AC;

$\frac{1}{2}g = 16\frac{1}{12}$;

$A = $ horizontal section of the ditches;

$x = $ AG;

z : AE.



Then $\surd\frac{1}{2}g : \surd$EG $:: g : \surd\frac{1}{2}g(x-z)$ the velocity of the water through AEFB; and

$\surd\frac{1}{2}g : \surd$EG $:: \frac{2}{3}g : \frac{2}{3}\surd\frac{1}{2}g(x-z)$ the mean vel. through EGHF,

theref. $bz\surd\frac{1}{2}g(x-z)$ is the quantity per sec. through AEFB; and $\frac{2}{3}b(x-z)\surd\frac{1}{2}g(x-z)$ is the same through EGHF;

$bz\surd[\frac{1}{2}g(x-z)] + \frac{2}{3}b(x-z)\surd[\frac{1}{2}g(x-z)]$

$= (bz + \frac{2}{3}b(x-z)\surd[\frac{1}{2}g(x-z)] =$

$\frac{1}{3}b(3z + 2x - 2z) = \frac{1}{2}g(x-z) = \frac{1}{3}b(2x+z)\surd[\frac{1}{2}g(x-z)]$,

conseq. $\frac{1}{3}b\surd\frac{1}{2}g \times (2x+z)\surd(x-z)$ is the whole through AGHB per second. This quantity divided by the surface A, gives $\frac{b\surd\frac{1}{2}g}{3A} \times (2x+z)\surd(x-z) = v$ the velocity per second with which EF, or the surface of the water in the ditches, rises. Therefore

$$v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{3A}{b\surd\frac{1}{2}g} \times \frac{\dot{z}}{(2x+z)\surd(x-z)}.$$

But, as GH rises uniformly 1 foot in 30′ or 1800″, therefore $1 :$ AG $:: 1800'' : 1800x = t$ the time of the tide rising through AG; conseq. $\dot{t} = 1800\dot{x} = \frac{3A}{b\surd\frac{1}{2}g} \times \frac{\dot{z}}{(2x+z)\surd(x-z)}$, or $m\dot{z} = (2x+z)\surd(x-z) \cdot \dot{x}$ is the fluxional equa. expressing the relation between x and z; where $m = \frac{A}{1200b\surd\frac{1}{2}g} = \frac{3200}{231}$ or $13\frac{197}{231}$ when $A = 200000$ square feet.

Now to find the fluent of this equation, assume $z = Ax^{\frac{5}{2}} + Bx^{\frac{8}{2}} + Cx^{\frac{11}{2}} + Dx^{\frac{14}{2}}$, &c. So shall

$$\surd(x-z) = x^{\frac{1}{2}} - \frac{A}{2}x^{\frac{4}{2}} - \frac{A^2+4B}{8}x^{\frac{7}{2}} - \frac{A^3+4AB+8C}{16}x^{\frac{10}{2}}, \text{ \&c,}$$

$$2x + z = 2x + Ax^{\frac{5}{2}} + Bx^{\frac{8}{2}} + Cx^{\frac{11}{2}}, \text{ \&c.}$$

$$(2x+z)\surd(x-z)\dot{x} = 2x^{\frac{3}{2}}\dot{x} - \frac{3A^2}{4}x^{\frac{9}{2}}\dot{x} - \frac{A^3+6AB}{4}x^{\frac{12}{2}}\dot{x}, \text{ \&c,}$$

and $m\dot z = \frac{5}{2} m\text{A} x^{\frac{3}{2}} \dot x + \frac{8}{2} m\text{B} x^{\frac{6}{2}} \dot x + \frac{11}{2} m\text{C} x^{\frac{9}{2}} \dot x + \frac{14}{2} m\text{D} x^{\frac{12}{2}} \dot x$, &c.
Then equate the coefficients of the like terms,

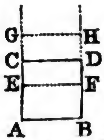
so shall	and consequently
$\frac{5}{2} m\text{A} = 2,$	$\text{A} = \frac{4}{5m},$
$\frac{8}{2} m\text{B} = 0,$	$\text{B} = 0,$
$\frac{11}{2} m\text{C} = -\frac{3}{4}\text{A}^2,$	$\text{C} = \frac{24}{275m^3},$
$\frac{14}{2} m\text{D} = -\frac{3}{4}\text{A}^3 - \frac{3}{2}\text{AB},$	$\text{D} = -\frac{16}{875m^4},$
&c.;	&c.

Which values of A, B, C, &c. substituted in the assumed value of z, give

$$z = \frac{4}{5m} x^{\frac{5}{2}} * - \frac{24}{275m^3} x^{\frac{11}{2}} - \frac{16}{875m^4} x^{\frac{14}{2}}, \text{ \&c.};$$

$$\text{or } z = \frac{4}{5m} x^{\frac{5}{2}} \text{ very nearly.}$$

And when $x = 3 = \text{AC}$, then $z = \cdot 886$ of a foot, or $10\frac{2}{3}$ inches, $= \text{AE}$, the height of the water in the ditches when the tide is at CD or 3 feet high without, or in the first hour and half of time.



Again, to find the time, after the above, when EF arrives at CD, or when the water in the ditches arrives as high as the top of the sluice.

The notation remaining as before,
then $bz\sqrt{\frac{1}{2}g(x-z)}$ per sec. runs through AF,
and $\frac{2}{3}b(3-z)\sqrt{\frac{1}{2}g(x-z)}$ per sec. thro' ED nearly;
therefore $\frac{2}{3}b\sqrt{\frac{1}{2}g} \times (12+z)\ \sqrt{(x-z)}$ is the whole per second through AD nearly.

conseq. $\frac{2b\sqrt{\frac{1}{2}g}}{5\text{A}} \times (12+z)\ \sqrt{(x-z)} = v$ is the velocity per second of the point E; and therefore

$$v : \dot z :: 1'' : \dot t = \frac{\dot z}{v} = \frac{5\text{A}}{2b\sqrt{\frac{1}{2}g}} \times \frac{\dot z}{(12+z)\sqrt{(x-z)}} = 1800\dot x, \text{ or}$$

$$m\dot z = (12+z)\sqrt{(x-z)}\,.\,\dot x, \text{ where } m = \frac{\text{A}}{720b\sqrt{\frac{1}{2}g}} = 23\tfrac{2}{23} \text{ nearly.}$$

Assume $z = \text{A}x^{\frac{3}{2}} + \text{B}x^{\frac{4}{2}} + \text{C}x^{\frac{5}{2}} + \text{D}x^{\frac{6}{2}}$, &c. So shall

$$\sqrt{(x-z)} = x^{\frac{1}{2}} - \frac{\text{A}}{2}x^{\frac{2}{2}} - \frac{\text{A}^2+4\text{B}}{8}x^{\frac{3}{2}} - \frac{\text{A}^3+4\text{AB}+8\text{C}}{16}x^{\frac{4}{2}}, \text{ \&c.}$$

$12+z=12+Ax^{\frac{3}{2}}+Bx^{\frac{4}{2}}+Cx^{\frac{5}{2}}$, &c. ;

$(12+z)\cdot\sqrt{(x-z)}\cdot\dot{x}=12x^{\frac{1}{2}}\dot{x}-6Ax^{\frac{2}{2}}\dot{x}-(\frac{3}{2}A^2+6B)x^{\frac{3}{2}}\dot{x}$, &c.;

$m\dot{z}=\frac{3}{2}mAx^{\frac{1}{2}}\dot{x}+\frac{4}{2}mBx^{\frac{2}{2}}\dot{x}+\frac{5}{2}mCx^{\frac{3}{2}}\dot{x}$, &c.

Then, equating the like terms, &c. we have

$A=\frac{8}{m}$, $B=-\frac{24}{m^2}$, $C=\frac{96}{5m^3}$, $D=\frac{64}{3m^4}$ nearly, &c.

Hence $z=\frac{8}{m}x^{\frac{3}{2}}-\frac{24}{m^2}x^2+\frac{96}{5m^3}x^{\frac{5}{2}}+\frac{64}{3m^4}x^3$, &c.

Or $z=\frac{8}{m}x^{\frac{3}{2}}$ nearly.

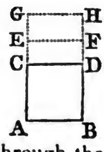
But, by the first process, when $x=3$, $z=\cdot886$; which substituted for them, we have $z=\cdot886$, and the series $=1\cdot63$; therefore the correct fluents are

$$z-\cdot886=-1\cdot63+\frac{8}{m}x^{\frac{3}{2}}-\frac{24}{m^2}x^2\text{, \&c.}$$

$$\text{or } z+\cdot741-\frac{8}{m}x^{\frac{3}{2}}-\frac{24}{m^2}x^2\text{, \&c.}$$

And when $z=3=AC$, it gives $x=6\cdot369$ for the height of the tide without, when the ditches are filled to the top of the sluice, or 3 feet high; which answers to 3h 11′ 4″.

Lastly, to find the time of rising the remaining 3 feet above the top of the sluice; let



$x=CG$ the height of the tide above CD,
$z=CE$ ditto in the ditches above CD;
and the other dimensions as before.
Then $\sqrt{\frac{1}{2}g}:\sqrt{EG}::g:\sqrt{\frac{1}{2}g(x-z)}=$ the velocity with which the water runs through the whole sluice AD; conseq. $AD\times\sqrt{\frac{1}{2}g}\,(x-z)=9\sqrt{\frac{1}{2}g(x-z)}$ is the quantity per second running through the sluice, and $\frac{9\sqrt{\frac{1}{2}g}}{A}\sqrt{(x-z)}=v$ the velocity of z, or the rise of the water in the ditches, per second; hence $v:\dot{z}::1'':$ $\dot{t}=\frac{\dot{z}}{v}=\frac{A}{9\sqrt{\frac{1}{2}g}}\times\frac{\dot{z}}{\sqrt{(x-z)}}=1800\dot{x}$, and $m\dot{z}=\dot{x}\sqrt{(x-z)}$ is the fluxional equation; where $m=\frac{2A}{180^1\sqrt{\frac{1}{2}g}}=\frac{3200}{2079}$.

To find the fluent,

Assume $z = \text{A}x^{\frac{3}{2}} + \text{B}x^{\frac{4}{2}} + \text{C}x^{\frac{5}{2}} + \text{D}x^{\frac{6}{2}}$, &c.

Then $x - z = x - \text{A}x^{\frac{3}{2}} - \text{B}x^{\frac{4}{2}} - \text{C}x^{\frac{5}{2}}$, &c.

$$\dot{x}\sqrt{(x-z)} = x^{\frac{1}{2}}\dot{x} - \frac{\text{A}}{2}x^{\frac{2}{2}}\dot{x} - \frac{\text{A}^2+4\text{B}}{8}x^{\frac{3}{2}}x, \text{ \&c.}$$

$$m\dot{z} = \tfrac{3}{2}n\text{A}x^{\frac{1}{2}}\dot{x} + \tfrac{4}{2}n\text{B}x^{\frac{2}{2}}\dot{x} + \tfrac{5}{2}n\text{C}x^{\frac{3}{2}}\dot{x}, \text{ \&c.}$$

Then equating the like terms, gives

$$\text{A} = \frac{2}{3n}, \ \text{B} = \frac{-1}{6n^2}, \ \text{C} = \frac{1}{90n^3}, \ \text{D} = \frac{-1}{810n^4}, \text{ \&c.}$$

Hence $z = \dfrac{2}{3n}x^{\frac{3}{2}} - \dfrac{1}{6n^2}x^2 + \dfrac{1}{90n^3}x^{\frac{5}{2}} - \dfrac{1}{810n^4}x^3$, &c.

But, by the second case, when $z = 0$, $x = 3{\cdot}369$, which being used in the series, it is, $1{\cdot}936$; therefore the correct fluent is $z = -1{\cdot}936 + \dfrac{1}{3n}x^{\frac{3}{2}} - \dfrac{1}{6n^2}x^2$, &c. And when $z = 3$, $x = 7$; the heights above the top of the sluice, answering to 6 and 10 feet above the bottom of the ditches. That is, for the water to rise to the height of 6 feet within the ditches, it is necessary for the tide to rise to 10 feet without, which just answers to 5 hours; and so long it would take to fill the ditches 6 feet deep with water, their horizontal area being 20000 square feet.

Further, when $x = 6$, then $z = 2{\cdot}117$, the height above the top of the sluice; to which add 3, the height of the sluice, and the sum $5{\cdot}117$ is the depth of water in the ditches in 4 hours and a half, or when the tide has risen to the height of 9 feet without the ditches.

Note. In the foregoing problems, concerning the efflux of water, it is taken for granted that the velocity is the same as that which is due to the whole height of the surface of the supplying water: a supposition which agrees with the principles of the greater number of authors: though some make the velocity to be that which is due to the half height only: and others make it still less.

Also in some places, where the difference between two parabolic segments was to be taken, in estimating the mean velocity of the water through a variable orifice, I have used a near mean value of the expression; which makes the operation of finding the fluents much more easy, and is at the same time sufficiently exact for the purpose in hand.

We may further add a remark here concerning the method

of finding the fluents of the three fluxional forms that occur in the solution of this problem, viz. the three forms $m\dot{z} = (2x+z)\sqrt{(x-z)}\dot{x}$, and $m\dot{z} = (12+z)\sqrt{(x-z)}\dot{x}$, and $m\dot{z} = \sqrt{(x-z)}\dot{x}$, the fluents of which are found by assuming the fluent mz in an infinite series ascending in terms of x with indeterminate coefficients, A, B, C, &c. which coefficients are afterwards determined in the usual way, by equating the corresponding terms of two similar and equal series, the one series denoting one side of the fluxional equation, and the other series the other side. By similar series, is meant when they have equal or like exponents; though it is not necessary that the exponents of all the terms should be like or pairs, but only some of them, as those that are not in pairs will be cancelled or expelled by making their coefficients = 0 or nothing. Now the general way to make the two series similar, is to assume the fluent z equal to a series in terms of x, either ascending or descending, as here

$$z = x^r + x^{r+s} + x^{r+2s}, \text{ \&c. for ascending,}$$
$$\text{or } z = x^r + x^{r-s} + x^{r-2s}, \text{ \&c. for a descending}$$

series, having the exponents r, $r \pm s$, $r \pm 2s$, &c. in arithmetical progression, the first term r, and common difference s; without the general coefficients A, B, C, &c. till the values of the exponents be determined. In terms of this assumed series for z, find the values of the two sides of the given fluxional equation, by substituting in it the said series instead of z; then put the exponent of the first term of the one side equal to that of the other, which will give the value of the first exponent r; in like manner put the exponents of the two 2d terms equal, which will give the value of the common difference s; and hence the whole series of exponents r, $r \pm s$, $r \pm 2s$, &c. becomes known.

Thus, for the last of the three fluxional equations above mentioned, viz. $m\dot{z} = \sqrt{(x-z)}\dot{x}$, or only $\dot{z} = \sqrt{(x-z)}\dot{x}$; having assumed as above $z = x^r + x^{r+s}$ &c. and taking the fluxion, then $\dot{z} = x^{r-1}\dot{x} + x^{r+s-1}\dot{x} +$ &c. omitting the coefficients; and the other side of the equation $\sqrt{(x-z)}\dot{x} = \sqrt{(x - x^r - x^{r+s} \text{ \&c.})} = x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}\dot{x}$, &c. Now the exponents of the first terms made equal, give $r - 1 = \frac{1}{2}$, theref. $r = 1 + \frac{1}{2} = \frac{3}{2}$; and those of the 2d terms made equal, give $r+s-1 = r-\frac{1}{2}$, theref. $s-1 = -\frac{1}{2}$, and $s = 1 - \frac{1}{2} = \frac{1}{2}$; conseq. the whole assumed series of exponents r, $r+s$, $r+2s$, &c., becomes $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, &c. as assumed above in p. 555.

Again for the 2d equation $m\dot{z}$ or $\dot{z} = (12+z)\sqrt{(x-z)}\dot{x}$

$=(a+z)\sqrt{(x-z)}\dot{x}$; assuming $z = x^r + x^{r+s}$, &c. as before, then $\dot{z} = rx^{r-1}\dot{x} + x^{r+s-1}\dot{x}$, &c., and $\sqrt{(x-z)}\dot{x} = x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}\dot{x}$ &c., both as above; this mult. by $a+z$ or $a+x^r+x^{r+s}$, &c. gives $ax^{\frac{1}{2}}\dot{x} - ax^{r-\frac{1}{2}}\dot{x}$, &c.: then equating the first exponents gives $r-1=\frac{1}{2}$ or $r=\frac{3}{2}$, and $r+s-1=r-\frac{1}{2}$, or $s=1-\frac{1}{2}=\frac{1}{2}$; hence the series of exponents is $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, &c. the same as the former, and as assumed in p. 556.

Lastly, assuming the same form of series for z and $\dot{z}$ as in the above two cases, for the 1st fluxional equation also, viz. $m\dot{z} = (2x+z)\sqrt{(x-z)}\dot{x}$: then $\sqrt{(x-z)}\dot{x} = x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}x$, &c. which mult. by $2x+z$, gives $2x^{\frac{3}{2}}\dot{x} - x^{r+\frac{1}{2}}\dot{x}$, &c.: here equating the first exponents gives $r-1=\frac{3}{2}$ or $r=\frac{5}{2}$, and equating the 2d exponents gives $r+s-1=r+\frac{1}{2}$, or $s=\frac{3}{2}$; hence the series of exponents in this case is $\frac{5}{2}$, $\frac{8}{2}$, $\frac{11}{2}$, &c. as used for this case in p. 553. Then, in every case, the general coefficients, A, B, C, &c. are joined to the assumed terms x^r, x^{r+s}, &c., and the whole process conducted as in the three pages just referred to.

Such then is the regular and legitimate way of proceeding, to obtain the form of the series with respect to the exponents of the terms. But, in many cases we may perceive at sight, without that formal process, what the law of the exponents will be, as I indeed did in the solutions in the pages above referred to; and any person with a little practice may easily do the same.

PROBLEM LXI.

To determine the fall of the water in the arches of a bridge.

The effects of obstacles placed in a current of water, such as the piers of a bridge, are, a sudden steep descent, and an increase of velocity in the stream of water, just under the arches, more or less in proportion to the quantity of the obstruction and velocity of the current: being very small and hardly perceptible where the arches are large and the piers few or small, but in a high and extraordinary degree at London-bridge, and some others, where the piers and the sterlings are so very large, in proportion to the arches. This is the case, not only in such streams as run always the same way, but in tide rivers also, both upward and downward, but much less in the former than in the latter. During the time of flood, when the tide is flowing upward, the rise of the water is against the under side of the piers; but the differ-

ence between the two sides gradually diminishes as the tide flows less rapidly towards the conclusion of the flood. When this has attained its full height, and there is no longer any current, but a stillness prevails in the water for a short time, the surface assumes an equal level, both above and below bridge. But, as soon as the tide begins to ebb or return again, the resistance of the piers against the stream, and the contraction of the waterway, cause a rise of the surface above and under the arches, with a fall and a more rapid descent in the contracted stream just below. The quantity of this rise, and of the consequent velocity below, keep both gradually increasing, as the tide continues ebbing, till at quite low water, when the stream or natural current being the quickest, the fall under the arches is the greatest. And it is the quantity of this fall which it is the object of this problem to determine.

Now, the motion of free running water is the consequence of, and produced by the force of gravity, as well as that of any other falling body. Hence the height due to the velocity, that is, the height to be freely fallen by any body to acquire the observed velocity of the natural stream, in the river a little way above bridge, becomes known. From the same velocity also will be found that of the increased current in the narrowed way of the arches, by taking it in the reciprocal proportion of the breadth of the river above, to the contracted way in the arches; viz. by saying, as the latter is to the former, so is the first velocity, or slower motion, to the quicker. Next, from this last velocity, will be found the height due to it as before, that is, the height to be freely fallen through by gravity, to produce it. Then the difference of these two heights, thus freely fallen by gravity, to produce the two velocities, is the required quantity of the waterfall in the arches; allowing, however, in the calculation, for the contraction, in the narrowed passage, at the rate as observed by Sir I. Newton, in prop. 36 of the 2d book of the Principia, or by other authors, being nearly in the ratio of 25 to 21. Such then are the elements and principles on which the solution of the problem is easily made out as follows.

Let b = the breadth of the channel in feet;
v = mean velocity of the water in feet per second;
c = breadth of the waterway between the obstacles.

Now $25 : 21 :: c : \frac{21}{25}c$, the waterway contracted as above.

And $\frac{21}{25}c : b :: v : \frac{25b}{21c}v$, the velocity in the contracted way.

Also $32^2 : v^2 :: 16 : \frac{1}{64}v^2$, height fallen to gain the velocity v.

And $32^2 : (\frac{25b}{21c}v)^2 :: 16 : (\frac{25b}{21c})^2 \times \frac{1}{64}v^2$, ditto for the vel. $\frac{25b}{21c}v$.

Then $(\frac{25b}{21c})^2 \times \frac{v^2}{64} - \frac{v^2}{64}$ is the measure of the fall required.

Or $[(\frac{25b}{21c})^2 - 1] \times \frac{vv}{64}$ is a rule for computing the fall.

Or rather $\frac{1 \cdot 42b^2 - c^2}{64c^2} \times v^2$ very nearly, for the fall.

EXAM. 1. *For London-bridge.*

By the observations made by Mr. Labelye in 1746,
The breadth of the Thames at London-bridge is 926 feet;
The sum of the waterways at the time of low-water is 236 ft.;
Mean velocity of the stream just above bridge is $3\frac{1}{6}$ ft. per. sec.
But under almost all the arches are driven into the bed great numbers of what are called dripshot piles, to prevent the bed from being washed away by the fall. These dripshot piles still further contract the waterways, at least $\frac{1}{6}$ of their measured breadth, or near 39 feet in the whole; so that the waterway will be reduced to 197 feet, or in round numbers suppose 200 feet.

Then $b = 926$, $c = 200$, $v = 3\frac{1}{6} = \frac{19}{6}$.

Hence $\frac{1 \cdot 44b^2 - c^2}{64c^2} = \frac{1217616 - 40000}{64 \times 40000} = \cdot 46$.

And $v^2 = \frac{19^2}{6^2} = 10\frac{1}{36}$.

Theref. $46 \times 10\frac{1}{36} = 4 \cdot 683$ ft. $= 4$ ft. $8\frac{1}{5}$ in. the fall required. By the most exact observations made about the year 1736, the measure of the fall was 4 feet 9 inches.

EXAM. 2. *For Westminster-bridge.*

Though the breadth of the river at Westminster-bridge is 1220 feet; yet at the time of the greatest fall, there is water through only the 13 large arches, which amount to but 820 feet; to which adding the breadth of the 12 intermediate piers, equal to 174 feet, gives 994 for the breadth of the river at that time; and the velocity of the water a little above the bridge, from many experiments, is not more than $2\frac{1}{4}$ ft. per second.

Here then $b = 994$, $c = 820$, $v = 2\frac{1}{4} = \frac{9}{4}$.

Hence $\frac{1{\cdot}42b^2-c^2}{64c^2} = \frac{1403011-672400}{64\times672400} = {\cdot}01722$.

And $v^2 = \frac{81}{16} = 5\frac{1}{16}$.

Theref. $\cdot01722 \times 5\frac{1}{16} = \cdot0872$ ft. $= 1$ in. the fall required; which is about half an inch more than the greatest fall observed by Mr. Labelye.

And, for Blackfriars-bridge, the fall will be much the same as that of Westminster, or rather less.

See farther on this subject, *Gregory's Mathematics for Practical Men*, p. 308, &c.

APPENDIX.

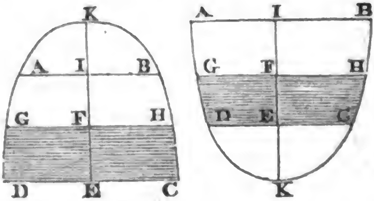
A dissertation on the times of exhausting vessels of a fluid, through holes or apertures in their bottoms, parallel to the horizon.

THEOREM I.

Supposing ABCD to be any vessel containing a fluid; and putting $m = 32\frac{1}{6}$ feet $= 386$ inches, $n =$ the area of the aperture in the bottom DC, $x =$ EF the altitude of the surface of the fluid above the bottom, and $z =$ the area of the descending surface GH; then, the time of exhausting the fluid CDGH will be equal to the fluent of

$$\frac{z\dot{x}}{n\sqrt{mx}}.$$

Demonstration. By Sir I. Newton's Principia, lib. ii. prop. 36, the velocity of the issuing fluid at E, is equal to that which is acquired by a body in falling through $\frac{1}{2}$EF



or $\frac{1}{2}x$; but the velocities of falling bodies are as the roots of the spaces fallen, and m is the velocity (per second) acquired by falling through the space $\frac{1}{2}m$; hence $\sqrt{\frac{1}{2}m} : \sqrt{\frac{1}{2}x} :: m : \sqrt{mx} =$ the velocity of the issuing fluid at E; but the velocities at the orifice and of the descending surface GH will be inversely as their areas (for since the quantity or solidity is the same, the velocities, or altitudes of the equal column will be inversely as the sections); therefore $z : n :: \sqrt{mx} : \frac{n\sqrt{mx}}{z} =$ the velocity (per second) of the descending surface

GH, or the space through which GH would uniformly descend in one second with the velocity it hath at the altitude x. Now in descending the space $\dot{x}$, the velocity may be considered as uniform; and uniform descents are as their times, wherefore $\frac{n\sqrt{mx}}{z} : \dot{x} :: 1 \text{ second} : \frac{z\dot{x}}{n\sqrt{mx}}$ = the time of descending $\dot{x}$ space, or the fluxion of the time of exhausting. Q. E. D.

Scholium.

In the above investigation, the resistance made by the air to the issuing fluid, is neglected, the exhaustion being supposed to be made in a vacuum; which will occasion a small difference (though scarce perceptible) between the calculations and experiments made in the air. But this consideration must by no means be neglected, when the density of the medium, into which the fluid issues, bears any considerable proportion to that of the fluid.

It will make no difference in the account, in whatever part of the base the aperture is placed, the altitude of the surface above it being the only consideration; nor is it material what the figure of it is, whether circular, triangular, square, &c. regular, or irregular, the area of it alone being the only necessary consideration.

As we have above found the fluxion of the time, so by taking the fluent of it we obtain the time itself, substituting first the value of z instead of it, as found in terms of x from the equation of the figure of the vessel, as in the following problems, except when it is a prism, for then z is constant, and does not affect the fluent.

PROBLEM I.

To find the time of emptying a prism.

When the vessel is a cylinder or any other prism, then the section GH or z is constant, and considering it as such, the fluent of $\frac{z\dot{x}}{n\sqrt{mx}}$ is $\frac{2z}{n}\sqrt{\frac{x}{m}}$, which is the time of exhausting a prism whose base is z, and altitude x.

Cor. 1. If the apertures of two prisms therefore be equal, the times of emptying them, will be as their bases drawn into the square roots of their altitudes. If their bases be also equal, the times will be as the square roots of their heights. But when their altitudes are equal, the times will be as their bases.

Cor. 2. Putting a for the altitude EI of the prisms ABCD, and b for the area of the base CD; then the time of emptying the part ABHG at E or D is $\frac{2b}{n} \times \frac{\sqrt{a}-\sqrt{x}}{\sqrt{m}}$. For, by the problem, the time of emptying ABCD is $\frac{2b}{n}\sqrt{\frac{a}{m}}$; and the time of GHCD is $\frac{2b}{n}\sqrt{\frac{x}{m}}$; their difference is the time of ABHG as above.

Cor. 3. By the problem, the time of exhausting ABHG at G or F is $\frac{2b}{n}\sqrt{\frac{a-x}{m}}$; comparing this therefore with the last corollary, it appears that the times of emptying the same part ABHG, at the depth D and G, will be respectively as $\sqrt{a} - \sqrt{x}$ to $\sqrt{(a-x)}$.

Cor. 4. By writing, in the general theorem $\left(\frac{z\dot{x}}{n\sqrt{mx}}\right)$, b for z, a for x, and $a-x$ for $\dot{x}$, we obtain $\frac{a-x}{n\sqrt{ma}} \times b$ for the time in which AB would descend to GH with the first velocity, or the time in which a quantity equal to ABHG would run out supposing the vessel to be kept always full by a supply from without; comparing this therefore with corollary 2, it will appear that the time of emptying the part ABHG at D, when there is no supply from without, is to the time in which an equal quantity would run out when the vessel is kept always full, as $\sqrt{a} - \sqrt{x}$ to $\frac{a-x}{2\sqrt{x}}$, or as $2 \times (a - \sqrt{ax})$ to $a-x$. And when x is nothing, or when ABHG becomes the whole prism, the proportion is that of $2a$ to a or 2 to 1.

PROBLEM II.

To determine the time of exhausting any pyramidal vessel.

Putting, as before, $a =$ the altitude EI, $b =$ the base or area of the bottom DC, $t =$ the area of the top AB; and $n =$ area of the aperture, $m = 32\frac{1}{6}$ feet; also $x =$ any other altitude EF, and z the area of the section GH.

Then, by the nature of the figure, whether the greater or less end is uppermost, we shall have as IE : EF : : the difference between the lines AB, CD, to the difference between the lines GH, CD; that is $a : x : : \sqrt{b} - \sqrt{t} : \sqrt{b} - \sqrt{z} = \frac{\sqrt{b}-\sqrt{t}}{a} \times x$; hence $\sqrt{z} = \sqrt{b} - \frac{\sqrt{b}-\sqrt{t}}{a} \times x$, and $z = b -$

$\frac{b-\sqrt{bt}}{a} \times 2x + \frac{b-2\sqrt{bt}+t}{aa} \times x^2$: this being substituted for z in $(\frac{z\dot{x}}{n\sqrt{mx}})$ the fluxion of the time, in the theorem, and the fluent taken, we obtain $\frac{2\sqrt{x}}{n\sqrt{m}} \times : b - \frac{b-\sqrt{bt}}{3a} \times 2x + \frac{b-2\sqrt{bt}+t}{5aa} \times x^2$ for the time of emptying the part CDGH, for either end up.

Cor. 1. When EF = EI, or $x = a$, the above expression becomes $\frac{2\sqrt{a}}{n\sqrt{m}} \times : b - \frac{2b-2\sqrt{bt}}{3} + \frac{b-2\sqrt{bt}+t}{5}$, or $\frac{2\sqrt{a}}{n\sqrt{m}} \times \frac{8b+4\sqrt{bt}+3t}{15}$ for the time of emptying the whole frustum ABCD.

Cor. 2. When t and b are equal to each other, the above expression in corollary 1 becomes $\frac{2b\sqrt{a}}{n\sqrt{m}}$ for the time in the prism, the same as in the last problem.

Cor. 3. When t is = nothing, the expression in the first corollary becomes $\frac{16b\sqrt{A}}{15n\sqrt{m}}$ for the time of emptying the whole pyramid DKC by a hole at the base; A being here the whole altitude EK.

Cor. 4. When b is = nothing, the expression in the first corollary becomes $\frac{2t\sqrt{A}}{5n\sqrt{m}}$ for the time of emptying the whole pyramid KAB at the vertex K; A being the whole altitude KI, and t the base or the same with b in the last corollary. And hence the first corollary to the last problem will hold also in this.

Cor. 5. By comparing the last three corollaries together, it appears that the time of emptying the prism, the pyramid at its base, and at its vertex, are to one another as 15, 8, and 3; the apertures, altitudes, and bases being equal in each figure.

Cor. 6. Putting a = the axe EK of the pyramid, with either end up, or the distance of the vertex from DC, and the other letters as in corollary 1: Then, by the nature of the pyramid, $A : A \mp a :: \sqrt{b} : \frac{A \mp a}{A} \times \sqrt{b} = \sqrt{t}$; which being substituted for it in the first corollary, we have $\frac{2b\sqrt{a}}{n\sqrt{m}} \times$

$\frac{15AA \mp Aa + 3aa}{15AA}$ for the time of exhausting ABCD at CD; viz. either — or + according as the greater or less end is downward.

But if A represent KI when the vertex is downward, and KE when upward, or the distance from the vertex to the greater end: then, as above, $\frac{2b\sqrt{a}}{n\sqrt{m}} \times \frac{15AA - 10Aa + 3aa}{15AA}$ will be the time of emptying ABCD with the greater end down. But when the less end is down, then $A : A - a :: \sqrt{t} : \frac{A-a}{A} \times \sqrt{t} = \sqrt{b}$; which being written for it in corollary 1, we have, $\frac{2t\sqrt{a}}{n\sqrt{m}} \times \frac{15AA - 20Aa + 8aa}{15AA}$ for the time of emptying ABCD at the less end; where A and a have the same value as at the other end, and t here $= b$ there.

Cor. 7. By taking the difference between two times for two different altitudes, there will be obtained the time of emptying the part next the top of the vessel whose altitude is equal to the difference of the other two next the bottom.

By problem 1 and its 4th corollary $\frac{b\sqrt{a}}{n\sqrt{m}}$ is the time in which a prism of water would run out with the first velocity, and the prism being to the frustum of a pyramid (whose base is b, top t, and altitude a) as b to $\frac{b + \sqrt{bt} + t}{3}$, therefore $\frac{\sqrt{a}}{n\sqrt{m}} \times \frac{b + \sqrt{bt} + t}{3}$ is the time in which a quantity would run out equal to the frustum of a pyramid when it is kept full by a supply at the top. And by comparing this with corollary 1, it will appear that the time of emptying the frustum of a pyramid when it has no supply from without, is to the time in which an equal quantity would run out when kept always full, as $16b + 8\sqrt{bt} + 6t$ to $5b + 5\sqrt{bt} + 5t$.

Cor. 8. By supposing either b or t to vanish in these last expressions, we shall obtain the proportions of the times when there is no supply from without to that when there is, for the whole pyramid. Thus, when $t = 0$, the expressions become as 16 to 5 by running out at the base; and when $b = 0$, they are as 6 to 5 at the vertex: so that they are to one another as 16, 6, and 5.

THEOREM II.

If x be the altitude EF of any part of a vessel formed from a line of the second order, viz. a conic section, $m=32\frac{1}{6}$ feet, n = the area of the aperture in the bottom, and A, B, C constant quantities: then the time of emptying the part CDGH whose altitude is x will be

$$\frac{2\sqrt{x}}{n\sqrt{m}} \times (\mathrm{A} + \tfrac{1}{3}\mathrm{B}x + \tfrac{1}{5}\mathrm{C}x^2).$$

Demonstration. By theorem 1, the fluxion of the time is $\frac{z\dot{x}}{n\sqrt{mx}}$; and by p. 222 of my Mensuration z is as $\mathrm{A}+\mathrm{B}x+\mathrm{C}x^2$, or z may be supposed $= \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2$: substituting therefore this value of z instead of it, and taking the fluent, we obtain the time as is expressed in the theorem.

Scholium. The particular values of A, B, and C will be determined from the nature of the curve in question, and from the situation of x with regard to the vertex or to the centre of the curve. In general, however,

Corollary. When x begins at the vertex of the curve, then A is $= 0$, and the above expression becomes $\frac{\sqrt{x}}{n\sqrt{m}} \times (\frac{1}{3}\mathrm{B}x+\frac{1}{5}\mathrm{C}x^2)$ for the time in the whole solid.

PROBLEM III.

To find the time of emptying a parabolic vessel, or any part of a paraboloid.

In the paraboloid, putting, as before, t for the area of the top AB of the vessel, b for that of the bottom DC, a for its altitude EI, x = EF, and z = the section GH. Then, by the property of the figure, $a : x : : t-b : z-b$; hence $z = b + \frac{t-b}{a} \times x$: but the general value of z is $\mathrm{A} + \mathrm{B}x + \mathrm{C}x^2$: comparing these two therefore together, we have $\mathrm{A} = b$, $b = \frac{t-b}{a}$, and $\mathrm{C} = 0$. By substituting therefore these values in theorem 2, we obtain $\frac{2\sqrt{x}}{n\sqrt{m}} \times (b + \frac{t-b}{3a}) \times x = \frac{2\sqrt{x}}{n\sqrt{m}}$

$\times \frac{3ab+tx-br}{3a}$ for the time of emptying GHCD, whether the greater or less end be upward.

Other expressions for the time might be found, by bringing into the form the whole axe of the parabola and its parameter.

Cor. 1. When x is $= a$, the above form becomes $\frac{2\sqrt{a}}{n\sqrt{m}} \times \frac{2b+t}{3}$ for the time of emptying the whole frustum ABCD.

And the difference between this and the former would give the time of emptying ABHG at DC.

Cor. 2. When t is $= 0$, the last becomes $\frac{4b\sqrt{A}}{3n\sqrt{m}}$ for the time of emptying the whole paraboloid at the base; A being the whole axe.

Cor. 3. When b is $= 0$, the first corollary becomes $\frac{2t\sqrt{A}}{3n\sqrt{m}}$ for the time of emptying the same at the vertex; t in this being equal b in the last, and A = the whole axe, as in the last corollary. So that the times are as 2 to 1. Any time is universally as the base into the root of the altitude, &c. in every particular, as in corollary 1 to problem 1.

Cor. 4. When $b = t$, corollary 1 gives $\frac{2b\sqrt{a}}{n\sqrt{m}}$ for the prism, as in the first problem.

Cor. 5. A parabolic frustum whose base is b, top t, and altitude a, is to a prism of = base and altitude, as $\frac{b+t}{2}$, to b; hence, by proceeding as in corollary 8 to problem 2, we shall obtain $\frac{\sqrt{a}}{n\sqrt{m}} \times \frac{b+t}{2}$ for the time in which a quantity equal to the whole frustum ABCD would run out, when it is kept full by a supply at the top; and is therefore to the time, in corollary 1, of emptying the same when there is no supply, as $\frac{b+t}{4}$ to $\frac{2b+t}{3}$, or as $3b+3t$ to $8b+4t$.

Cor. 6. When $t = 0$, the last proportion becomes as 3 to 8 for the times at the base: and when $b = 0$, it becomes that of 3 to 4 at the vertex. So that all the three times for the whole paraboloid, viz. time at the base, and time at the

vertex when there is no supply from without, and the time of running an equal quantity at either end when it is kept full by a supply from without, are respectively as the numbers 8, 4, and 3.

PROBLEM IV.

To find the time of emptying any part of the sphere.

Putting r for the radius of the sphere, $p = 3{\cdot}14159$, &c. q for the distance of the centre of the sphere above the bottom of the vessel, x for the altitude of the vessel from the bottom; and z, m, n, as before. Then, by the nature of the circle, $z = (r+q-x) \times (r - q+x) \times p = [r^2 - (q-x)^2] \times p = (rr - qq + 2qx-xx) \times p$: comparing this with the general expression $z = \text{A}+\text{B}x+\text{C}x^2$, we have $\text{A} = (rr-qq) \times p$, $\text{B} = 2pq$, and $\text{C} = -p$; which values being substituted in theorem 2, we get $\dfrac{2p\sqrt{x}}{n\sqrt{m}} \times (rr-qq+\frac{4}{3}qx-\frac{1}{5}xx)$ for the time of emptying the spherical vessel whose altitude is x.

The above supposes the bottom of the vessel to be below the centre, but the expression will hold in every case; for when the bottom is above the centre, then q is negative, and the term $+ \frac{4}{3}qx$ will become $- \frac{4}{3}qx$; and when the bottom passes through the centre, then $q = 0$, and the terms qq and $+ \frac{4}{3}qx$ vanish.

Cor. 1. When $q = 0$, the expression becomes $\dfrac{2p\sqrt{x}}{n\sqrt{m}}$ $(rr-\frac{1}{5}xx)$ for the time of emptying at the base any altitude x of a hemisphere.

Cor. 2. And when $x = r$ the whole altitude, the last expression becomes $\dfrac{8prr\sqrt{r}}{5n\sqrt{m}}$ for the time of emptying the hemisphere at the base $= \dfrac{8b\sqrt{r}}{5n\sqrt{m}}$, putting $b = prr$ the area of the base or greatest circle.

Cor. 3. When the bottom of the vessel or the aperture is at the vertex of the figure, then q is $= r$, and the expression in the problem becomes $\dfrac{2p\sqrt{x}}{n\sqrt{m}} \times (\frac{2}{3}rx - \frac{1}{5}xx)$ for the time of emptying at the vertex any altitude x of the hemisphere.

Cor. 4. When x is $= r$, the last expression becomes

$\frac{14prr\sqrt{r}}{15n\sqrt{m}} = \frac{14b\sqrt{r}}{15n\sqrt{m}}$ for the time of emptying the hemisphere at the vertex.

Cor. 5. When $x = 2r$, the expression in corollary 3 becomes $\frac{16prr\sqrt{2r}}{15n\sqrt{m}} = \frac{16b\sqrt{2r}}{15n\sqrt{m}}$ for the time of emptying the whole sphere.

Cor. 6. Subtracting the time in cor. 4, from that in cor. 5, we obtain $\frac{2prr\sqrt{r}}{n\sqrt{m}} \times \frac{8\sqrt{2}-7}{15} = \frac{2b\sqrt{r}}{n\sqrt{m}} \times \frac{8\sqrt{2}-7}{15}$ for the time of emptying the upper hemisphere at the vertex of the lower one.

Cor. 7. By comparing cor. 1, with cor. 3, it appears that the times of emptying equal altitudes of a hemisphere at the base and vertex, are to each other respectively as $3rr - \frac{3}{5}xx$ to $2rx - \frac{3}{5}xx$, or as $15rr - 3xx$ to $10rx - 3xx$.

And when x becomes equal to r, the times of emptying the whole hemisphere at the base and vertex, are as 12 to 7.

Cor. 8. By cor. 4, to prob. 1, $\frac{2prr\sqrt{r}}{3n\sqrt{m}}$ or $\frac{2b\sqrt{r}}{3n\sqrt{m}}$ is the time of running out a quantity equal to the hemisphere with the first velocity, or when it is kept full by a continual supply from without; by comparing this with cor. 2 and 4, we find that the above time and the times of emptying at the vertex and base when there is no supply, are as the numbers 5, 7, 12.

PROBLEM V.

To determine the time of emptying any spheroidal vessel.

Putting r for the revolving and f for the fixed semi-axe, and the other quantities as in the last problem. Then in this problem will be two cases, viz. one with the revolving and the other with the fixed axe horizontal.

CASE I.

When the revolving axe is horizontal.

In this case the horizontal sections are circles, and the circle z is $= \frac{prr}{ff} \times (ff - qq + 2qx - xx)$; and then pro-

ceeding as in the last problem, we shall obtain $\frac{2prr\sqrt{x}}{ffn\sqrt{m}} \times (ff - qq + \frac{4}{3}qx - \frac{1}{2}xx)$ for the time of emptying any part whose altitude is x.

Cor. 1. When the bottom of the vessel passes through the centre of the spheroid, then $q = 0$, and the above expression becomes $\frac{2prr\sqrt{x}}{ffn\sqrt{m}} \times (ff - \frac{1}{5}xx)$ for the time of emptying at the circular base any altitude x of a hemi-spheroid.

Cor. 2. And when $x = f$ the whole altitude, the last expression becomes $\frac{8prr\sqrt{f}}{5n\sqrt{m}} = \frac{8b\sqrt{a}}{5n\sqrt{m}}$ for the time of emptying the hemi-spheroid at the base; putting $b = prr$ the base, and a for the whole altitude or vertical semi-axe.

Cor. 3. When the water issues at the vertex, then $q = f$ and the expression in the problem becomes $\frac{2prr\sqrt{x}}{ffn\sqrt{m}} \times (\frac{2}{3}fx - \frac{1}{5}xx)$ for the time of emptying at the vertex any altitude x.

Cor. 4. When x is $= f$, the last expression becomes $\frac{14prr\sqrt{f}}{15n\sqrt{m}} = \frac{14b\sqrt{a}}{15n\sqrt{m}}$ for the time of emptying the semi-spheroid at the vertex.

Cor. 5. And when $x = 2f$ the whole vertical axe, then it becomes $\frac{16prr\sqrt{2f}}{15n\sqrt{m}} = \frac{16b\sqrt{A}}{15n\sqrt{m}}$ for the time of emptying the whole spheroid, A being the whole vertical axe.

CASE II.

When the fixed axe is horizontal.

In this case the horizontal sections are ellipses, and the section z is $= \frac{pf}{r} \times (rr - qq + 2qx - xx)$; and therefore $\frac{2pf\sqrt{x}}{rn\sqrt{m}} \times (rr - qq + \frac{4}{3}qx - \frac{1}{5}xx) =$ the time of emptying the frustum whose altitude is x.

Cor. 1. When $q = 0$, the above expression becomes $\frac{2pf\sqrt{x}}{rn\sqrt{m}} \times (rr - \frac{1}{3}xx)$ for the time of the frustum whose altitude is x, and its bottom the greatest ellipse or section through the centre.

Cor. 2. And when $x=r$, the last becomes $\frac{8pfr\sqrt{r}}{5n\sqrt{m}} = \frac{8b\sqrt{a}}{5n\sqrt{m}}$ for the time of emptying the semi-spheroid at the base; b and a being as in the last case.

Cor. 3. When $q=r$ the general expression becomes $\frac{2pf\sqrt{x}}{rn\sqrt{m}} \times (\frac{4}{3}rx - \frac{1}{5}xx) =$ the time of the segment at the vertex.

Cor. 4. And when $x = r$, this last becomes $\frac{14pfr\sqrt{r}}{15n\sqrt{m}} = \frac{14b\sqrt{a}}{15n\sqrt{m}}$ for the time of the semi-spheroid at the vertex.

Cor. 5. Also when $x=2r$, the same becomes $\frac{16pfr\sqrt{2r}}{15n\sqrt{m}} = \frac{16b\sqrt{A}}{15n\sqrt{m}}$ for the time of the whole spheroid, A being the whole vertical axe.

And the times for the hemi-spheroid when there is no supply is to that when there is, the very same as the times for the hemisphere in the 8th cor. to the last problem, viz. as the numbers 12, 7, 5.

PROBLEM VI.

To determine the time of emptying any part of a hyperboloid.

Putting t for the semi-transverse, c the semi-conjugate, x for any altitude from the bottom of the vessel, z the section at the top of x, q for the altitude of the centre of the generated hyperbola above the bottom of the vessel, and which therefore must be negative when the less end of the vessel is downward, and m, n, p, as before.

Then by the nature of the hyperbola, $tt : cc :: p \times (q - x + t) \times (q - x - t) : \frac{pcc}{tt} \times [(q - x)^2 - t^2] = \frac{pcc}{tt} \times (qq - tt - 2qx + xx)$. Hence, as in the last problem, $\frac{2pcc\sqrt{x}}{ttn\sqrt{m}}$

$\times\ (qq - tt - \frac{2}{3}qx + \frac{1}{5}xx)$ will be the time of emptying any frustum whose altitude is x.

Cor. 1. In a whole hyperboloid with the vertex upward, q is $= t + x$; which being substituted for it in the above expression, we obtain $\frac{8pcc\sqrt{x}}{15ttn\sqrt{m}} \times (5tx + 2xx)$ for the time of emptying the whole hyperboloid at the base whose altitude is x.

But since the base b is $= pcc \times \frac{2tx+xx}{tt}$, the time of emptying the same will also be $\frac{8b\sqrt{x}}{15n\sqrt{m}} \times \frac{5t+2x}{2t+x}$.

Cor. 2. And when $x = t$, the above expression becomes $\frac{56cc\sqrt{t}}{15n\sqrt{m}} = \frac{56b\sqrt{a}}{45n\sqrt{m}}$ for the time of emptying a whole hyperboloid at the base when the altitude is equal to the semi-transverse; b being the base, and a the altitude.

Cor. 3. When the smaller end is downward, then q is negative, and the general expression becomes $\frac{2pcc\sqrt{x}}{ttn\sqrt{m}} \times (qq - tt + \frac{2}{3}qx + \frac{1}{5}xx)$ for the time in a frustum with the less end downward.

Cor. 4. And when q is $= t$, this last expression becomes $\frac{2pcc\sqrt{x}}{ttn\sqrt{m}} \times (\frac{2}{3}tx + \frac{1}{5}xx) = \frac{2b\sqrt{x}}{15n\sqrt{m}} \times \frac{10t+3x}{2t+x}$ for the time of emptying a whole hyperboloid at the vertex, the altitude being x, and the area of the end b.

Cor. 5. And when x is $= t$, the last form becomes $\frac{26pcc\sqrt{t}}{15n\sqrt{m}} = \frac{26b\sqrt{t}}{45n\sqrt{m}} = \frac{26b\sqrt{a}}{45n\sqrt{m}}$ for the time of emptying at the vertex the hyperboloid whose altitude is equal to the semi-transverse.

Cor. 6. Since $\frac{b\sqrt{x}}{n\sqrt{m}} =$ the time of running out a quantity equal to the prism bx with the first velocity; and, by page 384 of my Mensuration, $bx \times \frac{t+\frac{1}{3}x}{2t+x}$ being the content of the hyperboloid; wherefore $\frac{b\sqrt{x}}{3n\sqrt{m}} \times \frac{3t+x}{2t+x}$ will be the time of running out a quantity equal to the hyperboloid with the first

velocity, or when it is kept full by a continual supply from without. Comparing therefore this with cor. 1 and 4, it appears that the time above and the times of emptying at the vertex and base when there is no supply from without, are to one another as the three quantities, $5 \times (3t + x)$, $2 \times (10t + 3x)$, and $8 \times (5t + 2x)$; and when the altitude x is equal to t, the same times are as the numbers 10, 13, and 28.

General Scholium. In all the foregoing complete solids, viz. the prism, the whole pyramid, the whole paraboloid and hyperboloid whose altitude is equal to the semi-transverse, and hemisphere and hemispheroid, it appears that the time of emptying each, either at the base or vertex, is always as the base drawn into the square root of the altitude, and that all that is said in cor. 1 to prob. 1, belongs in common to them all. And, collecting all the rules for the times of exhausting at the base and vertex, and the time of running out an equal quantity with the first velocity, or when the vessel is kept always full by a continual supply from without, we shall have the time for each equal to $\frac{\textit{base} \times \sqrt{(\textit{altitude})}}{\textit{apert.} \times \sqrt{(32\frac{1}{6}\textit{feet})}}$ drawn into these corresponding numbers:

	At the base.	At the vertex.	With the 1st. veloc.
For the prism	2	2	1
Hemisphere and Hemispheroid .	$\frac{8}{5}$	$\frac{14}{15}$	$\frac{2}{3}$
Paraboloid	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{1}{2}$
Hyperboloid	$\frac{56}{45}$	$\frac{28}{45}$	$\frac{4}{9}$
Pyramid	$\frac{16}{15}$	$\frac{2}{5}$	$\frac{1}{3}$

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ADDITIONS,

BY THE EDITOR, R. ADRAIN.

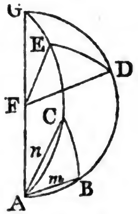
New Method of determining the Angle contained by the chords of two sides of a Spherical Triangle.

See prob. v. page 79 this vol.

THEOREM.

If any two sides of a Spherical Triangle be produced till the continuation of each side be half the supplement of that side, the arc of a great Circle joining the extremities of the sides thus produced will be the measure of the angle contained by the chords of those two sides.

DEMONSTRATION.



Let the two sides AB, AC of the spherical triangle ABC be produced till they meet in G, and let the supplements BG, CG, be bisected in D and E, also let the chords A*m*B, A*n*C of the arcs AB, AC be drawn; and the great circular arc DE will be the measure of the rectilineal angle contained by the chords A*m*B, A*n*C.

Let the diameter AG be the common section of the planes of ABG, ACG, and F the centre of the sphere, from which draw the straight lines FD, FE.

Since, by hypothesis, GE is the half of GC, therefore the angle at the centre GFE is equal to the angle at the circumference GA*n*C (theo. 49. Geom.) and therefore A*n*C and FE, being in the same plane, are parallel: in like manner, it is shown that FD and A*m*B are parallel, and therefore the rectilineal angles BAC and DEF, are equal, and consequently, since DE is the measure of the angle DFE, it is also the measure of the angle contained by the chords A*m*B, A*n*C. Q. E. D.

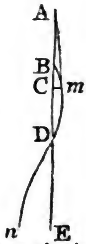
New method of Determining the Oscillations of a Variable Pendulum.

The principles adopted by Dr. Hutton in the solution of his 51st problem, page 541 this vol. are, in my opinion, erroneous. He supposes the number of vibrations made in a given particle of time to depend on the length of the pendulum only, without considering the accelerative tension of the thread; so that by his formula we have a finite number of vibrations performed in a finite time by the descending weight, even when the ascending weight is infinitely small or nothing. Besides, the stating by which he finds the fluxion of the number of vibrations, is referred to no geometrical or mechanical principle, and appears to be nothing but a mere hypothesis. The following is a specimen of the method by which such problems may be solved according to acknowledged principles.

PROBLEM.

If two unequal weights m and m′, connected by a thread passing freely over a pulley, are suspended vertically, and exposed to the action of common gravity, it is required to investigate the number of vibrations made in a given time by the greater weight m, supposing it to descend from the point of suspension, and to make indefinitely small removals from the vertical.

SOLUTION.



Let the summit A of a vertical ABCDE be the point from which m descends, B any point in AE taken as the beginning of the plane curve BmDn described by m, which is connected with m' by the thread Am. Let mC be at right angles to AE, and put AC$=x$, C$m=y$, A$m=r$; also let τ, t and T be the times of the descent of m through the vertical spaces AB, AC and BC; $g = 32\frac{1}{6}$ feet, = the measure of accelerative gravity; f = the measure of the retarding force which the tension of the thread exerts on m in the direction mA, and c = the indefinitely small horizontal velocity of m at B.

As $r : x :: f : \dfrac{fx}{r}$ = the vertical action of the tension on m;

and theref. $g - \dfrac{fx}{r}$ = the true accelerative force with which m is urged in a vertical direction.

Again, $r : y :: f : \frac{fy}{r}$ = the horizontal action on m produced by the tension of the thread AM. Thus the whole accelerative forces by which m is urged in directions parallel to x and y, are $g - \frac{fx}{r}$, and $\frac{fy}{r}$, the former of those forces tending to increase x, and the latter to diminish y; and therefore by the general and well-known theorem of variable motions (See *Mec. Cel.* B. 1, Chap. 2), we have the two equations

$$\frac{\ddot{x}}{\dot{t}^2}=g-\frac{fx}{r} \text{ and } \frac{\ddot{y}}{\dot{t}^2}=-\frac{fy}{r}.$$

But by hypothesis, the angle mAC is indefinitely small, we have therefore $\frac{x}{r}=1$, and $f=\frac{2m'g}{m+m'}$ = a given quantity; our first fluxional equation therefore becomes

$$\frac{\ddot{x}}{\dot{t}^2}=g-f,$$

of which the proper fluent is $x=\frac{1}{2}(g-f)t^2$: and by substituting for x the value just found, our second fluxional equation becomes

$$\frac{\ddot{y}}{\dot{t}^2}=-\frac{2f}{g-f}\,\frac{y}{t^2} \text{ or } \frac{t^2\ddot{y}}{\dot{t}^2}+py=0, \text{ (putting } p=\frac{2f}{g-f}=\frac{4m'}{m-m'}\text{)}.$$

Now when p is less than $\frac{1}{4}$, let $q=\sqrt{\frac{1}{4}-p}$, and in this case the correct fluent of the equation $\frac{t^2\ddot{y}}{\dot{t}^2}+py=0$, is easily found to be

$$\frac{y}{c}=\frac{t^{\frac{1}{2}}\tau^{\frac{1}{2}}}{2q}\cdot\left\{\left(\frac{t}{\tau}\right)^{q}-\left(\frac{t}{\tau}\right)^{-q}\right\};$$

from which equation it is manifest that as t increases y also increases, so that m never returns to the vertical, and there are no vibrations. Again, when $p=\frac{1}{4}$, the correct fluent of the same fluxional equation is

$$\frac{y}{c}=\sqrt{t\tau}\text{ hyp. log. }\left(\frac{t}{\tau}\right).$$

So that in this case also, when t increases y increases, and the body m never returns to the vertical. Since in this case $p=\frac{4m'}{m-m'}=\frac{1}{4}$, therefore $17m'=m$, and therefore by this case and the preceding, there are no vibrations performed by the descending weight m when it is equal to or greater than 17 times the ascending weight m'.

But when p is greater than $\frac{1}{4}$, put $n=\sqrt{p-\frac{1}{4}}$, and in this case the correct equation of the fluents is

$$\frac{y}{c}=\frac{t^{\frac{1}{2}}\tau^{\frac{1}{2}}}{n}\ .\ \text{sin.}\ (n.\ \text{hyp. log.}\ \frac{t}{\tau}).$$

This equation shows us that we shall have $y=0$, as often as n. hyp. log. $\frac{t}{\tau}$ becomes equal to any complete number of semi-circumferences: if therefore $\pi=3\cdot1416$, and N=any number in the series 1, 2, 3, 4, 5, &c. we can have $y=0$ only when n. hyp. log. $\frac{t}{\tau}=N\pi$, from which we have $t=\tau\,.\,e^{\frac{N\pi}{n}}$, supposing hyp. log. $e=1$, and therefore

$$T=\tau\,.\,\left\{e^{\frac{N\pi}{n}}-1\right\},$$

which shows the relation between the number of vibrations N and the time T in which they are performed.

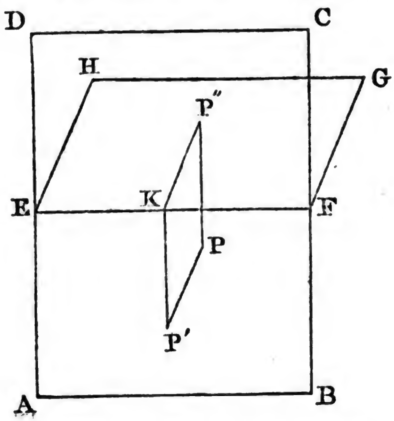
Hence it is manifest that the times or durations of the several successive vibrations constitute a series in geometrical progression.

DESCRIPTIVE GEOMETRY.

CHAPTER I.

Containing the First Principles of Descriptive Geometry, with Illustrations.

Descriptive Geometry is the art of determining by constructions performed on one plane the various points of lines and surfaces which are in different planes. The principle on which this art is founded, consists in projecting the points of any line or surface on two given planes at right angles to each other. These two planes are usually denominated the horizontal and vertical planes, or the fundamental or primitive planes, or the planes of projection. In the constructions the vertical plane is supposed to have revolved about the line of their common intersection, and to be coincident with the horizontal plane; and it is by means of this coincidence that both the projections on the horizontal and vertical planes are effected by constructions performed on the horizontal plane.

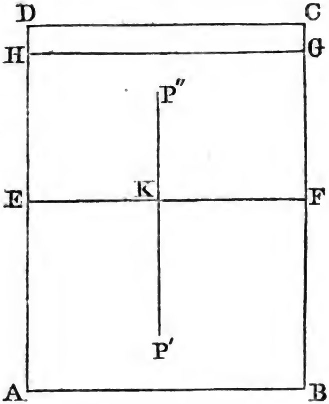


To illustrate this, let ABCD be the horizontal plane, and EFGH the vertical plane at right angles to it, and meeting it in

the straight line of their common section EF. Suppose P to be any point in space, from which on the plane ABCD let fall the perpendicular PP′, meeting that plane in P′; and on the vertical plane EFGH, let fall from P the perpendicular PP″ meeting it in P″; then P′ and P″ are the projections of the point P on the primitive planes: and it is obvious that the projections of any other point besides P cannot be coincident with both the points P′ and P″, and therefore, when the points P′ and P″ are given, there is but one point P of which they are the projections.

From the point P′ draw in the horizontal plane the straight line P′K at right angles to the common section EF, and join P″K. It is obvious that P′K is at right angles to the plane EFGH, and by supposition PP″ is at right angles to the same plane, consequently PP″, P′K are parallels, and therefore in one plane; and since the angle PP′K is a right angle, therefore P′PP″ is also a right angle: and because PP″K is a right angle; it follows that P′KP″ is likewise a right angle; thus it appears that the plane figure PP′ KP″ is a rectangle, and the two distances P′K, P″K, are equal to the two projecting perpendiculars PP″ and PP′. Those perpendiculars PP″ and PP′, or their equals P′K, P″K, are called the ordinates of the point P.

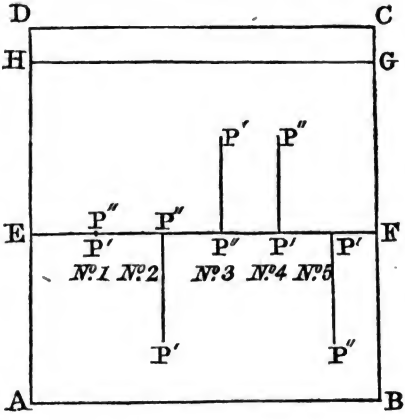
Suppose now, after the points P′ and P″ are determined, that the plane EFGH revolves about its intersection EF from its position at right angles to ABCD, until it coincides with the horizon-



tal plane: during this revolution the straight line KP′ in its motion continues at right angles to the common section EF; and when the vertical plane EFGH, has coincided with the

horizontal plane ABCD, the line P″K of the former plane evidently falls in the continuation of P′K; so that P′K, KP″, make one straight line at right angles to EF, and lying in the horizontal plane; the distances P′K, P″K, being the ordinates or co-ordinates to P, the point in space.

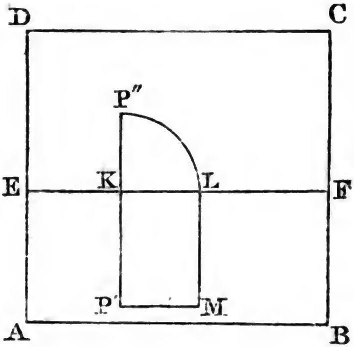
The point P in space is said to be given, when the two perpendiculars or ordinates, P′K and P″K, are given in magnitude and position; and a point sought, P is said to be found when the two ordinates P′K and P″K have been found. The various positions of the projections P′ and P″ corresponding to the different situations of the point P in space, should be clearly conceived by the learner: on this account the following varieties of position deserve attention: and it is particularly to be noted, that the horizontal projection of the point in space is marked with one accent, and the vertical projection with two accents, by means of which the several points of the horizontal and vertical planes will be easily distinguished.



If the point P which is to be projected, be in the ground line or common intersection EF of the fundamental planes, its projections P′, P″, must evidently coincide with the point itself as in N°. 1.

If the point P be in one of the fundamental planes but not in the other, let it first be in the horizontal plane at P′, as in N°. 2, N°. 3. In each of which the vertical projection P″ falls on the ground line: in N°. 2, the point P′ is before

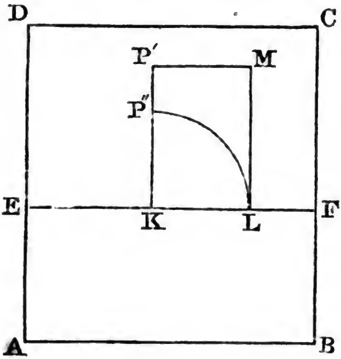
the vertical plane, and in N°. 3, the point P′ is behind the vertical plane. Next let the point P be in the vertical but not in the horizontal plane, as at P″ N°. 4, No. 5. In each of these cases the horizontal projection is manifestly on the ground line at P′. In N°. 4, the point P or P″ is in the vertical plane directly above the point P′ of the ground line, and by the revolution of the vertical plane EFGH into a horizontal position, the point P″ falls behind the ground line EF. In N°. 5, the point P or P″ is directly below the point P′ of the ground line in the continuation of the vertical plane EFGH below the horizontal plane ABCD; and by the same revolution of EFGH as before, the point P″ of the vertical plane immediately below P′, is brought up to the horizontal plane; so that in this last case the point P″ is before the ground line: and therefore the points P″ and P″ of N°. 4 and N°. 5, fall on opposite sides of EF on the horizontal plane by the revolution of the vertical plane. When the point of space P is in neither of the primitive planes, there are four different situations in which it may be found, that require to be particularly distinguished from one another.



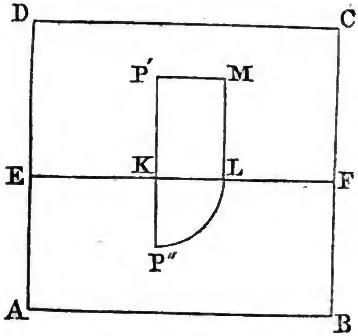
1. When the point P is above the horizontal plane and before the vertical plane. In this case the horizontal projection P′ falls before the ground line EF, and the vertical projection P″ falls behind EF; the horizontal and vertical ordinates being KP′ and KP″. To conceive distinctly the place of the point P, take in EF the distance KL equal to KP″, and on the horizontal plane complete the rectangle KLMP″. Imagine now that the rectangle KLMP′ revolves about its fixed side KP′ from a horizontal to a vertical position by the ascent of the rectangle above the horizontal plane; and when the

rectangle KLMP′ is in this vertical position, its angular point M will coincide with the point P of which the projections are P′ and P″ : and the angular point L, after having described a quadrant of a circle on the primitive vertical plane, will coincide with that point of it which is the vertical projection of P, and which is denoted by the point P″.

2. When the point P is above the horizontal plane, and behind the vertical plane. In this case the projections P′ and P″ both fall behind the ground line EF, in the same straight line KP″P′. Having made KL equal to KP″, and completed the rectangle, suppose it to revolve about its side KP′ which remain fixed by ascending from a horizontal to a vertical position, and the point M will coincide with the point P, which is conceived to be directly above P′, and at an altitude equal to KP″ or P′M.

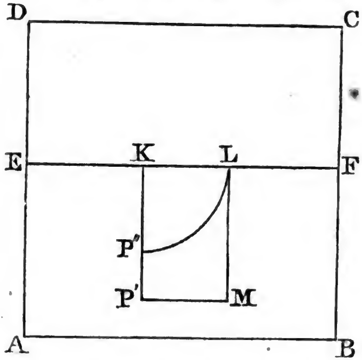


3. When the point P is below the horizontal plane and behind the vertical plane. In this case the point P′ of the horizontal projection of P falls behind the ground line EF, and the vertical projection P″ falls before it. To determine the situation of the point P, corresponding to the projections P′ and P″, complete the rectangle KLMP′ as before ; and suppose it to revolve about the fixed side P′K, from a horizontal to a vertical position by the descent of the side LM so that the point M may be directly below the point P′ of the horizontal plane. Then will the point M coincide with the point P, of which the horizontal and vertical projections are P′ and P″. When the rectangle KLMP′ is in the vertical position, and M coinciding with P, MP′ is the projecting line or ordinate upwards from P on the horizontal plane, and ML is the projecting line from P on the vertical plane ; the point L being in the vertical plane direct-



ly below K. The point L immediately below K is that which is denoted by the point P″; it is brought into the point P″ coinciding with the horizontal plane by the revolution of the vertical plane about EF, from a vertical to a horizontal position, the upper part of the vertical plane falling behind EF toward CD, and the lower part rising so as to coincide with the horizontal plane EF towards AB.

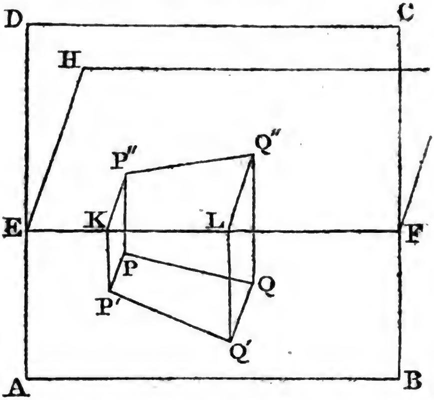
4. When the point P is below the horizontal plane and before the vertical plane. In this case both points of projection P′ and P″ are before the ground line EF. The exact situation of P



corresponding to P′ and P″, may be known by constructing the rectangle KLMP′ as before, and conceiving it to revolve about KP′, by descending from a horizontal to a vertical position. In this vertical situation of the rectangle, the point M coincides

with the point P, which is directly below P′, and MP′, ML are the two projecting lines by which P is represented at P′ on the horizontal plane, and at L directly below K in the vertical plane. By the revolution of the vertical plane about EF, the point of projection L directly below K is brought upwards into the point P″ on the horizontal plane.

As a farther elucidation of the general principle, let us consider the projections of straight lines.



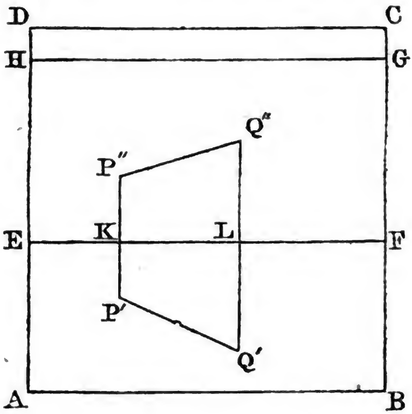
Let ABCD be the horizontal plane, and EFGH the vertical plane at right angles to the former, and meeting it in their common intersection EF; and let PQ be any straight line in space. From P and Q any two points of the straight line PQ imagine two straight lines PP and QQ′ to be drawn at right angles to the horizontal plane ABCD and meeting it in P′ and Q′; and from the same points P and Q two other straight lines PP″ and QQ″ to be drawn at right angles to the vertical plane EFGH, meeting it in the points P″ and Q″. Draw P′K, Q′L at right angles to the ground line EF; join P″K, Q″L; and we have as before the rectangle PP′KP″, of which the sides PP″ and PP′, or their equals P′K and P″K, are the ordinates of the point P: and in like manner QQ″ and QQ′, or their equals Q′LQ″L, are the ordinates of the point of space Q.

Suppose now a plane to pass through the line in space PQ, and either of the perpendiculars PP′, and QQ′; and it is easy to perceive that it will pass through the other perpendicular, and meet the horizontal plane in the straight P′Q′ which joins the points P′ and Q′. It is also evident that all the perpendi-

culars let fall on the horizontal plane from the several points of the line in PQ will meet the horizontal plane in the straight line P′Q′; the straight line P′Q′ is therefore called the horizontal projection of the straight line PQ. From this construction it is plain that the projection of a straight line on a plane is a straight line on the plane passing through the projections on the same plane of any two points of the proposed straight line; or which amounts to the same thing, the projection of a straight line on a plane is the common intersection of this plane and another plane at right angles to the former, and passing through the straight line.

From this definition it is manifest that P″Q″ is the projection of PQ on the vertical plane; so that P′Q′ and P″Q″ are the horizontal and vertical projections of the straight line PQ.

Conceive now that after the projections of PQ are thus made, the vertical plane EFGH revolves about the common section EF from a vertical position till it coincide with the horizontal plane, the higher part of the vertical plane being supposed to fall behind the common section EF; the straight

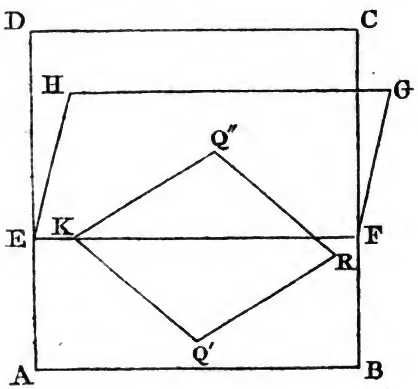


lines which are at right angles to EF will fall in continuation of P′K and Q′L: so that the projections of PQ will now obtain the positions P′Q′, P″Q″ on the same plane; the ordinate P′K, KP″ making one straight line, and Q′L, KQ′ also making one straight line.

The various positions of the projections P′Q′, P″Q″ will be fully exemplified in the subsequent problems; it is sufficient

to observe here that a straight line PQ is said to be given in space when its projections P′Q′ and P″ Q″ are given; and a straight line PQ is said to be found when its projections PQ′, P″Q″ are found. To which we may add that the two planes passing through PQ and each of the projections P′Q′ and P″Q″ are called the projecting planes of PQ; of course a straight line will also be given in position, when we have the intersection of its projecting planes.

When a plane exists in space it is referred to the planes of projection by means of its two intersections with those two planes. Let ABCD and EFGH be the horizontal and vertical planes; and let KQ′RQ″ be any other plane: this plane will in general cut both the planes of projection; the horizontal plane in the straight line KQ′ and the vertical plane in the straight line KQ″. These intersections KQ′, KQ″ are called the traces of the plane KQ′ RQ″; the former KQ′ being the horizontal trace of the plane, and KQ″, its vertical trace. When the vertical plane EFGH revolves about the ground line from a vertical to a horizontal position, the vertical trace KQ″ will be in the horizontal plane, and the two traces will then be in the same horizontal plane, meeting each other in the point K in which the plane KQ′RQ″ cuts the ground line EF. A plane is said to be given in position when its horizontal and vertical traces are given.



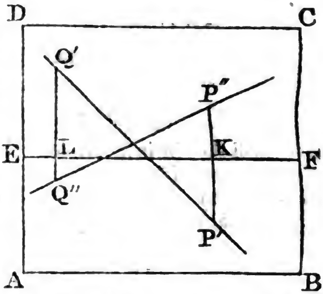
The various positions of the traces of a plane according to the situation of the plane will be exhibited in some of the problems in the following chapter.

CHAPTER II.

Containing Fundamental Problems.

PROBLEM I.

If a straight line be given by its projections, it is required to find its height above the horizontal plane at any point of its projection on that plane.



Let ABCD be the horizontal plane, and EFCD the vertical plane, which by revolution about their common intersection or ground line EF, is brought into a horizontal position.

Let P'Q' and P''Q'' be the two projections of the given line, the former P'Q' on the horizontal plane, the latter P''Q'' on the vertical plane; it is required to find the altitude of the given line above any point P' of the horizontal projection P'Q'.

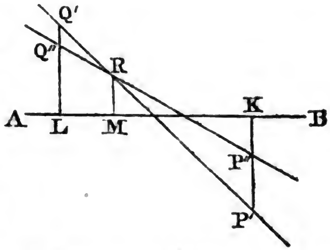
Draw P'K at right angles to the ground line EF, and produce P'K if necessary to meet P''Q'' the vertical projection in P'': and KP'' will be the height of the given straight line above the point P'.

Because P'Q' is the horizontal projection of a straight line, therefore the point P' is the horizontal projection of some point P of that line; but the two projections of a point are always in the same straight line at right angles to the ground line, therefore the vertical projection of P is in P'KP''; and because P''Q'' is the vertical projection of the given line, the vertical projection of P must be in P''Q'', therefore the vertical projection of P is in the point P'', which is the intersection of P'KP'' and P''Q''. Therefore P'K, P''K are the horizontal and vertical ordinates of the point P, and KP'' is equal to the height of the given line above the point P'.

If Q' be any other point in the horizontal projection P'Q'; draw as before Q'LQ'' at right angles to EF, and LQ'' will be the height required. In this second case the point Q'' falls before

the ground line; and therefore, agreeably to the illustrations of the projections of a point given in the first chapter, the distance LQ″ is a depression below the horizontal plane: that is, the point Q, of which Q′ and Q″ are the horizontal and vertical projections, is directly below the point Q′ of the horizontal plane, its distance below Q′ being equal to LQ″. Thus it appears that the point of the given line, of which P and P′ are the projections, is above the horizontal plane and before the vertical plane, but that the point of this line of which Q′ and Q″ are the projectors, is below the horizontal and behind the vertical plane.

Again, let P′ Q′, and P″ Q″ be the horizontal and vertical projections of a straight line, and AB the ground line. In this figure the point of the given line Q, of which Q′ and Q″ are the projections, is above the point Q′ of the horizontal plane, at a height equal to the ordinate L Q″; and the point of the line P, of which P′ and P″ are the projections, is below the point P′ of the horizontal plane, the depression of the point being equal to the ordinate KP″: whence it appears that the point of the given line projected into Q′ and Q″ is above the horizontal and behind the vertical plane; and that the point projected into P′ and P″ is below the horizontal and before the vertical plane.

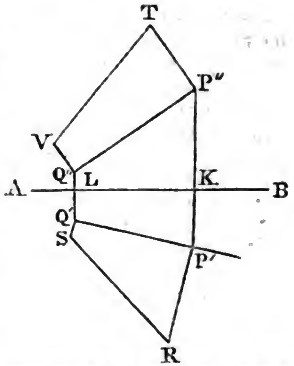


If the projections, P′ Q′ and P″ Q″ intersect in R′, the ordinate RM is common to both the horizontal and vertical projections.

It is evident that if the point P″ of the vertical projection were given, we proceed as before to find the point P, and consequently the distance KP′ of the point from the vertical plane.

PROBLEM II.

If the projecting planes of a given straight line be supposed to revolve about the projections of the straight line, till they coincide with the planes of projection, it is required to find the positions of the straight line on the horizontal and vertical planes.



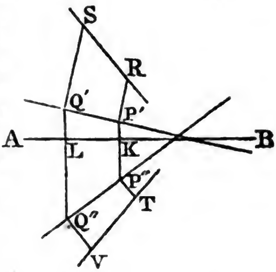
Let AB be the ground line, and P′ Q′, P″ Q″ the horizontal and vertical projections of the lines, which are therefore given. From any two points P′, Q′ of the horizontal projection P′ Q′ draw P′KP″, and Q′LQ″, at right angles to the ground line AB, meeting the vertical projection P″ Q″ in P″ and Q″; also draw P′R, Q′S at right angles to P′Q′, and P″T, Q″V at right angles to P″Q″; and make P′R, Q′ S equal to KP″, LQ″, and P″ T, Q″ V equal to KP′, LQ′: join RS, TV, which will be the positions required.

For the altitudes at P′ and Q′ are, by prob. 1, equal to KP″ and LQ″, and at right angles to P′Q′: these altitudes therefore must by their revolution about P′Q′ coincide with the straight lines P′R, and Q′S, and consequently the straight line itself must coincide with RS. Exactly in the same way it is shown that the straight line will coincide with the line TV.

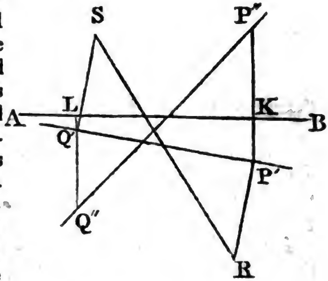
Hence we have a simple method of determining the position of a straight line in space when we have its projections: we have only to find the position of the line on the horizontal plane by this problem, as SR: and then, supposing the trapezoid P′RSQ′ to revolve about its side P′Q′ from a horizontal to a vertical position, the line RS will coincide with the straight line of which the projections are P′Q′ and P″ Q″.

In like manner, if we suppose the trapezoid P″ TVQ″ to revolve about the side P″Q″ from a horizontal to a vertical position, it will then be at right angles to the vertical plane, which in the construction coincides with the horizontal plane: and if now the vertical plane resume its vertical position, VT will coincide with the given line. Thus RS, TV, will coincide, and P′Q′SR, KP′Q″L will be the projecting planes of the given line.

In this construction the ordinates KP″, LQ″ are both altitudes above the horizontal plane, and therefore the perpendiculars P′R, Q′S are drawn on the same side of P′Q′: and because the points P′, Q′ are both before the ground line, the perpendiculars P″T, Q″V, are both on the same side of P″ Q″: and in this case the part of the given line with which RS coincides is directly above the projection P′Q′.



If the points P′ and Q′ of the horizontal projection are both behind the ground line AB, and P″, Q″ the points of vertical projection, both before AB; the construction of the positions RS, TV are made as before: but in this case the trapezoid P′RSP″ must descend by revolving about P′Q′, in order that RS may coincide with the given line, which lies directly below P′Q′.



When the two points P′ and Q′ of the horizontal projection are both on one side of the ground line, and the corresponding points P″ and Q″ of the vertical projections are on different sides of ACB. In this case we place the perpendiculars P′R and Q′S on different sides of P′ Q′; and RS being drawn will be the horizontal position of the given line.

If we suppose the plane P′RSQ′ to revolve about P′Q′ from a horizontal to a vertical position by the ascent of the point R′, and consequently by the descent of S, the straight line RS will then coincide with the given line.

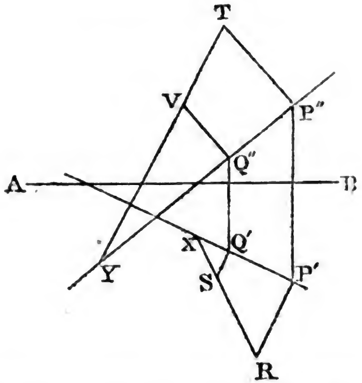
It is evident therefore, that the segment of the given straight line with which RS coincides, has one part of it above the horizontal plane and another part below it.

From these varieties already considered, it appears that when the vertical projections P″, Q″ are both on the same side of the ground line AB, the perpendiculars P′R, Q′S must be placed on the same side of PQ′; and when P″, Q″ are on different sides of AB, the perpendiculars P′R, Q′S must be placed on different sides of P′Q′.

PROBLEM III.

To find the points in which a given straight line meets the planes of projection.

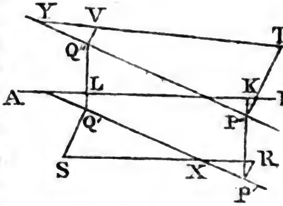
Let AB be the ground line, and P′Q′, P″ Q″ the horizontal and vertical projections of the given line, which projections are



therefore given. Construct by prob. 2, the positions RS, TV, of the given line on the primitive planes by revolving about the projections $P'Q'$, $P''Q''$: produce RS, TV if necessary to meet the projections $P'Q'$, $P''Q''$ in X and Y; and X and Y will be the points in which the given line meets the horizontal and vertical planes.

If RSX, TVY revolve about $P'Q'$ and $P''Q''$, they will (the vertical plane being supposed at right angles to the horizontal) each coincide with the given line, and therefore the points X and Y are in the given line. The point X belongs to the horizontal plane, but the point Y, though determined by a construction on the horizontal plane, is not a point of this plane through which the given straight line passes; it is a point of the vertical plane, and by the revolution of the vertical plane from a horizontal to a vertical position, it falls below the ground line AB.

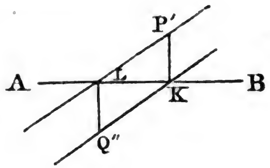
In determining the points X and Y by this method, particular attention must be paid to the positions of the points P', Q', P'', Q'', with respect to the ground line AB. In the annexed figure the points P'' and Q'' fall on different sides of AB, and therefore the perpendiculars $P'R$ and $Q'S$ equal to KP'' and LQ'' must be set on different sides of $P'Q'$, and the point X in which the given line meets the horizontal plane, is between the points P' and Q': but because the points P' and

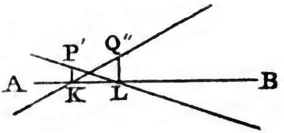


Q′ are both on the same side of AB, the perpendiculars P″T and Q″V equal to KP′ and LQ′, must be placed on the same side of the projection P″ Q″.

Another Construction.

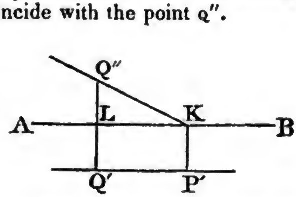
Let AB be the ground line, and LP′, KQ″, the horizontal and vertical projections of the given line, which meet the ground line AB in K and L. From K and L draw KP′, LQ″ at right angles to AB, meeting the projections LP′ and KQ″ in P′, Q″: the points P′ and Q″ are those in which the given straight line meets the horizontal and vertical planes.





The truth of this construction appears from prob. 1. For the point P′ of the horizontal projection has P′K for its distance from the vertical plane, which is also the distance of the point P, of which P′ is the horizontal projection from the same plane; but corresponding to this horizontal distance KP′, there is no vertical distance or ordinate of P, and therefore the point P must be in the horizontal plane, and consequently coincident with P′. In like manner it appears that the vertical ordinate Q″L has no corresponding horizontal ordinate, and therefore the point Q, of which Q″ is the vertical projection, must be in the vertical plane, and must therefore coincide with the point Q″.

If one of the projections, as the horizontal projection P′Q′, be parallel to the ground line AB, let the other projection, viz. the vertical projection Q″K meet the ground line AB in K; from K draw KP′ at right angles to AB meeting PQ′ in P′; and P′ will be the point in which the given line meets the horizontal plane.

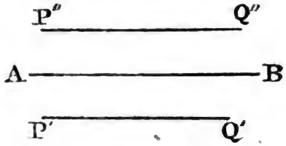


Because P′Q′ is parallel to AB, it is evident that a plane passing through P′Q′ and perpendicular to the horizontal plane, and which is therefore a projecting plane of the given line, must be parallel to the vertical plane: thus it appears that when the horizontal projection P′Q′ is parallel to the ground

line AB, the given line is in a plane parallel to the vertical, and that consequently all its horizontal ordinates are equal.

The given line meets the horizontal plane in P′, and at any point Q′ of the horizontal projection has an altitude equal to the ordinate LQ″.

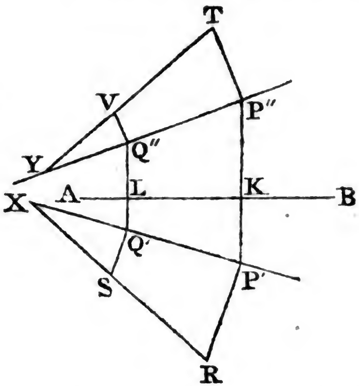
If both the projections P′Q′ and P″ Q″ are parallel to the ground line AB, the given line is in a vertical plane passing through P′Q′, and in a horizontal plane passing through P″Q″, and consequently the given line is a parallel to the ground line AB.



This also is readily deduced from finding the positions of the given line on the horizontal or vertical plane by rotation about the projections by prob. 2.

PROBLEM IV.

To find the angles which a given straight line makes with the planes of projections.



Let AB be the ground line, and P′Q′, P″ Q″ the horizontal and vertical projections of the given line. Find by prob. 2, the positions of the given line RS, TV on the horizontal and vertical plane which produced if necessary meet the projections P′Q′, P″Q″ in X and Y; and RXP′, TYP″ are the angles which the given line makes with the horizontal and vertical planes.

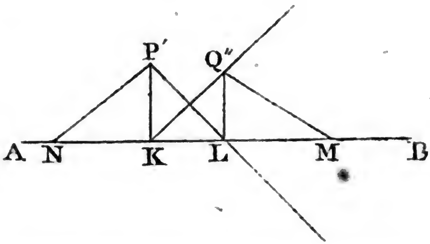
If the triangle RXP′ revolve from a horizontal into a verti-

cal position, it will by prob. 2, coincide with the given line, and the angle RXP′ will therefore coincide with the angle which the given line makes with the horizontal plane; in the same manner it is shown that TVP″ is equal to the angle which the given line makes with the vertical plane.

If the vertical projection P″ Q″ be parallel to the ground line AB, it is evident that RS will be parallel to P′Q′ : of course the angle X will vanish, and the given straight line is parallel to the horizontal plane. In like manner, if P′ Q′ be parallel to AB, TV will be parallel to P″ Q″, the angle Y will vanish, and the given straight line is parallel to the vertical plane.

Another Construction.

Let AB be the ground line, and P′L, Q″K the horizontal and vertical projections of the given straight line which meet AB in L and K. Draw KP′, LQ″ at right angles to AB, and meeting the projections in P′ and Q″. Make LM equal to LP′, and KN equal to KQ″, and the angles LMQ″, KNP″ are equal to the angles which the given line makes with the horizontal and vertical planes.

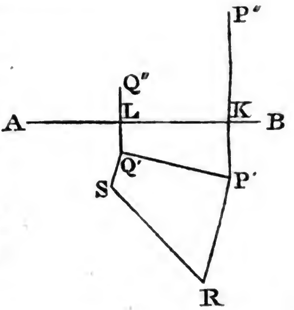


For P′ is the point in which the given line meets the horizontal plane, by prob. 3, and LQ″ is the altitude of the given line at L; it is evident therefore, that the angle which the given line makes with the horizontal plane, is equal to the angle at the base of a right angled triangle of which the base is LP′, or LM, and perpendicular LQ″, and is consequently equal to the angle Q″ML.

PROBLEM V.

To find the distance between two points given by their projections.

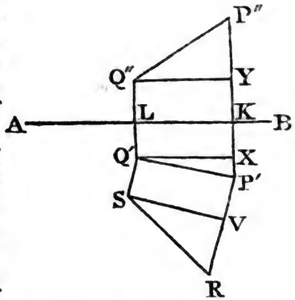
Let AB be the ground line, P′, Q′, the horizontal projections of the two given points, and P″, Q″, their vertical projection: join P′ P″, and Q′ Q″, and the straight lines P′P″, Q′ Q″, will cut the ground line at right angles in K and L. Draw P′ Q′, and find by prob. 2, the horizontal position RS of the line by means of the ordinates KP″, LQ″, and RS will be the required distance.



The truth of this construction is easily perceived; for if the trapezoid P′Q′SR assume a vertical position on the base P′Q′, the straight line RS will evidently coincide with the required line.

A similar construction may be made on the vertical plane by joining the points P″ and Q″.

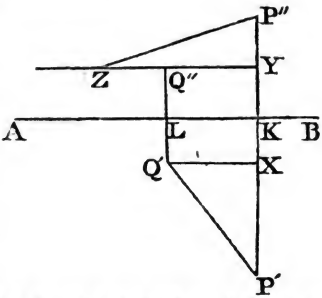
From this construction we deduce the following theorem. The square of the distance between two points, is equal to the sum of the squares of the interval between their ordinates; of the difference of the horizontal ordinates, and of the difference between their vertical ordinates. For if Q″Y, Q′X be parallel to AB, and SV be parallel to Q′ P′, the square of RS is equal the squares of SV and VR; but the square of SV is equal to the square of Q′P′, which is equal to the squares of QX, XP′, or of LK, XP′, and the square of VR is equal to the square of YP″; therefore the square of RS is equal to the squares of KL, XP′ and YP″.



Another Construction of prob. 5.

Let AB be the ground line, and P′, Q′, P″, Q″, the horizontal and vertical projections of the given points as before, and consequently P′P″, Q′Q″, at right angles to AB meeting it in K and L. Through Q″ draw YQ″Z parallel to AB, in which take YZ equal to P′Q′; join P″Z which will be the distance required.

To demonstrate this draw $Q'X$ parallel to AB: and because YZ is equal to $P'Q'$, therefore the square of YZ is equal to the squares of $P'X$, $Q'X$; and therefore the square of $P''Z$ is equal to the squares of KL, XP', and YP''; and therefore, by the preceding theorem, $P''Z$ is equal to the distance between the given points.

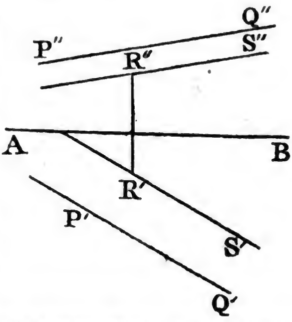


A similar construction may be made by drawing the parallel through Q' instead of Q'', and using the distance $P''Q''$ instead of $P'Q'$.

PROBLEM VI.

Through a given point to draw a straight line parallel to a given straight line.

Let AB be the ground line, $P'Q'$ and $P''Q''$ the horizontal and vertical projections of the given line, and R', R'' the horizontal and vertical projections of the given point. Through R' and R'' draw $R'S'$ and $R''S''$ parallel to PQ' and $P''Q''$; and $R'S'$, $R''S''$, will be the horizontal and vertical projections of the required line.



For let two parallel planes pass through $P'Q'$, $R'S'$, at right angles to the horizontal plane; and let two other parallel planes pass through $P''Q''$, $R''S''$ at right angles to the vertical plane which is supposed coincident with the horizontal plane. Conceive the vertical plane to revolve about AB from its horizontal to its vertical position, while the parallel planes through $P''Q''$, $R''S''$ continue at right angles to it; then the straight line of which $P'Q'$ and $P''Q''$ are the projections, will be the common section of the planes passing through $P'Q'$, and $P''Q''$; and the line of which $R'S'$ and $R''S''$ are the projections, will be in the parallel planes passing through these lines; but when two parallel planes are cut by two other parallel planes, the common sections are parallel;

therefore, the straight line of which R′S′ and R″S″ are the projections, is parallel to the given straight line of which the projections are P′Q′ and P″Q″.

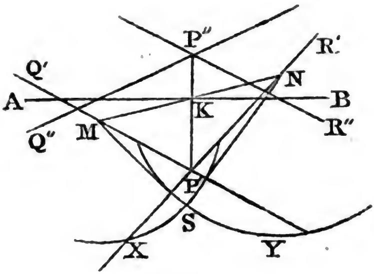
And because the projections R′S′ and R″S″ pass through the projections R′ and R″, therefore, the straight line of which R′S′ and R″S″ are the projections, passes through the given point.

Corol. If the projections of two straight lines on the horizontal plane be parallel, and also their projections on the vertical plane, the straight lines themselves are parallel. To which may be added, that if two straight lines be parallel, their projections on the horizontal plane are parallel, and also their projections on the vertical plane.

PROBLEM VII.

To find the angle contained by two given straight lines meeting each other.

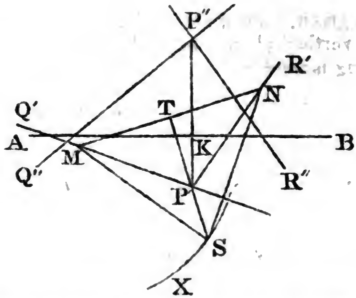
Let AB be the ground line, P′Q′ and P″Q″ the horizontal and vertical projections of one of the given lines, and P′R′, P″R″ the horizontal and vertical projections of the other; the horizontal projections intersecting in P′, and the vertical in P″, which are evidently the projections of the point, in which the given straight lines cut each other, and therefore P′KP′ is a straight line meeting the ground line AB at right angles in K.



Find by prob. 3, the points M and N in which the given straight lines meet the horizontal plane; and by prob. 5, find the distances from the point of the intersection of the given lines to each of the points M and N; with the centres M and N and distances just found, describe two arcs SX, SY intersecting in S; join MS, NS, and the angle MSN will be the angle required.

To demonstrate this draw MN, and we have two triangles on the same base MN, one of which has its vertex in S, and the other its vertex in the point in which the given lines intersect each other; and because the two sides of one of these triangles is by construction equal to the two sides of the other, and the base MN common, therefore the angles at their vertices must be equal; therefore MSN is equal to the angle contained by the given straight lines.

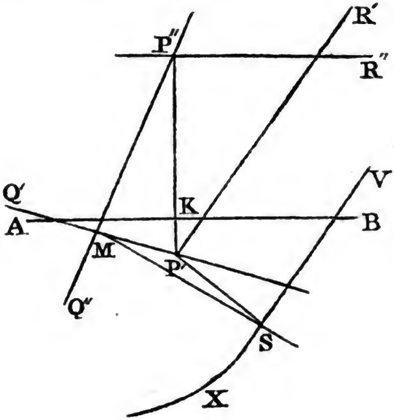
Corol. Since the triangle MP′N is the horizontal projection of the triangle MSN, when S coincides with the point of intersection of the given lines; it follows that the perpendiculars P′T and ST must be in the same straight line; and therefore, the required angle may be found as follows: having M, and N as before, join MN, and through P′ draw TP′S at right angles to MN, with the centre M and distance MS equal to the distance from M to the point of intersection of the given lines, describe an arc SX cutting TP′S in S; join SN, and MSN will be the angle contained by the given lines. From this construction it is evident that if the point N go off to infinity, the straight line MN will become parallel to P′N, and consequently TP′S, which is always at right angles to MN, will then be at right angles to P′R′.



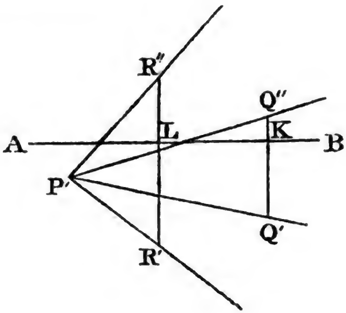
Hence the following construction of the case when one of the projections P″R″ is parallel to the ground line AB.

Let AB be the ground line, P′Q′, and P″Q″ the horizontal and vertical projections of one of the given lines, and P′R′, P″R″ those of the other, the vertical projection of P″R″ being parallel to AB. Draw P′S at right angles to P′R′; find M the point in which the straight line of which P′Q′ and P″Q″ are the projections meets the horizontal plane; also find the distance from M to the point in which the given lines meet each other; with this distance as radius and centre M describe an arc SX meet-

ing P′S in S; joins MS; lastly, draw SV parallel to P′R′, and MSV will be equal to the angle contained by the given straight lines.



When the given lines meet each other in a point of the horizontal or vertical plane, the problem may be constructed in the following manner:

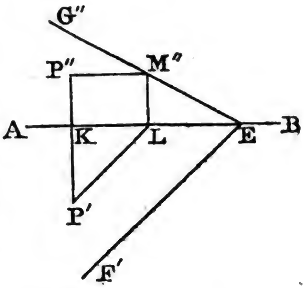


Let AB be the ground line; P′Q′, and P″Q″, the projections of one of the given lines, and P′R′, P″R″ of the other; the given lines meeting in the point P′ of the horizontal plane. Draw

any two perpendiculars to the ground line Q′Q″ and R′R″; and Q′, Q″ will be the projections of a point in one of the given lines, and R′, R″ those of a point in the other. Find by prob. 5, the two distances from P′ to each of the two points of which Q′, Q″, R′, R″ are the projections, and also the distance between these two points: construct a triangle of which the three sides are equal to these three distances, and its angle contained by the first two distances will be equal to the angle sought.

PROBLEM VIII.

If a plane be given by its traces or intersections with the horizontal and vertical planes, it is required to determine its altitude above any given point of the horizontal plane.



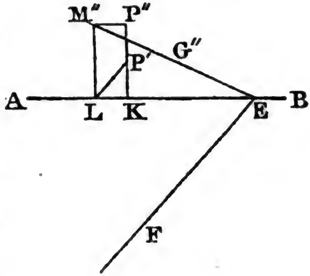
Let AB be the ground line, EF′ the horizontal trace of the given plane, and EG″ its vertical trace; and let P′ be any point in the horizontal plane: it is required to find the altitude of the plane above the point P′.

Draw P′L parallel to F′E meeting the ground line in L; and LM″ at right angles to AB meeting the vertical trace EG″ in M″; from M″ draw M″P″ parallel to AB, meeting P′K produced if necessary in P″; and P″K will be the height of the plane above the horizontal plane at the point P′.

Conceive a vertical plane to pass through P′L; this plane will meet the given plane in a straight line parallel to EF′, because LP″ and EF′ are parallel; and therefore, the common intersection of this vertical plane and the given plane is parallel to P′L, and consequently the altitude of the given plane above the point P′ is equal to its altitude above the point L; but since the trace EM″G″ represents a straight line drawn on the vertical plane, therefore LM″ is the height of the point M″ of the given plane, that is, LM″ is the altitude of the given plane above the point L; therefore, LM″, and consequently its equal KP″, is the altitude of the given plane above the given point P′.

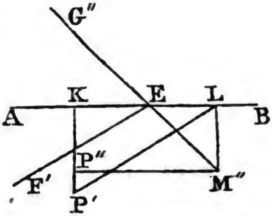
Cor. It appears from the demonstration, that the altitude of the given plane above every point of P′L is equal to KP″ or LM″; and therefore P′L and P″ M″ are the horizontal and ver-

tical projections of the straight line in which the vertical plane passing through $P'L$ intersects the given plane.

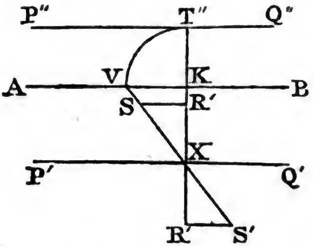


When the point P' is beyond the ground line AB, the straight line $P'L$ must be drawn in a direction opposite to that in the preceding figure in order to meet the ground line.

For the point P' in these two figures, we have found an altitude of the given plane, or the plane passes over the given point of the horizontal plane.



But in the annexed figure the perpendicular from L meets EG'' produced below the ground line AB, and therefore, the given plane is below the horizontal plane at the point P', the depth below P' being equal to KP''; however, the position of P'' shows universally whether the distance from P' at right angles to the horizontal plane be an elevation or depression; for when P'' is behind the ground line, it denotes an elevation: and when before the ground line, it denotes a depression.



When the traces of the plane are parallel to the ground line, the construction is as follows:

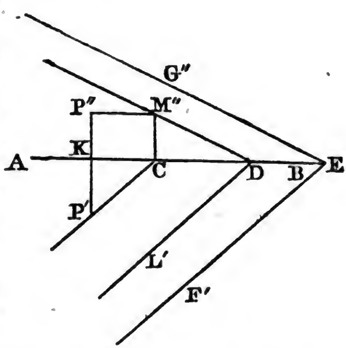
Let AB be the ground line; $P'Q'$, $P''Q''$ the horizontal and vertical traces given which are parallel to AB; and R' any given point in the horizontal plane. Through R' draw $R'X\,KT''$ meeting $P'Q'$, AB, $P''Q''$ at right angles in X, K, and T''; make KV equal to KT'', join VX, and through R' draw $R'S$ parallel AB, meeting VX produced if necessary in S; and $R'S'$ will be the altitude or depression required.

If the triangle VXK revolve about the side KX from a horizontal position to a vertical, it is evident that the point V will coincide with the point T″ of the given plane, and VK will be the altitude at K: therefore R′s, which also becomes perpendicular to the horizontal plane, is the altitude or depression of the given plane at the point R′.

In a similar manner, when a point is given in the vertical plane, we may determine the horizontal distance of the point from the given plane.

PROBLEM IX.

Through a given point to draw a plane parallel to a given plane.

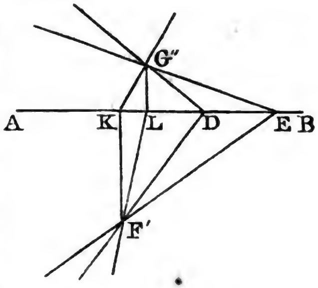


Let AB be the ground line; EF′, EG″ the horizontal and vertical traces of the given plane; and P′, P″ the projections of the given point. Through the point P′ draw the straight line P′C parallel to the horizontal trace F′E, meeting the ground line AB in C: draw from C the straight line CM″ at right angles to AB, and through P″ the straight line P″M″ parallel to AB: through M″ draw M″D parallel to G″E, and DL′ parallel to EF′; and DL′, DM″ will be the horizontal and vertical traces of the plane required.

For, since the plane of which the traces are DL′, DM″, has CM″ for its altitude at C; therefore, by cor. to prob. 8, the altitude at P′ is also equal to CM′ or its equal KP″; and therefore, the point of which the projections are P′ and P″ is in the plane L′DM″. Suppose now the vertical plane with its parallel lines DM′ and EG″ to revolve about AB from a horizontal to a vertical position; and because, in this situation the two straight lines L′D, DM″ are respectively parallel to F′E, EG″, therefore the plane passing through L′D, DM″ is parallel to the plane passing through F′E, EG″; and consequently, L′D, DM″ are the traces of the plane required.

PROBLEM X.

To find the straight line which is the intersection of two given planes.

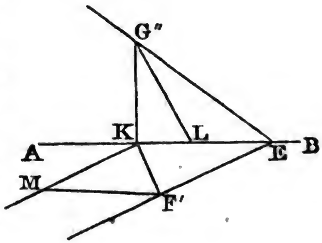


Let DF′, DG″ be the horizontal and vertical traces of one of the given planes, and EF′, EG″ those of the other. Let the horizontal traces of the given planes meet in F′, and their vertical traces in G″. From F′ and G″ draw F′K and G″L at right angles to the ground line AB; join F′L, KG″, and F′L, KG″ will be the horizontal and vertical traces of the intersection required.

For since each of the given planes passes through the point F′, therefore F′ is a point in the horizontal plane through which the required line must pass: in like manner it is shown that the required line must pass through the point G″ of the vertical plane, therefore the intersection of the given planes must pass through the points F′ and G″; but F′L and KG″ are evidently the horizontal and vertical projections of a straight line passing through F′ and G″, and therefore F′L and KG′ are the horizontal and vertical projections of the straight line required.

PROBLEM XI.

To find the angles which a given plane makes with the fundamental planes.



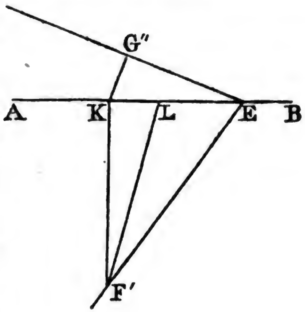
Let AB be the ground line, and EF′, EG″ the horizontal and vertical traces of the given plane. In the horizontal trace EF take any point F′, from which draw F′K at right angles to it, meeting the ground line AB in K; draw KG″ at right angles to AB, meeting EG″ in G″; make KL equal to KF′, join G′L, and the angle G″LK will be

equal to the angle which the given plane makes with the horizontal plane. Or draw KM at right angles to KF′ and equal to KG″; join MF′, and MF′K will be equal to the angle which the given plane makes with the horizontal plane.

For, if the right-angled triangle F′KM revolve about its base F′K from a horizontal to a vertical position, the straight line F′M will coincide with the given plane; because KM is now at right angles to the horizontal plane, and equal to KG″ which is the altitude of the given plane above the point K. And because the straight line EF′ is at right angles to KF′, the common intersection of the horizontal plane, and of KF′M in its vertical position, therefore EF′ is at right angles to MF′, as well as to KF′, and consequently KF′M is the inclination of the given plane to the horizontal plane, or KF′M is the angle required.

Again, since G″K, KL are equal to MK, KF′, and the angles G″KL, MKF′ are equal, being right angles; therefore, the angle G″LK is equal to MF′K, and is therefore equal to the angle contained by the given plane and the horizontal plane.

In like manner we may proceed to find the angle contained by the given plane and the vertical plane.



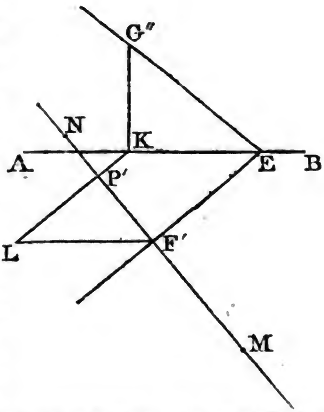
Let AB be the ground line, EF′ and EG″ the horizontal and vertical traces of the given plane. From any point G″ of EG″ draw GK″ at right angles to EG″, and KF′ at right angles to AB; make KL equal to KG″, join F′L and F′LK will be the angle contained by the given plane and the vertical plane.

PROBLEM XII.

If a given plane revolves about its intersection with the horizontal plane till these two planes coincide, it is required to find on the horizontal plane, the position of any given point of the given plane.

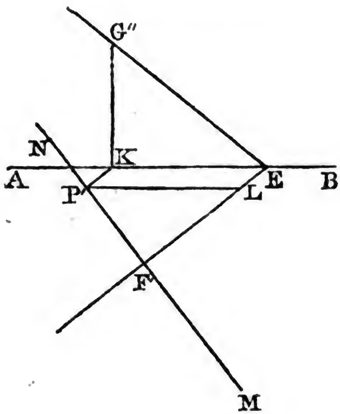
Let AB be the ground line; EF′ and EG″ the horizontal and vertical traces of the given plane; and F′ the horizontal projection of the given point in the plane. Draw through the given point F′ the straight line F′F′ at right angles to EF′, and KF′L at right angles to F′F; from the point K of the ground line draw KG″ at right angles to AB; make F′L equal to KG″; join F′L,

and take in $F'P'$ produced if necessary the distances $F'M$, and $F'N$, each equal to $F'L$, and M and N will be the point required on the horizontal plane.



For, if the triangle $P'F'L$ revolve from a horizontal to a vertical position about the base $P'F'$, till it become vertical, the straight line $F'L$ will coincide with the given plane, and L with the given point: because KG'', to which $P'L$ is equal, is the altitude of the plane above the point P' of the horizontal plane. And because, when the triangle $LF'P'$ is vertical, *the straight line* LF' is at right angles to EF, the intersection of the horizontal plane and the given plane; therefore, if the given plane revolve about EF', the given point with which L has become coincident, must fall on the straight line $P'F'$ which is at right angles to EF': and since LF' is equal to $F'M$, or $F'N$, therefore the given point must fall on one of the points M and N.

Another Construction.



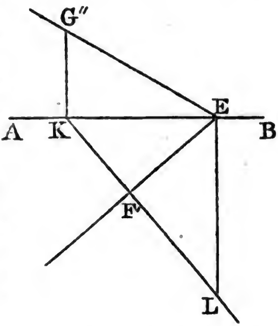
Let AB be the ground line; EF′ and EG″ the horizontal and vertical traces of the given plane; and P′ the horizontal projection of the given point. Having drawn as before MF′P′N at right angles to EF′, make F′L equal to the altitude KG″ of the plane above P′; join P′L, and make F′M, FN each equal to P′L, and M or N will be the point required.

For, because the square of P′L is equal to the squares of P′F′ and F′L, that is, to the squares of P′F′, and the altitude of the given point above P′; therefore, P′L is the distance in the given plane from F′ to the given point. And since the given point is in a vertical plane passing through MN, therefore by the revolution of the given plane about EF′, the point given will describe the circumference of a circle on the vertical plane passing through MN; and this circumference must meet the horizontal plane in the points M and N; because F′M and F′N are each equal to P′L the distance of the given point from F′, consequently M and N are the points required.

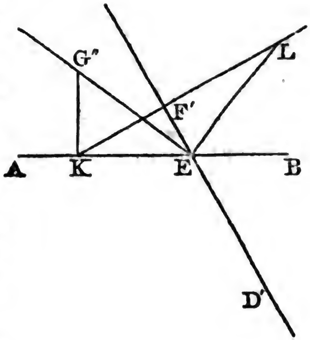
PROBLEM XIII.

To find the angle contained by the traces of a given plane.

Let AB be the ground line; EF′, EG″ the horizontal and vertical traces of the given plane. Take K any given point in AB, draw KG″ at right angles to AB, meeting the vertical trace EG″ in G″, and G″ will be a given point in the given plane, and K its horizontal projection: find by prob. 12, the position L of the point G″ by revolution of the given plane till it coincide with the horizontal plane; join EL, and F′EL will be the angle sought.



For, since the point G″ falls on the point L of the horizontal plane, therefore the straight line or trace EG″ will coincide with EL; but F′E is the horizontal trace, therefore F′EL is the angle contained between the traces of the given plane.

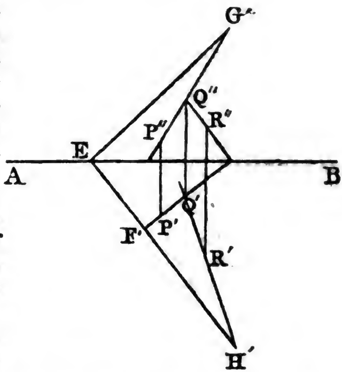


When one of the angles AEG″ is acute, and the other AED′ obtuse, take as before any point K in the ground line AB, make KG″ perpendicular to AB, meeting the vertical trace EG″ in G″; and consequently, G″ is a point of the vertical plane, of which K is the horizontal projection: find L the position of G″ on the horizontal plane: join EL; and LED′ will be the angle contained between the traces of the given plane.

If the given horizontal trace ED′, instead of being given before the ground line AB, is given *behind it* as EF′, the construction is exactly the same as before; and LEF′ is the angle contained between the horizontal trace EF′ and the vertical trace EG″.

PROBLEM XIV.

To describe a plane through three given points.



Let AB be the ground line; P′, Q′, R′ the horizontal projections of the three given points; and P″, Q″, R″, their corresponding vertical projections. Through P′, Q′, and P″, Q″, draw Q′P′F′, and P″Q″G″, the horizontal and vertical projections of a straight line passing through two of the given points; and through Q′, R′; Q″R″ draw Q′R′H′ and Q″R″ the horizontal and vertical projections of a straight line passing through other two of the given points. Find by prob. 3, the points F′ and H′ in which these two straight lines meet the horizontal

plane, and G″ the point in which one of them meets the vertical plane.

Draw H′F′E meeting the ground line AB in E; join EG″; and EH′, EG″ will be the horizontal and vertical traces of the required plane.

The straight line of which P′Q′ and P″Q″ are the horizontal and vertical projections, passes through two of the given points; therefore, that straight line must lie in the required plane; but that line passes through the point F′ of the horizontal plane; therefore, the required plane passes through F′: for the same reason the required plane must pass through H′; and consequently, EF′H′ is the horizontal trace of the required plane: and because the vertical trace must pass through the points E and G″ of the vertical plane, therefore EH″ is the vertical trace; and consequently, EF′ and EG″ are the horizontal and vertical traces of the required plane.

Instead of finding two points in the horizontal and one in the vertical plane, we may find two in the vertical and one in the horizontal plane: or we may find two points in the horizontal plane, and two in the vertical; and the two straight lines drawn, these points on the horizontal and vertical planes will be the traces of the required plane. Sometimes the point E in which the traces meet the ground line is too remote to be conveniently found in practice: this happens when one or both of the traces are nearly parallel to the ground line: in this case, it is proper to find two points through which the horizontal trace must pass, and in a similar manner, two points through which the vertical trace must pass in the vertical plane.

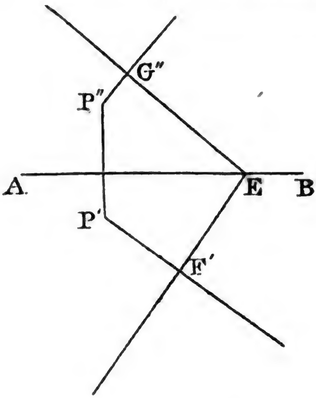
PROBLEM XV.

Through a given point to draw a straight line perpendicular to a given plane.

Let AB be the ground line; EF′, EG″ the horizontal and vertical traces of the given plane; and P′, P″ the horizontal and vertical projections of the given point. Through the projections P′ and P″, draw the straight lines P′F′, P″G″ at right angles to the given traces EF′ and EG″ and P′F′, P″G″ will be the horizontal and vertical projections of the line required.

Conceive a vertical plane to meet the horizontal plane in P′F′; and because EF′ is drawn in one of these planes at right angles to their common section, therefore EF′ is perpendicular to the vertical plane passing through P′F′; and consequently, the given plane which passes through EF′, is at right angles to

the vertical plane passing through P′F′ : but the given point is in this vertical plane, therefore a straight line drawn from the given point at right angles to the given plane, must lie in this vertical plane, and therefore, P′F′ is the horizontal projection of the required straight line. In a similar manner, it may be shown that P″G″ is the vertical projection of the required line.



Corol. From the demonstration of this construction, it appears that when a straight line and a plane are at right angles to each other, the projections of the straight line are at right angles to the corresponding traces of the plane.

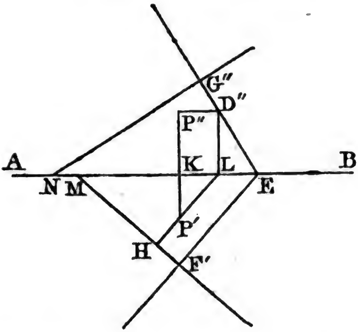
PROBLEM XVI.

Through a given point, to draw a plane at right angles to a given straight line.

Let AB be the ground line; MF′, NG″ the projections of the given straight line; and P′, P″ the projections of the given point.

Through the point P′ draw HP′L at right angles to MF′ meeting the ground line in L; draw from L the straight line LD″ at right angles to AB, meeting the straight line P″ D″ parallel to AB in D″. Through D″ draw ED″G″ at right angles to the vertical projection NG″, meeting AB in E: from E draw EF′ at right angles to the horizontal projection MF′, and EF′, EG″ are the traces of the plane required.

Because the traces EF′, EG″ are at right angles to the given projections MF′, NG″ of the straight line; therefore by the preceding corollary, the plane of which EF′, EG″ are the traces,



is at right angles to the given straight line. Again, because KP″ is equal to LD″, which is the height of the plane above the horizontal plane at L, and consequently to the height at P′ or any point in LP′H; therefore, the plane of which EF′, EG″ are the traces, passes through the given point, and is consequently the plane required.

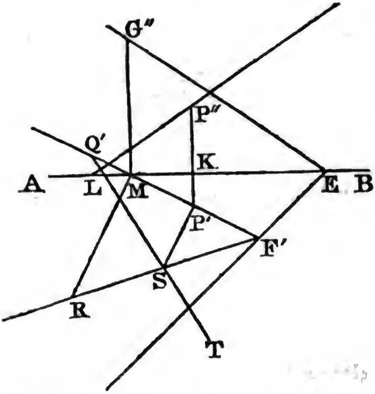
PROBLEM XVII.

It is required to find the point in which a given straight line intersects a given plane.

Let AB be the ground line; P′Q′ the horizontal, and P″L the vertical projection of the given line; and EF′, EG″ the horizontal and vertical traces of the given plane.

Let P′Q′ meet the ground line AB in M, draw MG″ at right angles to AB, and having drawn MR at right angles to P′Q′, make it equal to MG″; let P′Q′ meet EF′ in F, join F′R. Again, find by prob. 2, the position Q′T of the given line on the horizontal plane by revolution about the horizontal trace MF′, and let Q′T intersect F′R in S; draw SP′ at right angles to Q′F′, and P′KP″ at right angles to AB, then P′, P″ will be the horizontal and vertical projections of the point in which the given plane is intersected by the given straight line.

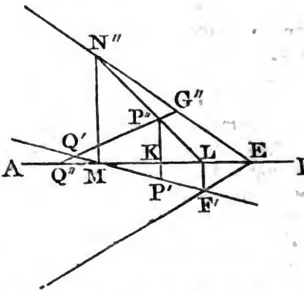
Because MR is equal to MG″, it is evident, that F′R is the horizontal position of the straight line in which the given plane is cut by a vertical plane, passing through MF′, and Q′T is the horizontal position of the given line; therefore, if the triangle Q′SF′ revolve about Q′F from a horizontal to a vertical position, the point S will be at once in the given straight line and given plane; and will therefore be in the point required.



And since SP′ is perpendicular to P′Q′, the straight line P′S will become perpendicular to the horizontal plane; and therefore, P′ is the horizontal projection of the required point.

Lastly, because P″ is the vertical projection of a point in the given line of which P′ is the horizontal projection; therefore, P′ and P″ are the projections of the required point on the horizontal and vertical planes.

Another Construction.



Let AB be the ground line; P′Q′, P″Q″ the horizontal and vertical projections of the given line; and EF′, EG″ the horizontal and vertical traces of the given plane.

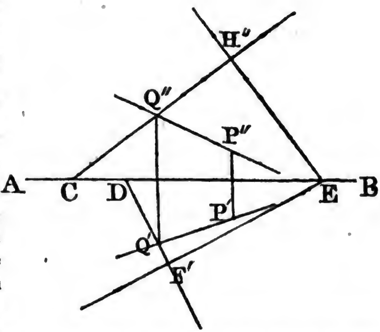
Let P′Q′ meet AB in M, and EF′ in F′; and from M and F′ draw MN, FL at right angles to the ground line AB, meeting EG″, and AB in N and L; join LN″ meeting Q″G″ produced if necessary in P″. Draw P″KP′ at right angles to AB meeting Q′F′ in P′, and P′, P″ will be the horizontal and vertical projections of the point required.

For MN″ is the altitude of the given plane above the point M, and N″ is the vertical projection of the point of the plane in which it is intersected, by the perpendicular to the horizontal plane at M: also L is the vertical projection of the point

F′, and therefore, LN″ is the vertical projection of the straight line which is the common section of the given plane, and a vertical plane passing through MF′: but the given line is in this vertical plane; and therefore meets the given plane in some point of the line, in which the given plane is intersected by the vertical plane passing through MF′; therefore, the vertical projection of the required point must be in LN″, and it must also be in Q″G′; therefore it is in P″, the intersection of LN″, Q″G″; but P′ is the corresponding horizontal projection of that point in the given line of which Q″ is the vertical projection, and consequently P′, P″ are the horizontal and vertical projections of the point required.

PROBLEM XVIII.

To draw through a given point a straight line perpendicular to a given straight line.



Let AB be the ground line; DF′ CH″ the projections of the given straight line, and P′, P″ the projections of the given point. Find by prob. 16, EF′ and EH″ the horizontal and vertical traces of a plane at right angles to the given straight line, and passing through the given point of which the projections are P′ and P″; again by prob. 17, find Q′ and Q″, the projections of point in which the given line meets the plane of which EF′ and EH″ are the traces; lastly, through P′, Q′, P″, Q″, draw P′Q′, P″Q″, which will be the horizontal and vertical projections of the line required.

For, because the plane of which EF′, EH″ are the traces is at right angles to the given line, and passes through the given point of which P′, P″ are the projections; therefore, the perpendicular on the given line from the given point must be in the plane F′EH″, and must pass through the point in which the given line passes through this plane, that is, through the point of which Q′ and Q″ are the horizontal and vertical projections: the line required must also pass through the given point of which P′, Q″, are the projections; consequently, P′ Q′, P″ Q″ are the horizontal and vertical traces of the line required.

PROBLEM XIX.

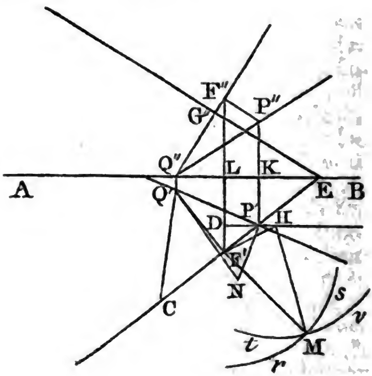
To find the angle contained between a given plane and a given straight line.

The construction may be given in general terms as follows: Take any point in the given straight line, from which by prob. 15, draw a straight line perpendicular to the given plane. Find by prob. 7, the angle contained by these two straight lines, which will be the complement of the angle required.

For, if we draw a straight line in the given plane, from the point in which the given line meets it to the point in which the forementioned perpendicular meets it, we shall have a right-angled triangle in which one of the acute angles is the angle contained by the given plane, and given straight line; and the other acute angle, is the angle contained by the given straight line and the perpendicular on the plane; and therefore, this latter angle is evidently the complement of the angle required.

I shall now illustrate this general construction by a particular example in which all the operations are exhibited.

Let AB be the ground line, EF′, EG″ the horizontal and vertical traces of the given plane: and P′Q′, P″Q″ the horizontal and vertical projections of the given straight line. From Q″ the point in which the vertical projection P″Q″ meets the ground line AB; draw Q″Q′ at right angles to AB, and Q′ is the point in which the given line meets the horizontal plane. from Q′, Q″ draw Q′F′, Q″G″ at right angles to EF′, EG″, and Q′F′, Q″G″ will, by prob. 15, be the projections of a straight line drawn from the point of which Q′, Q″ are the projections at right angles to the given plane.



Now, make F'C equal to LF" which is the height of the perpendicular to the plane at F', and Q'C is the distance of the point Q' from the point of which the projections are F' and F"; again, make P'N equal to KP", which is the height of the given line at P', and consequently Q'N is the distance at the point Q' from the point of the given line of which P' and P" are the projections. Through P' draw DP'H parallel to AB, make DH equal to P"F", and F'H will be the distance between the two points of which the horizontal projections are P', F', and their vertical projections P" and F". Lastly, with the centres F' and H, and distances F'M and HM equal to Q'C, and Q'N, describe two arcs intersecting in M, and F'MH is the angle contained by the given straight line, and the perpendicular to the plane having Q'F' and Q"G" for its horizontal and vertical projections, and is therefore the complement of the angle required.

PROBLEM XX.

To find the angle contained by two given planes.

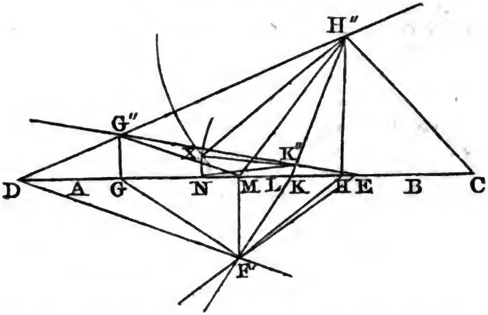
General Construction.

Find by prob. 10, the straight line which is the common section of the given planes, through any point of which draw by prob. 16, a plane at right angles to it: by prob. 10, find the common sections of this plane with each of the given planes; and lastly, by prob. 7, find the angle contained by these two intersections, which will be the angle required.

Because a plane cuts the common section of the given planes at right angles, it is evident, that the intersections of this plane with the given planes are at right angles to the common section of the given planes: and consequently, the angle contained by these intersections is equal to the required angle contained by the given planes.

In the figure annexed, all the lines are drawn that are wanted in the construction; let AB be the ground line; DF', DG" the horizontal and vertical traces of one of the given planes; and EF', EG" those of the other.

Draw F'M, G"G at right angles to AB, and F'G, MG" are the horizontal and vertical projections of the common section of the given planes: draw F'L, LH" at right angles to GF', G"M, and F'LLH" are the traces of a plane at right angles to the common section of the given planes. Draw H"H at right angles to AB, and F'H, MH" are the projections of the intersection of the plane F'LH" with the given plane F'DH": and HC being made equal to HF', H"C is the distance in this intersection from F' to H"; in like manner, by making KN equal to KF', we have K"N



equal to the distance from F′ to K″ in the intersection of the plane F′LH″; and the given plane F′EG″, and H″K″ is the distance between the points H″ and K″ which are both in the vertical plane; now with the centres H″, K″ and distances H″C, K″N describe two arcs intersecting in X, and K″XH″ will be the angle required.

PROBLEM XXI.

Through a given straight line to draw a plane parallel to another given straight line.

Through any point of the line through which the plane must pass, draw by prob. 6. a straight line parallel to the other given line. Find by prob. 3, the points of the horizontal plane in which this parallel and the first-mentioned line pass through it; through these two points draw a straight line, which will be the horizontal trace of the plane required. In like manner, by finding the two points in which the parallel and first-mentioned line meet the vertical plane, and drawing a straight line through them, we have the vertical trace of the plane required.

Because the plane thus constructed passes through the points in which the first-mentioned line and parallel meet the fundamental planes; therefore, these lines are in the plane; but when two straight lines are parallel, any plane passing through one of them is parallel to the other.

Construction in which are drawn all the necessary lines.

Let AB be the ground line; P′F′, P″G″ the horizontal and vertical projections of the straight line through which the

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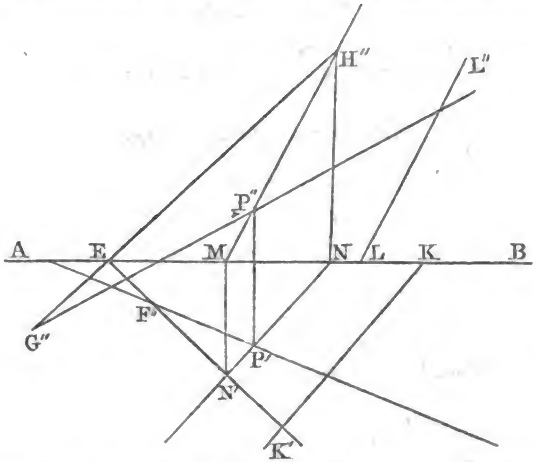
Note. The line KK″ at right angles to AB is wanting in the figure.

plane must pass, and KK′, LL″ the horizontal and vertical traces of the straight line to which the plane must be parallel.



Draw any straight line P′ P″ at right angles to AB, meeting the horizontal and vertical projections F′ F″ and P″G″ in P′ and P″; and P′, P″ will be the projections of a point in the first-mentioned line. Through P′ and P″ draw NP′N″, MP″H″ parallel to the projections KK′, LL″ of the second given line, and NN″, MH″, are the projections of the straight line passing through the point P′, P″ and parallel to the straight line KK′, LL″. Find F′ and N′ the points in which the line P′ F′, P″ G″, and the line NN′, MH″ meet the horizontal plane, and G″, H″ the points in which the same lines meet the vertical plane; and EF′N′, G″EH′ being drawn, will be the horizontal and vertical traces of the plane required.

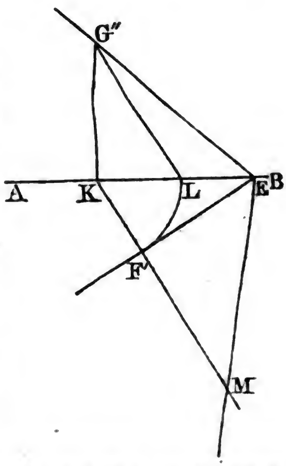
CHAPTER III.

Construction of the Cases in Spherics.

Right-angled Spherical triangles.

CASE I.

Given the two legs of a right-angled spherical triangle to find the angles and hypothenuse.



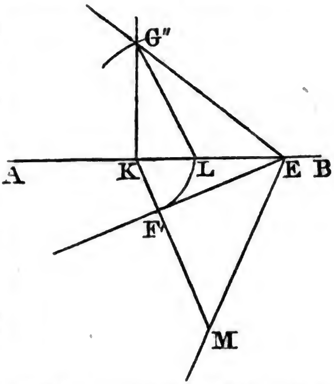
Let AB be the ground line, AEF′, AEG″ two angles of which the measures are the given sides. Find by prob. 11, the angles which the plane having EF′ and EG″ for its horizontal and vertical traces makes with the horizontal and vertical planes, and these angles will be the angles required.

Thus KLG″ is the angle of the spherical triangle which is opposite to the side that measures the angle AEG″. Again, find by prob. 12, the position M on the horizontal plane of the point G″ in the vertical trace EG″, and F′EM will be the angle of which the hypothenuse is the measure.

For, conceiving KEG to be at right angles to the horizontal plane AEF′ in the common section AB, and that a plane passes through EF′ and EG″, we shall evidently have a solid angle at E, of which the three plane sides are KEF′, KEG″, and the angle of which the sides are EF′, EG″, and which by construction is equal to the angle F′EM. This solid angle at E has its sides and the inclinations of these sides equal to the sides and angles of a right-angled spherical triangle: the right angle contained by the horizontal and vertical planes, being the right angle of the spherical triangle, and the inclination of the plane passing through EF′ and EG″, to the horizontal and vertical planes being the oblique angles of the triangle.

CASE II.

Given one leg, and the hypothenuse of a right-angled spherical triangle to find the three remaining parts.



Let AB be the ground line; AEF′, F′EM the given leg and hypothenuse. Draw from any point K of the ground line AB the straight line KF′M at right angles to EF; make KG equal to KF′; draw KG at right angles to AB, and make LG equal to F′M; then shall EF′, EG″ be the horizontal and vertical traces of the plane of the hypothenuse, AEG″ the leg required, and G″LK the angle adjacent to the given side or leg AEF′.

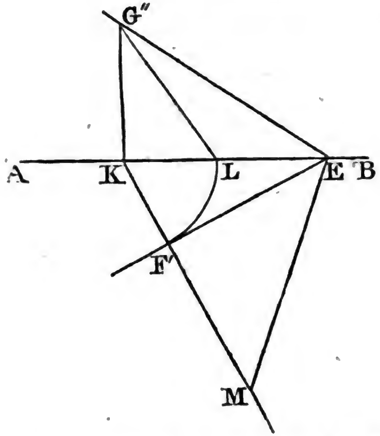
This construction is merely the converse of that in case 1. Because LG″ is by construction equal to F′M, therefore M is the horizontal position of the point G″ by the rotation of the plane of the hypothenuse about EF′; and therefore, the angle contained by F′E and EG″, when considered in the vertical plane, is equal to the given hypothenuse F′EM. Also, by prob. 11, G″LK is the inclination of the plane of the hypothenuse to the horizontal plane. By the same prob. we may determine the inclination of this plane of the hypothenuse to the vertical plane, which will be the angle of the spherical triangle adjacent to the side AEG″.

CASE III.

Given one leg and the adjacent angle of a right-angled triangle to determine the remaining parts.

Let AB be the ground line; and AEF′ the given leg on the horizontal plane. From any point K in the ground line AB draw KF′ at right angles to EF′; make KL equal to KF′; at L make the angle ALG″ equal to the given angle, and let LG″ meet the straight line KG″ at right angles to AB in G″; join EG″, and EF′, EG″ will be the horizontal and vertical traces of the plane of the hypothenuse; also, if F′M be made equal to LG″ the angle FEM will be the hypothenuse.

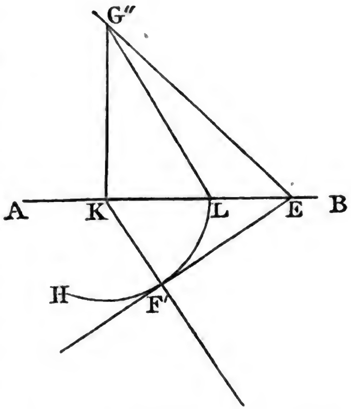
For, since KG″ is at right angles to AB, KF′ to EF′, and KL equal to KF′; therefore the given angle KLG″ is by prob. 11, the inclination to the horizontal plane of the plane of which EF′ and EG″ are the horizontal and vertical traces, which is the



spherical angle contained by the hypothenuse and base. Therefore KEG″ is the remaining leg: and, since M is the horizontal position of the point G″ by rotation about EF′, therefore F′EM is the hypothenuse.

CASE IV.

Given one leg and the opposite angle to find the remaining parts of the triangle.



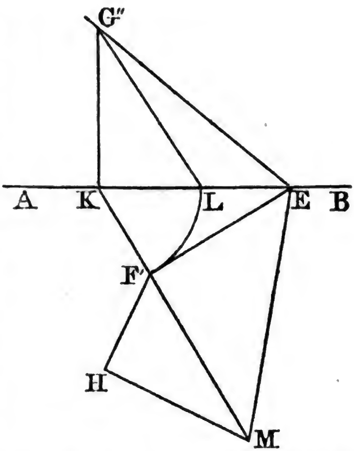
Let AB be the ground line; and AEG″ the angle of which the given leg of which the given leg is the measure. From K, any point in AB draw KG″ at right angles to AB, and make KG″L equal to the complement of the given angle: With the centre K, and distance KL describe a circular arc LF′H and draw EF′ to touch the arc LF′H in F′, then EF′, EG″ are the horizontal and vertical traces of the plane of the hypothenuse, and of course the angle AEF′ has for its measure the remaining leg of the triangle.

For, if we join KF′, the angle KF′E is a right angle, and therefore KLG″ is the angle made with the horizontal plane by the plane of which the horizontal and vertical traces are EF′, EG″.

The hypothenuse may be determined as before.

CASE V.

Given the hypothenuse and one angle of a spherical triangle, to determine the remaining parts.



Let F′EM be the given hypothenuse, and from any point F′ in the side EF′ draw F′M at right angles to F′E : at F′ make MF′H equal to the given angle, and draw MH at right angles to F′H.

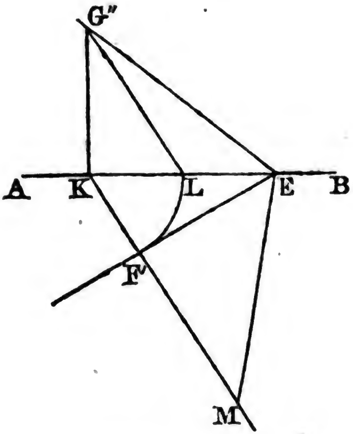
In MF′ produced take F′K equal to F′H ; join EK, and draw KG″ at right angles to EK ; make KL equal to KF′, and LG″ equal to F′M ; join EG″, and KEF′, KEG″ will be the angles of which the required legs are the measures.

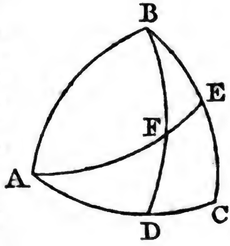
For, if AB be the ground line ; and EF′, EG″ the horizontal and vertical traces of a plane, it is plain that the inclination of this plane to the horizontal plane is KLG″, which by the construction is equal to MF′H : and the hypothenuse or angle contained by EF′ horizontal and EG″ in the vertical plane is equal to the given angle F′EM.

CASE VI.

Given the angles of a right-angled spherical triangle to determine the legs and the hypothenuse.

Let AB be the ground line : with one of the given angles KLG″ as an angle and the complement KEG″ of the other given angle as an opposite side, construct a spherical triangle by case 4, and the complements of its remaining side AEF′, of its hypothenuse F′EM, and of its remaining angle, will be the hypothenuse and two legs of the triangle required.





Let ABC be an equilateral, equiangular, rectangular spherical triangle, all the sides being quadrants and all the angles right angles ; and let any two great circles BFD, AFE pass through B and A, and BD, AE will be quadrants, and the angles at D and E right angles ; and therefore ADF, BEF are right-angled triangles, having the angles at F equal ; and the remaining parts of each are complements of those of the other: thus, BF is the complement of DF ; BE is the complement of CE or of the angle DAF, and the angle EBF or its measure DC is the complement of the side AD.

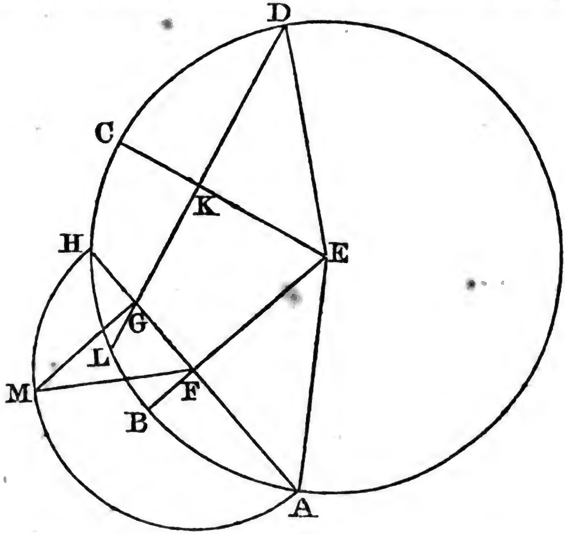
Suppose now that the angles DAF, DFA, are given to find the remaining parts of the triangle ADF. From what has just been shown, we have in the triangle BEF, the angle BFE and the opposite side BE which is the complement of the given angle DAF. It is evident, therefore, that when two angles are given in the triangle ADF, we have only to construct the triangle BEF of which the angle BFE and the opposite side BE are given ; which is effected as in case 4. And afterwards we obtain AF, AD, DF, by taking the complements of FE, the angle EBF, and the hypothenuse BF.

CASES OF OBLIQUE ANGLED SPHERICAL TRIANGLES.

CASE I.

Given the three sides of a spherical triangle to determine the angles.

With the centre E and any radius AE, describe on the horizontal plane a circle ABCD, in the circumference of which take AB, BC, CD equal to the three given sides: from A and D draw AFH and DKL at right angles to EB and EC, intersecting each other in G. On the chord AH as a diameter describe the semicircle AMH; draw GM at right angles to AH; join FM, and MFH will be the angle contained by the sides equal to AB and BC.



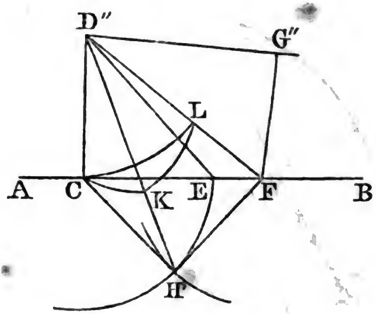
Supposing E to be the centre, and AE the radius of the sphere, to which the spherical triangle belongs; let the semicircle AMH revolve from its horizontal to a vertical position, and suppose the angle BEA to revolve about the side BE; it is plain, that the point A will describe the circumference AMH. In the same manner, the point D will describe the circumfer-

ence of a vertical circle on the diameter DL. It is evident therefore, that MG will thus become the common section of these two vertical semicircles, and M the point with which A and D coincide, when EA and ED are coincident. And because when the triangle GMF is vertical, MF will be at right angles to EB, the angle GFM will be the inclination of the plane of the angle BEA to the horizontal plane when the points A and M coincide.

By a similar construction, we may determine the remaining angles.

Another Construction.

Let AB be the ground line; in which take any point C, and draw CD″ at right angles to AB. At any point D″ in this perpendicular make the angles CD″E, CD″F, and FD″G″ equal to



the angles of which the given sides of the spherical triangle are the measures: make D″G″ equal to D″E: join FG″; and with the centres C and F, and distances CE, FG″ describe two arcs intersecting in H′; join CH′, and FCH′ will be the angle opposite to the side which measures the angle FD″G″.

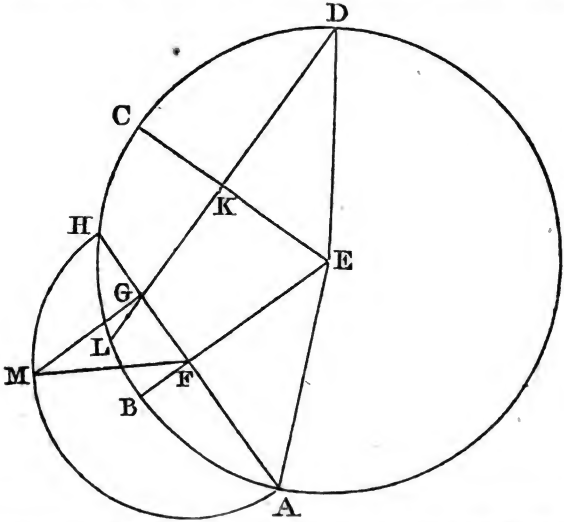
Conceive the vertical plane to stand at right angles to the horizontal, and in this situation suppose DH″ to be joined; also draw FH. Now CD″F being at right angles to the plane CFH′, it is evident that CFH′D″ is a triangular pyramid of which the base is CFH′, and vertex D″: and since CH′ is equal to CE, and the angles D″CH′, DCE equal being right angles, therefore the angle CD″H′ is equal to CD″E, and D″H′ to D″E. Also, because D″H′ is equal to D″E, that is, to D″G″, and FH′ to FG″, and D″F common to the two triangles FD″H′, FD″G″; therefore the angle FD″H is equal to the given angle FD″G″; and thus it is manifest,

that the three angles at the summit D″ of the pyramid, viz. CD″H′, CD″F, FD″H′ are equal to the three given angles CD″E′, CD″F, FD″G″; consequently, if with the centre D″ and radius D″E we describe a spherical surface, its intersections with the planes of the angles CD″H′; CD″F, FD″H, will be the spherical triangle, CLK having the given sides. And because D″C is at right angles to CF and CH′, therefore CF and CH′ are the tangents of the arcs CL and CK, and therefore the plane angle FCH′ is equal to the spherical angle LCK.

CASE II.

Given two sides and the contained angle to determine the remaining parts of the triangle.

With the centre E and any distance EA, describe on the horizontal plane, the circle ABCD; in the circumference of which take AB, BC equal to the two given sides. Make BH equal to AB; join AH, and on it as a diameter describe the se-

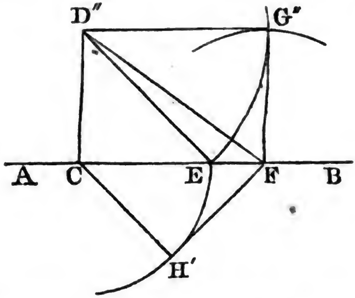


micircle AMH: at the centre F make the angle HFM equal to the given contained angle: draw MG at right angles to AH, and GKD at right angles to CE, and CD shall be the side of the spherical triangle that is opposite to the given angle.

The truth of this construction is evident from the demonstration of the first construction given to the preceding case.

Another Construction of Case 2.

Let AB be the ground line; in which take any point C, and draw CD″ of any length at right angles to AB. Make the angles CD″E, CD″F equal to the two given sides, and ECH′ equal to the given angle: with the centre C and distance CE describe the arc EH′, and join FH′; lastly, make D″G″, FG″ equal to DE and FH′ respectively, and the measure of the angle FD″G″ will be the side required.



This construction is evident from the demonstration of the second construction given to the preceding case.

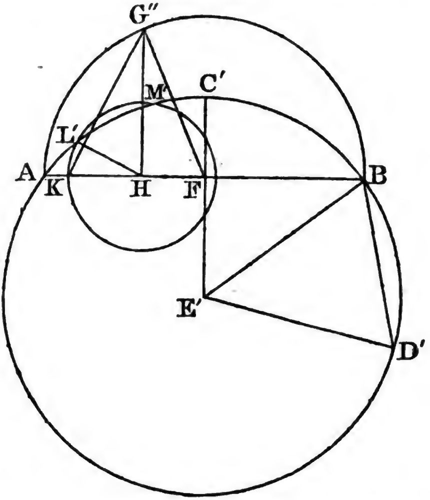
CASE III.

Given two sides and an angle opposite to one of them to find the remaining side.

With the centre E′ and any radius C′E′ describe on the horizontal plane a circle ABD′, in the circumference of which take C′B, BD′ equal to the given sides, and C′A equal to BC′. Join AB, on which as a diameter, describe the semicircle AG″B: at F the centre of AC′B make the angle AFG″ equal to the given angle; from G″ draw G″H at right angles to AB, and make GK equal to the chord BD′; with the centre H and radius KH describe the circle KLM, cutting ACB in L′ and M′, and C′L′, or C′M′ will be the required side of the triangle.

Conceive AB to be the ground line; and AG″B to be on the vertical plane, making right angles with the horizontal plane; also, suppose a sphere to be described with the centre E′ and

distance $E'C'$; then it is evident that the semicircumference $AG''B$, is in the surface of the sphere, that C' is one of its poles; and, consequently, that the distance from C' to G'' in the arc of a great circle of the sphere, is equal to the arc $C'B$.



Imagine a plane to pass through $C'E'$, and make with AF an angle equal to the given angle; this plane will evidently pass through the point G'' of the vertical plane, because the angle $G''FH$ was made equal to the given angle, and the arc of the great circle between C' and G'' lying in the forementioned plane, will make with $C'A$ an angle equal to the given angle.

Again, $G''H$ being at right angles to the horizontal plane, every point of the circumference $KL'M'$ is equally distant from G''; consequently the arc of the great circle passing from G'' to L' or M' has its chord equal to $G''H$, or the chord BD', and therefore the arc between G'' and L', or between G'' and M, is equal to the given side BD'.

Thus it is plain that C', G'' and L' or M' are the angular points of a spherical triangle having two of its sides equal to $C'B$ and BD', and its angle opposite the side $C'L'$ or $C'M'$ equal to the given angle; and therefore, $C'L'$ or $C'M'$ is the third side required.

The three remaining cases of oblique-angled spherical triangles are constructed exactly as the three preceding cases, by means of the polar triangle described, page 30, this vol. See the author's figure to theorem IV of Spherics.

CASE IV.

If the three angles of the spherical triangle DEF be given to find the sides; we take the supplements of the given angles, we have the three sides of the supplemental or polar triangle ABC; and, the angles of the triangle being found by case 1, their supplements will be the sides required in the triangle DEF.

CASE V.

If the side DE and the adjacent angles at D and E, be given to find the remaining parts of the triangle DEF; by taking the supplements of the given parts we have the two sides AB, AC and the contained angle BAC, to find the remaining parts of the triangle ABC by case 2.

CASE VI.

If the side DF and the two angles at D and E be given in the triangle DEF to find the remaining parts: we have in the triangle ABC the two sides AB, AC, and the angle ABC opposite to one of them, to find the remaining side by case 3.

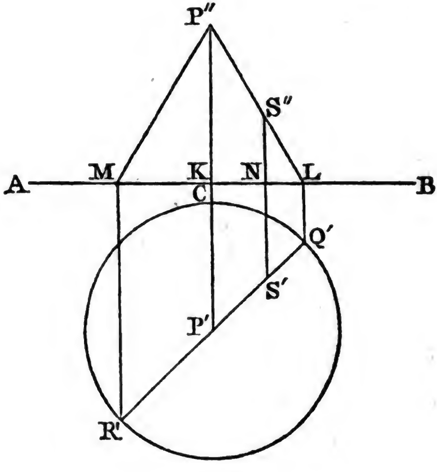
CHAPTER IV.

CONSTRUCTION OF THE CONIC SECTIONS.

PROBLEM I.

To construct a conic surface.

1. Draw the ground line AB, in the horizontal plane; take any point P′ for the centre of the circular base of the cone, and with the radius of the base describe about P′ as a centre, the circle CQ′R′ for the base of the cone. From P′ let fall on AB, the perpendicular P′K, in which produced take KP″ equal to the axis of the cone; that is to the distance between the vertex of the cone, and the centre P′ of the base; and P″ will be the vertical projection of the vertex of the cone.



Because the axis of the cone is at right angles to the base, it is evident that the horizontal projection of the axis is simply the point P′; and KP″ at right angles to the ground line AB is its vertical projection: and therefore, P′ and P″ are the horizontal and vertical projections of the vertex of the cone.

2. To project the slant side of the cone, take any point Q′ in the circumference of the base, through which and the centre P′ draw Q′P′R′ a diameter of the base; also, from Q′ draw Q′L at right angles to the ground line AB; and join P″L; then will P′Q′ and P″L be the horizontal projections of the slant side of the cone which passes through the point Q′.

This construction of the slant side is evident, because L being the vertical projection of the horizontal point Q′; therefore P′ and Q′ are the horizontal projections of two points of the slant side, and P″L, the corresponding vertical projections; and consequently, P′Q′, P″L are the required projections of the slant side passing through Q′.

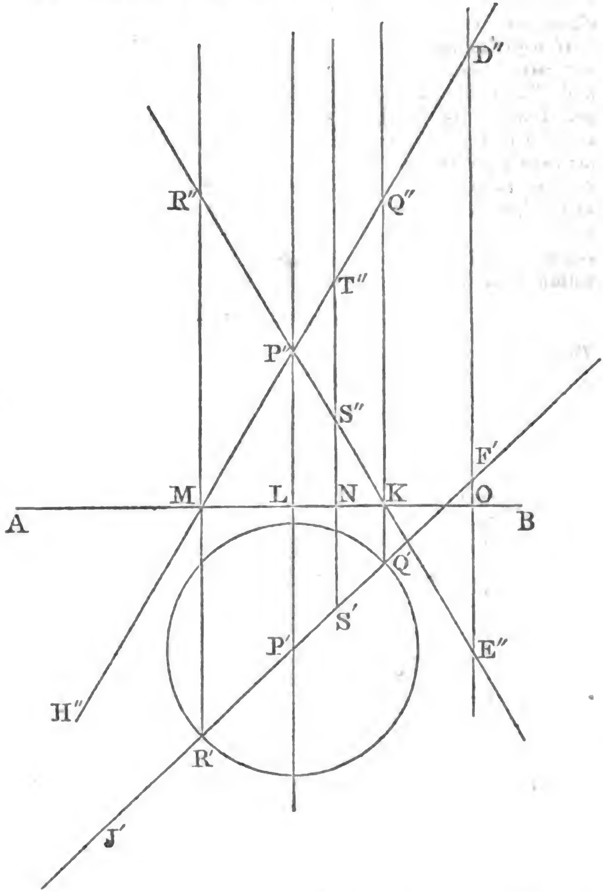
If we make a similar construction for the slant side passing through R′ we have the construction of the two slant sides of the cone, in which the curve surface of the cone is intersected by a vertical plane passing through the axis of the cone. Thus P″L and P″M are the vertical projections on the opposite slant sides passing through the extremities of the diameter Q′R′ of the base, and the opposite radii P′Q′, P′R′ are the corresponding horizontal projections.

3. To find the vertical projection of any point of the surface corresponding to any given horizontal projection. Let S′ be any given horizontal projection of a point of the curve surface of the cone; draw the radius P′S′Q′, and having constructed the slant side by its projections P′Q′, P″L, draw S′NS″ at right angles to the ground line AB, meeting the vertical projection P″L in S″, and S″ will be the vertical projection of that point of the conic surface which has S′ for its horizontal projection.

In the preceding construction we have considered only that part of the whole conic surface which is between the vertex and base; but as the conic surface may be extended indefinitely downwards below the base, and upwards above the vertex, it is plain that the horizontal projection R′P′Q′ of the opposite slant sides as well as the vertical projections P″K, P″M, should be produced indefinitely both ways; that is, R′Q′ towards F′ and J′; and P″K, P″M towards E″, R″; H″, D″.

Now, the vertical projections R″E″, D″H″ being both in a vertical plane passing through F′J′, if we produce S′S″ to meet D″H″ in T″, we shall have T″ for the vertical projection of the point in which a perpendicular to the horizontal plane at S′ meets the slant side which passes through the point R of the base; this perpendicular therefore meets the conic surface in two points, of which S″ and T″ are the vertical projections, the horizontal projections being coincident in the points S′. Also, the vertical ordinates of these two points being

NS″ and NT″, it is evident that the part of the perpendicular at S′ which is projected into S″T″, falls without the conic surface; the remaining parts of it falling within the upper and lower divisions of the conic surface.



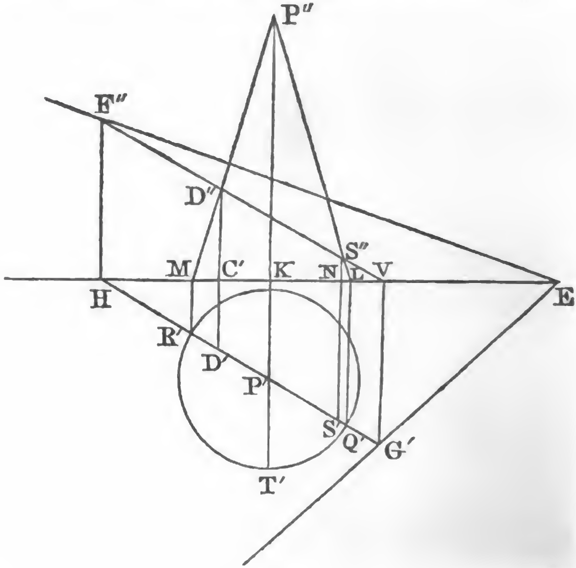
If we produce Q′K to meet D″H″ in Q″, the point Q″ will be the vertical projection of the point in which the perpendicular from Q′ to the horizontal plane, meets the upper division of the conic surface. In like manner, if R′M be produced to

R″, we have the vertical projection of the point in which the perpendicular at R′ meets the slant side of the cone passing through Q′ : and, because MR″ and KQ″ are equal, as is evident from the construction, it follows that R″ and Q″ are the vertical projections of two points diametrically opposite in a circular section of the upper conic surface parallel to the base.

If we take any point F′ in R′Q′ produced, and draw D′′F′, OE″ at right angles to the ground line AB, it is plain that D″ and E″ are the vertical projections of the points in the upper and lower divisions of the conic surface through which a straight line passes, that is, perpendicular to the horizontal plane at F′, so that OF′, OD″ are the horizontal and vertical ordinates of the points of intersection in the upper division, and DF′, OE″ in the lower. Also that part of the perpendicular at F′, that is represented by D″E″, falls without the conic surface : and the remaining parts above D″ and below E″ fall within the upper and lower divisions of the conic surface.

PROBLEM II.

To find the point in which a given plane is cut by the given slant side of a cone.

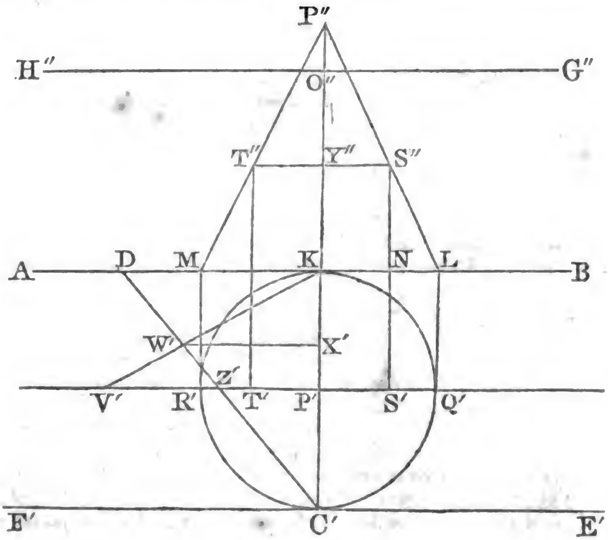


Let EK be the ground line ; P′ the centre of the base T′R′Q′ on the horizontal plane ; P″ the vertical projection of the vertex of the cone, Q′ any given point in the circumference of the base, and R′Q′, P″L the horizontal and vertical projections of the slant side passing through Q′ ; also, let EG′, EF″ be the horizontal and vertical traces of the given plane.

Since the slant side is given by its horizontal and vertical projections P′Q′, P″L, and the plane by its traces, we have only to find the projections of the required point by prob. 17, chap. 2. The operation is as follows : Produce Q′R′ to G′ and H ; draw G′V and HF″ at right angles to EH ; join VF′ cutting P″L in S″, and draw S″S′ at right angles to EH cutting EH and P′Q′ in N and S′, and NS′, NS″ will be the horizontal and vertical projections of the point required.

In a similar manner we find the horizontal and vertical projections D′ and D″ of the point in which the given plane is cut by the slant side which passes through R the other extremity of the diameter Q′R′.

If the point E be at an infinite distance, the traces EG′ and EF″ become parallel to the ground line ; and this circumstance produces a variation in the method of construction for some points that may require farther illustration.



Let AB be the ground line; KQ′C′R′ the base of the cone on the horizontal plane, P′ its centre, KP″ the vertical projection of the axis of the cone; and EF′, G″H″ the traces of the given plane, which are parallel to AB.

Suppose the horizontal projection Q′R′ of two opposite slant sides to be the diameter of the base parallel to AB; and therefore P″L, P″M the vertical projections of those sides. To determine the points in which the plane meets those slant sides we may proceed as follows:

Make KD equal to KO, join C′D cutting Q′R′ in Z′; make KY″ equal to P′Z′, and through Y″ draw S″T″ parallel to AB; then S″S′ and T″T′ being drawn perpendicular to AB, will give the horizontal projections of the required points; and S″, T″ the corresponding vertical projections.

Again, to determine the intersection of the plane and slant side passing through K: make P′V′ equal to KP″ which is the altitude of the cone; draw KV′ intersecting C′D in W′, make W′X′ parallel to AB, and X′ will be the horizontal projection of the required point, and X′W will be equal to the vertical ordinate, the horizontal ordinate being KX′.

PROBLEM III.

To construct the horizontal projection of the curve made by the intersection of a given plane with a given conic surface.

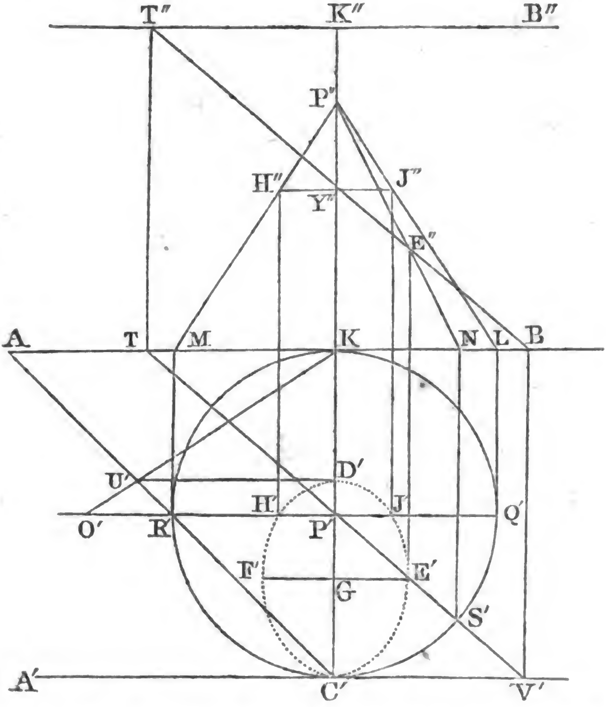
Let AB be the ground line; KQ′C′ the circumference of the base on the horizontal plane, touching the ground line in K; let P′ be the centre of the base, and at the same time the horizontal projection of the axis, and vertex; and KP″ the vertical projection of the axis.

Suppose the given intersecting plane to be parallel to the ground line, or which is the same in effect, let the horizontal and vertical traces A′V′, T″B″ of the given plane be parallel to AB, and suppose the horizontal trace A′V′ touch the base of the cone in C′.

To find the axis of the projection, make KA equal to KK″, and join C′A in P′R′O′ parallel to AB: take P′O′ equal to KP″; join KO′ intersecting AC′ in U′, and from U′ draw U′D′ parallel to AB, and C′D′ will be the axis of the projection.

Again, to find the points in which the curve to be projected cuts the diameter Q′R′ parallel to AB; from Q′ and R′ draw Q′L, R′M perpendicular to AB, and P″L, P″M will be the projections of the slant side passing through Q′ and R′: let C′A meet P′O′ in R′, and having made KY″ equal P′R′, draw H″Y″J″ parallel to AB, and J″J′, H″H′ parallel to P″P′ and J″H′ will be the points required in Q′R′.

To find the point in which the curve meets any other radius P′S′, draw S′N at right angles to AB; join P″N which is the vertical projection of the slant side passing through S′: produce P′S′ to meet A′V′ and AB in V′ and T; draw V′B, TT″ parallel to P′P″, and join BT″ cutting P″N in E″; draw E′E′ parallel to P′P′, and E′ will be the point in which P′S′ is intersected by the curve.



In a similar manner we may find any number of points in the required section C′E′D′F′.

When the points V and T become too remote to be consequently used in the construction, we may find the required points of the curve by the method used in determining the intersection of the plane by the slant side passing through K.

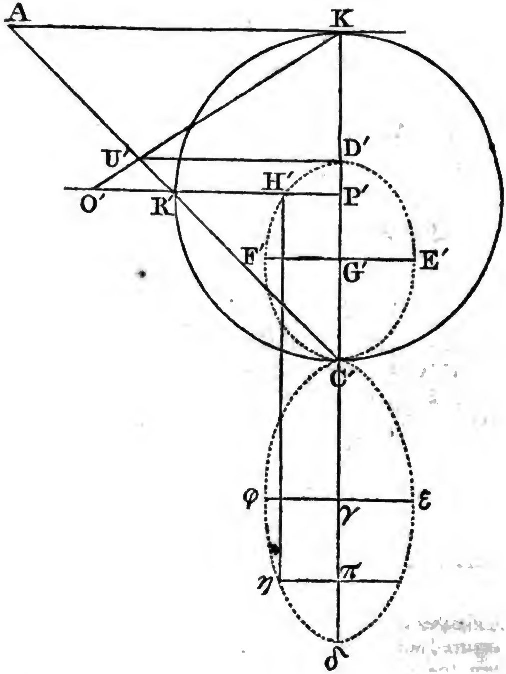
In this example, in which the cone is divided by the plane into upper and under parts of the conic section, is called an

ellipse; the curve C′E′D′F′ is therefore the projection of an ellipse, and consequently, C′E′D′F′ is also an ellipse.

PROBLEM IV.

To construct the ellipse of which the curve C′E′D′F′ in the preceding problem is the horizontal.

This problem is readily solved by finding by prob. 12, chap. II. the positions of the points of the curve on the horizontal plane by the revolution of the intersecting plane to a coincidence with the horizontal plane.



Or, we may proceed as follows, which is nearly equivalent. Because KO′ is the position on the horizontal plane of the slant side passing through H, and KC′A the angle of elevation of the given plane; it is plain, that CU′ is the transverse axis

of the ellipse section required. Make c′δ equal to c′v′, and δ will be the position on the horizontal plane of the vertex determined by v′.

In like manner, take c′π equal to cr′ and make the perpendicular πη equal to p′p′, and η will be a point in the required section. In a similar manner, we may find any number of points in the circumference of the required ellipse δec′φ.

It is evident, from this construction, that the ellipse δec′φ is derived from c′e′d′f′ by elongating each abscissa from c′ as cp′ in the constant ratio of c′d′ to c′v′; so that cp′ is to c′π as c′d′ to c′δ, while the semiordinate πη remains the same as p′h′.

And as the curve δec′φ is by the definitions of conic sections an ellipse, it is manifest from the constant ratio of the abscissas cp′ and c′π having a common semiordinate p′h′ or πη, that the projection c′e′d′f′ is also an ellipse.

PROBLEM V.

To construct the section of a cone by a plane parallel to the axis of the cone.

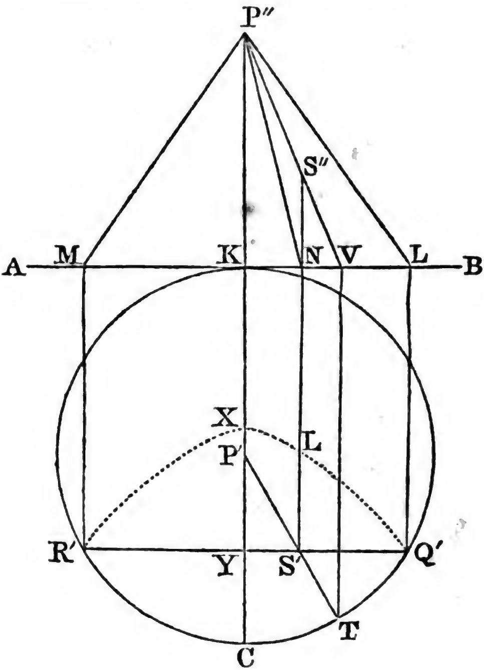
Let AB be the ground line; P′ the centre of the circular base KQ′R′ of the cone touching the ground line AB in K: produce the radius P′K to P″, and take KP″ equal to the axis of the cone which is supposed to be at right angles to the plane of its base, and consequently to the horizontal plane; then P″K is the vertical projection of the axis, and P″ of the vertex.

Suppose the cutting plane to be parallel to the vertical plane, and to intersect the horizontal plane in the straight line R′YQ′, which is therefore parallel to AB, and consequently perpendicular to the diameter CK.

Draw any radius P′S′T of the base, meeting R′Q′ in S″, and the circumference of the base in T. Find by prob. 1, chap. IV, the vertical projection P″V of the slant side passing through T, the corresponding horizontal projection of this slant side being P′T: through S′ draw S′SS″ at right angles to AB, and meeting VP″ in S″, and NS″ is the altitude of the conic surface at S′, because NS′, NS″ are evidently co-ordinates of a point of the slant slide passing through T.

And, since the cutting plane which passes through R′Q′ is perpendicular to the horizontal plane, it is evident that NS′, NS″ are the co-ordinates of the point in which the slant side

terminating in T penetrates the cutting plane; if therefore we make S′L equal to NS″, it is plain that L will be the position on the horizontal plane of the point denoted by S′, S″, by the revolution of the cutting plane about the intersection R′ Q′.



By a similar construction, we may determine any number of points in the curve Q′LXR′, which will be the section required.

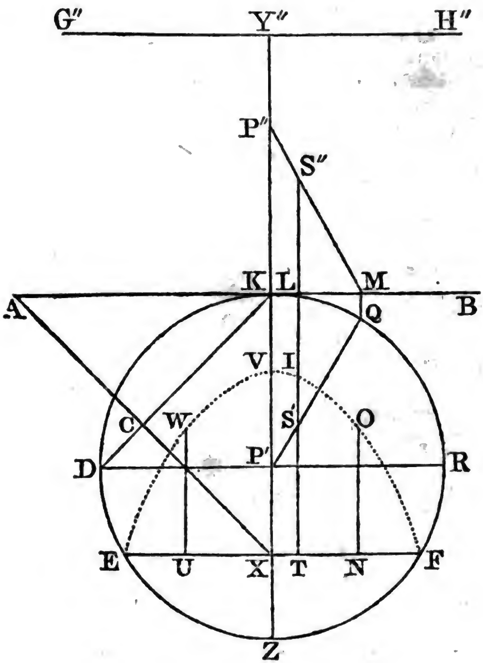
The curve required may be obtained still more simply by merely finding the perpendiculars NS″H, and describing the curve through L, S″, M, &c. without determining the corresponding points in Q′LXR′.

It is evident that the plane meeting the base at right angles in P′YQ′ must also meet the upper division of the conic surface, and produce another section equal and similar to Q′XR′. The curve determined by this construction is an hyperbola.

PROBLEM VI.

To construct the intersection of a conic surface by a plane parallel to one of the slant sides of the cone.

Let AB be the ground line; P′ the centre of the cone's base, which is supposed to be coincident with the horizontal plane; and let the base EFK touch the ground line in K: in P′K produced, take KP″ equal to the altitude or axis of the cone, and P″ is the vertical projection of the summit of the cone. Let the cutting plane be parallel to the ground line, and meet the



base in the horizontal trace EF, which will consequently be parallel to AB: in KP″ produced if necessary, take KY″ a fourth

proportional to the three straight lines KP′, KX, KP″, and the straight line G″Y″H″ parallel to AB, will be the vertical trace of the cutting plane.

The angle which the slant side passing through Z makes with the horizontal plane is evidently the acute angle at the base of a right-angled plane triangle of which the base is ZP′, and perpendicular equal to KP″; and the angle which the cutting plane makes with the horizontal plane is also the acute angle at the base of a right-angled triangle of which the base is XK and perpendicular KY″; and since these two triangles are in the same plane and have the bases and altitudes proportionals, it is plain that the acute angles at their bases are equal, and that the slant side passing through Z is parallel to the plane of which the traces are EF, G″H″.

To construct the curve of intersection draw any radius P′Q; from Q draw QM at right angles to AB, and join P″M, then P′Q, and P″M are the horizontal and vertical traces of the slant side passing through Q. Find by prob. 2. chap. IV. the horizontal and vertical projections S′ and S″ of the point in which this slant side meets the cutting plane; and by prob. 12. chap. II. find I the position on the horizontal plane of the point of which S′ and S″ are the projections by the rotation of the cutting plane about the intersection EF, and I is a point in the required curve.

In a similar manner we may proceed in determining any number of points in the required curve FIVE.

The ordinates NO, UW are obtained by the construction given in prob. 2, chap. IV. for the slant sides passing through the extremities of the diameter DR parallel to the ground line AB.

The vertex V is found by taking KA equal to KY″, and P′D equal to KP″; then drawing AX and KD, we have the position C′ of the vertex of the curve on the horizontal plane; and therefore making XV equal to XC, the point V will be the vertex of the curve.

It is obvious that the curve FIVE is a parabola.

THE END.